

Weierstrass type expression of curves of genus two and modular forms. KOMATSU Makoto¹

Abstract

The parameter space of versal deformation of curve singularity of type A_4 has a structure of $\mathbf{C}^*$ -bundle in a wider sense. The aim of this paper is to clarify the structure of the bundle by a period mapping.

1 Introduction.

For any positive integer n , zeros of a polynomial $-y^2 + x^{n+1}$ in $\mathbf{C}^2$ gives a curve singularity of type A_n . And the following polynomial

$$F_{A_n}(x, y, t) := -y^2 + x^{n+1} + t_2 x^{n-1} + \cdots + t_n x + t_{n+1}$$

expresses versal deformation of the above singularity. That is,

$$\Xi_{A_n} := \{(x, y, t) \in \mathbf{C}^2 \times \mathbf{C}^n \mid F_{A_n}(x, y, t) = 0\}$$

is called versal deformation of curve singularity of type A_n . The parameter space $\mathbf{C}^n$ is denoted by S_{A_n} :

$$S_{A_n} := \mathbf{C}^n (\ni t = (t_2, \dots, t_{n+1})).$$

On Ξ_{A_n} and S_{A_n} we define a $\mathbf{C}^*$ -action as

$$\lambda \cdot (x, y, t) := \begin{cases} (\lambda x, \lambda^{\frac{n+1}{2}} y, \lambda \cdot t) & (n : \text{odd}) \\ (\lambda^2 x, \lambda^{n+1} y, \lambda \cdot t) & (n : \text{even}) \end{cases}$$

where

$$\lambda \cdot t := \begin{cases} (\lambda^2 t_2, \dots, \lambda^{n+1} t_{n+1}) & (n : \text{odd}) \\ (\lambda^4 t_2, \dots, \lambda^{2n+2} t_{n+1}) & (n : \text{even}). \end{cases}$$

By the action, the parameter space S_{A_n} is regarded as the total space of a $\mathbf{C}^*$ -bundle in a wider sense. here we think of the following problem.

Problem 1.1 Clarify the structure of the above $\mathbf{C}^*$ -bundle S_{A_n} .

At the present time, only for $n = 1$ and $n = 2$, answer to the problem is already known. As for the case $n = 1$, the problem is trivial, because the base space $\mathbf{C}^* \backslash S_{A_1}$ consists of one point. As for the case $n = 2$, answer to the problem is a classical result, which we will see later (in subsection 2.4). In the present paper we think of the problem for $n = 4$. Here we avoid the problem for $n = 3$. If n is an odd integer and $n \geq 3$, there are some circumstances, in which, the case of A_n is rather different from that of A_2 . Therefore we cannot apply the way used in solving the problem of the case $n = 2$, simply, to the problem for the n . As for the problem for the n , we have no idea now. In the case $n = 2$, using a period mapping and applying a well-known frame of automorphic forms, we can see that the transition functions of the bundle S_{A_2} are given as a factor of automorphy. In the following section we review the frame of automorphic forms.

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2 A frame of automorphic forms.

In this section we review a well-known frame of automorphic forms.

2.1 Equivariant group action on a trivial bundle and a factor of automorphy.

Suppose X be a complex manifold, and G be a group acting on X discontinuously. Then the following (2-1-1), (2-1-2) are equivariant.

(2-1-1) To give a factor of automorphy $j : G \times X \rightarrow \mathbb{C}^*$.

(2-1-2) To give a G -action on $\mathbb{C}^* \times X$ which satisfies the following (i),(ii).

- (i) The G -action is commutative to the natural $\mathbb{C}^*$ -action on $\mathbb{C}^* \times X$.
- (ii) The G -action is equivariant to the natural projection $\mathbb{C}^* \times X \rightarrow X$.

In fact, if a factor of automorphy j is given, we can give a G -action on $\mathbb{C}^* \times X$ using j as follows:

$$\mathbb{C}^* \times X \ni (\lambda, x) \xrightarrow{\sigma} (j(\sigma, x)^{-1}\lambda, \sigma(x)) \in \mathbb{C}^* \times X \quad (\sigma \in G). \quad (1)$$

It can be easily seen that this G -action satisfy the above (i) and (ii). On the other hand, suppose that a G -action on $\mathbb{C}^* \times X$ satisfying (i) and (ii) is given. Then we define a map $j : G \times X \rightarrow \mathbb{C}^*$ by the following relation:

$$(1, x) \xrightarrow{\sigma} (j(\sigma, x)^{-1}, \sigma(x)) \quad (\sigma \in G, x \in X). \quad (2)$$

Then this j is a factor of automorphy. Those two procedures now explained are inverse to each other.

2.2 Invariant ring and ring of automorphic forms.

In general, when a group G is acting on a ring R , we denote by R^G the G -invariant subring of R . And for any complex analytic space Y , we denote by $\mathcal{O}(Y)$ the ring of all of holomorphic functions on Y .

When (2-1-1) (or, equivariantly, (2-1-2)) is satisfied, there are well known relations of four rings as follows:

$$\begin{aligned} & [\text{the ring of } (G, j)\text{-automorphic forms on } X] \\ & \cong \mathcal{O}(X)[\lambda, \lambda^{-1}]^G \subseteq \mathcal{O}(\mathbb{C}^* \times X)^G \cong \mathcal{O}((\mathbb{C}^* \times X)/G). \end{aligned} \quad (3)$$

It seems that in (3), only the first isomorphism is non-trivial (at least, to me). Here we explain the correspondence which gives the first isomorphism of (3). Suppose f be an element of $\mathcal{O}(X)[\lambda, \lambda^{-1}]^G$. We express f as Laurent polynomial in λ :

$$f(\lambda, x) = \sum_{k \in \mathbb{Z}} \lambda^k f_k(x) \quad (\text{finite sum}) \quad (4)$$

where $f_k \in \mathcal{O}(X)$. From the expansion, f satisfies the equality

$$f(j(\sigma, x)^{-1}\lambda, \sigma(x)) = \sum_k j(\sigma, x)^{-k} \lambda^k f_k(\sigma(x)) \quad (5)$$

for any $\sigma \in G$. Because f is G -invariant, (1), (4) and (5) imply that

$$f_k(\sigma(x)) = j(\sigma, x)^k f_k(x) \quad (\forall \sigma \in G, \forall x \in X, \forall k \in \mathbb{Z}).$$

That is, f_k is a (G, j) -automorphic form of weight k . On the other hand, for given finite set $\{f_k\}$ (where each f_k is a (G, j) -automorphic form of weight k), if we define f by (4), we can easily see that f is an element of $\mathcal{O}(X)[\lambda, \lambda^{-1}]^G$.

2.3 Our plan.

We denote by D_{A_n} the discriminant set of S_{A_n} :

$$D_{A_n} := \{t \in S_{A_n} | F_{A_n}(x, 0, t) \text{ has multiple roots.}\} \quad (6)$$

We treat $S_{A_n} - D_{A_n}$ rather than S_{A_n} itself. Suppose that there exist X and G which make the left hand side of the following diagram

$$\begin{array}{ccccccc} S_{A_n} - D_{A_n} & \xleftarrow{\sim} & (C^* \times X)/G & \xleftarrow{u} & C^* \times X & \ni & (1, x) \\ \downarrow & & \downarrow & \nwarrow u \circ s & \downarrow & \uparrow s & \uparrow s \\ C^* \backslash (S_{A_n} - D_{A_n}) & \xleftarrow{\sim} & X/G & \xleftarrow{\sim} & X & \ni & x \end{array} \quad \text{Diagram-1}$$

commutative, where u is a natural projection, and s is a global section of the trivial bundle $C^* \times X \rightarrow X$ defined as in the above diagram. Then by (3), the ring $\mathbb{C}[t_2, \dots, t_{n+1}]$ is regarded as a subring of $\mathcal{O}(X)[\lambda, \lambda^{-1}]^G$, and hence it is regarded as a subring of the ring of (G, j) -automorphic forms. Moreover, transition functions of the bundle $S_{A_n} - D_{A_n}$ is given as a factor of automorphy j . By the way, the G -actions on the total space and on the base space of the bundle $C^* \times X \rightarrow X$ are equivariant to the projection. Hence, by the relation (2) the section s satisfies

$$s(\sigma(x)) = j(\sigma, x) \cdot \sigma(s(x)) \quad (\forall \sigma \in G, \forall x \in X).$$

Moreover, the C^* -actions on $(C^* \times X)/G$ and on $C^* \times X$ are equivariant to the map u . And, in addition, u is G -invariant. Therefore, we have

$$(u \circ s)(\sigma(x)) = j(\sigma, x) \cdot (u \circ s)(x) \quad (\forall \sigma \in G, \forall x \in X).$$

Keeping the above frame in mind, we consider the problem 1.1 for $n = 4$ according to the following plan.

(2-3-1) First we find X and G which give Diagram-1.

(2-3-2) Next we investigate the effect of G -action on the map $u \circ s$ to obtain a factor of automorphy j explicitly.

2.4 Example. (A_2 -type curve singularity.)

As an example, we review the answer to the problem 1.1 for $n = 2$. In order to adapt the problem to the theory of Weierstrass' elliptic function, we modify the definition of F_{A_2} as follows:

$$F_{A_2} := -y^2 + 4x^3 - g_2x - g_3.$$

Then $S_{A_2} = \mathbb{C}^2$ and $D_{A_2} = \{g \in S_{A_2} | g_2^3 - 27g_3^2 = 0\}$. In this case, using the following multi-valued holomorphic mapping:

$$S_{A_2} - D_{A_2} \ni g \mapsto \left(\int_{A(g)} \frac{dx}{y}, \int_{B(g)} \frac{dx}{y} \middle/ \int_{A(g)} \frac{dx}{y} \right) \in \mathbb{C}^* \times \mathbb{H}, \quad (7)$$

we can apply the above frame to $S_{A_2} - D_{A_2}$, where $G = SL(2, \mathbb{Z})$ and $X = \mathbb{H} := \{\tau \in \mathbb{C} | \Im \tau > 0\}$. As a consequence, we obtain that $S_{A_2} - D_{A_2} \cong \mathbb{C}^* \times \mathbb{H} / SL(2, \mathbb{Z})$. Moreover, we have $j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) = c\tau + d$, and obtain the expression of g_i ($i = 2, 3$) as (G, j) -automorphic forms, which coincide to the well-known expressions as Eisenstein series.

3 Definition of period mapping.

We denote that $S := S_{A_4}$, $\Xi := \Xi_{A_4}$, and $D := D_{A_4}$. Discriminant of the polynomial $F_{A_4}(x, 0, t) \in (\mathbb{C}[t])[x]$ is as follows:

$$\begin{aligned} \Delta(t) := & 3125t_5^4 - 3750t_2t_3t_5^3 + 2000t_2t_4^2t_5^2 + 2250t_3^2t_4t_5^2 - 900t_2^3t_4t_5^2 + 825t_2^2t_3^2t_5^2 \\ & + 108t_2^5t_5^2 - 1600t_3t_4^3t_5 + 560t_2^2t_3t_4^2t_5 - 630t_2t_3^2t_4t_5 - 72t_2^4t_3t_4t_5 + 108t_3^5t_5 \\ & + 16t_2^3t_3^3t_5 + 256t_4^5 - 128t_2^2t_4^4 + 144t_2t_3^2t_4^3 + 16t_2^4t_4^3 - 27t_3^4t_4^2 - 4t_2^3t_3^2t_4^2. \end{aligned}$$

By (6), we have $D = \{t \in S \mid \Delta(t) = 0\}$. In addition, π denotes the natural projection $\Xi \ni (x, y, t) \mapsto t \in \hat{S}$, and X_t denotes $\pi^{-1}(t)$. We take a point $t_0 \in S - D$. t_0 is used as a base point of the fundamental group of $S - D$. Projection $\pi : \Xi - \pi^{-1}(D) \rightarrow S - D$ has the property of local triviality. Hence $\pi_1(S - D, t_0)$ acts on $H_1(X_{t_0}, \mathbb{Z})$. Moreover, this action preserves the intersection form $\langle \cdot, \cdot \rangle$ on $H_1(X_{t_0}, \mathbb{Z})$. Therefore we have the following representation:

$$\rho : \pi_1(S - D, t_0) \longrightarrow \text{Aut}(H_1(X_{t_0}, \mathbb{Z}), \langle \cdot, \cdot \rangle)$$

(monodromy representation), where $\text{Aut}(H_1(X_{t_0}, \mathbb{Z}), \langle \cdot, \cdot \rangle)$ denotes all of automorphisms of $H_1(X_{t_0}, \mathbb{Z})$ which preserve the intersection form $\langle \cdot, \cdot \rangle$. $\Gamma := \rho(\pi_1(X_{t_0}, \mathbb{Z}))$ is called as monodromy group. We take a symplectic basis of $H_1(X_{t_0}, \mathbb{Z})$ as in Figure-1. Then by the basis, the following group isomorphism holds:

$$\text{Aut}(H_1(X_{t_0}, \mathbb{Z}), \langle \cdot, \cdot \rangle) \cong Sp(4, \mathbb{Z}), \quad (8)$$

and by the isomorphism, Γ is regarded as a subgroup of $Sp(4, \mathbb{Z})$. Now we define a covering space of $S - D$ as follows:

$$\widehat{S - D} := (\text{universal covering space of } S - D) / (\text{kernel of } \rho).$$

$\widehat{S-D}$ is called as monodromy covering. Natural projection $\widehat{S-D} \rightarrow S-D$ is denoted by σ . Here we can define a period mapping.

$$P : \widehat{S-D} \ni h \mapsto \begin{pmatrix} \omega_{11}(h) & \omega_{12}(h) & \omega_{13}(h) & \omega_{14}(h) \\ \omega_{21}(h) & \omega_{22}(h) & \omega_{23}(h) & \omega_{24}(h) \end{pmatrix} \in \mathbb{C}^{2 \times 4},$$

$$\omega_{ij}(h) := \int_{A_j(h)} \frac{x^{i-1} dx}{y},$$

where $A_1(h), A_2(h), A_3(h) = B_1(h), A_4(h) = B_2(h)$ are symplectic basis of $H_1(X_{\sigma(h)}, \mathbb{Z})$ and depend on h "continuously". That is, each $A_j(h)$ is a local system. Note that, On some $h_0 \in \sigma^{-1}(t_0)$, we take $A_j(h_0)$ ($j = 1, 2, 3, 4$) as in the Figure-1.

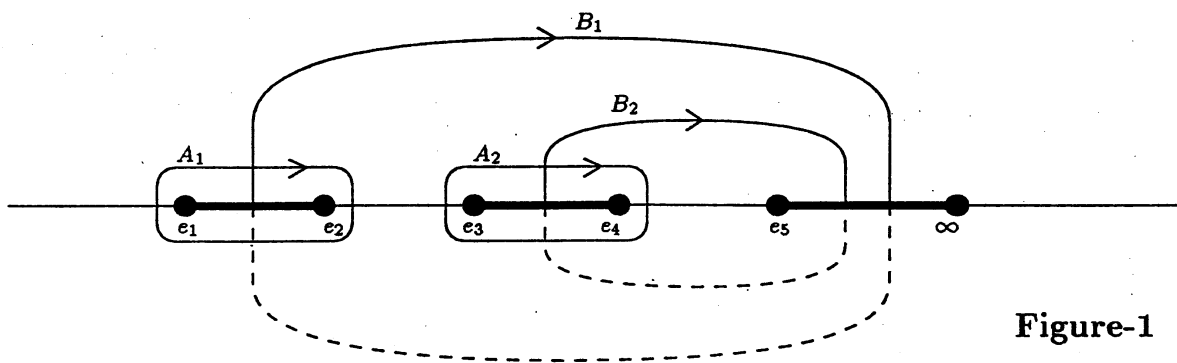


Figure-1

Remark. Each $A_j(t)$ is multi-valued on $S-D$. But, on $\widehat{S-D}$, each $A_j(t)$ is single-valued. In fact $\widehat{S-D}$ is the minimal covering on which each $A_j(t)$ is single-valued. Therefore the above period map P is single-valued.

By the definition of P , each $P(h)$ ($h \in \widehat{S-D}$) is a 2×4 matrix. We define a map φ as

$$\varphi : \text{Image}(P) \ni (\Omega_A \Omega_B) \mapsto (\Omega_A^{-1} \Omega_B) \in \mathbb{H}_2$$

where Ω_A, Ω_B denote the left 2×2 part, the right 2×2 part of the 2×4 matrix $P(h)$, respectively, and $\mathbb{H}_2$ denotes the Siegel upper half space of genus two.

4 Results(1) — towards (2-3-1).

The $\mathbb{C}^*$ -action on $S-D$ can be lifted to an action on $\widehat{S-D}$, where the $\mathbb{C}^*$ -action is fixed point free. Hence it can be easily seen that $\widehat{S-D}$ is regarded as the total space of a $\mathbb{C}^*$ -bundle in the strict sense. By the way, in the Problem for $n=2$, we can see that $S_{A_2-\widehat{D}_{A_2}}$ is isomorphic to the trivial bundle $\mathbb{C}^* \times \mathbb{H}$ via the period mapping (7). As for the Problem for $n=4$, we obtained the following theorem.

Theorem 4.1 ([4]) *The above period mapping P gives an isomorphism*

$$\widehat{S-D} \cong \mathbb{C}^* \times \mathbb{H}_2^0 \quad (9)$$

as $\mathbb{C}^*$ -bundles, where

$$\mathbb{H}_2^0 := \mathbb{H}_2 - A, \quad A := \{M \circ \tau \mid M \in Sp(4, \mathbb{Z}), \quad \tau \in \mathbb{H}_2, \tau \text{ is a diagonal matrix}\}. \quad (10)$$

(Definition of $M \circ \tau$ is given in subsection 5.2.)

(Outline of the proof.) To prove the theorem we give a global section of the bundle $\widehat{S-D} \rightarrow \mathbf{H}_2^0$ by using Rosenhain's formula [7]. The formula says that, in our situation, the following equalities

$$\frac{e_3 - e_1}{e_2 - e_1} = \frac{\vartheta_{0000}^2 \vartheta_{0100}^2}{\vartheta_{1000}^2 \vartheta_{1100}^2}, \quad \frac{e_4 - e_1}{e_2 - e_1} = \frac{\vartheta_{0100}^2 \vartheta_{0001}^2}{\vartheta_{1100}^2 \vartheta_{1001}^2}, \quad \frac{e_5 - e_1}{e_2 - e_1} = \frac{\vartheta_{0000}^2 \vartheta_{0001}^2}{\vartheta_{1000}^2 \vartheta_{1001}^2} \quad (11)$$

hold, where

- $e_1, \dots, e_5$ are five roots of $F(x, 0, \sigma(h))$,
- τ is a period matrix, which is obtained from $X_{\sigma(h)}$ with a basis $A_j(\sigma(h))$ $\{j = 1, 2, 3, 4\}$ of $H_1(X_{\sigma(h)}, \mathbf{Z})$,

and for any $\varepsilon = (\varepsilon' \varepsilon'') = (\varepsilon'_1 \dots \varepsilon'_g \varepsilon''_1 \dots \varepsilon''_g) \in \mathbf{Z}^{2g}$ and $\tau \in \mathbf{H}_g$,

$$\vartheta_\varepsilon = \vartheta_\varepsilon(\tau) := \sum_{n \in \mathbf{Z}^g} \exp \left[\pi i \left(n + \frac{\varepsilon'}{2} \right) \tau \left(n + \frac{\varepsilon'}{2} \right) + 2\pi i \left(n + \frac{\varepsilon'}{2} \right) \frac{\varepsilon''}{2} \right] \quad (12)$$

are theta constants of genus g , where $\mathbf{H}_g$ denotes Siegel upper half space of genus g . Here we use only theta constants of genus two.

In our situation, the equality $e_1 + \dots + e_5 = 0$ holds. Therefore, by using some formulas of theta constants, (11) imply that the following equality holds,

$$(e_1, \dots, e_5) = (\lambda^2 \alpha_1, \dots, \lambda^2 \alpha_5)$$

for some $\lambda \in \mathbf{C}^*$, where

$$\begin{aligned} \alpha_1 &:= \frac{1}{5} (-\vartheta_{1000}^2 \vartheta_{1100}^2 \vartheta_{1001}^2 - \vartheta_{0000}^2 \vartheta_{0100}^2 \vartheta_{1001}^2 - \vartheta_{1000}^2 \vartheta_{0100}^2 \vartheta_{0001}^2 - \vartheta_{0000}^2 \vartheta_{1100}^2 \vartheta_{0001}^2), \\ \alpha_2 &:= \frac{1}{5} (+\vartheta_{1000}^2 \vartheta_{1100}^2 \vartheta_{1001}^2 - \vartheta_{0010}^2 \vartheta_{0110}^2 \vartheta_{1001}^2 - \vartheta_{0011}^2 \vartheta_{0110}^2 \vartheta_{1000}^2 - \vartheta_{0011}^2 \vartheta_{0010}^2 \vartheta_{1100}^2), \\ \alpha_3 &:= \frac{1}{5} (+\vartheta_{0000}^2 \vartheta_{0100}^2 \vartheta_{1001}^2 + \vartheta_{0010}^2 \vartheta_{0110}^2 \vartheta_{1001}^2 - \vartheta_{0110}^2 \vartheta_{1111}^2 \vartheta_{0100}^2 - \vartheta_{0010}^2 \vartheta_{1111}^2 \vartheta_{0000}^2), \\ \alpha_4 &:= \frac{1}{5} (+\vartheta_{1000}^2 \vartheta_{0100}^2 \vartheta_{0001}^2 + \vartheta_{0011}^2 \vartheta_{0110}^2 \vartheta_{1000}^2 + \vartheta_{0110}^2 \vartheta_{1111}^2 \vartheta_{0100}^2 - \vartheta_{0011}^2 \vartheta_{1111}^2 \vartheta_{0001}^2), \\ \alpha_5 &:= \frac{1}{5} (+\vartheta_{0000}^2 \vartheta_{1100}^2 \vartheta_{0001}^2 + \vartheta_{0011}^2 \vartheta_{0010}^2 \vartheta_{1100}^2 + \vartheta_{0010}^2 \vartheta_{1111}^2 \vartheta_{0000}^2 + \vartheta_{0011}^2 \vartheta_{1111}^2 \vartheta_{0001}^2). \end{aligned}$$

Using those functions, we define a map $F : \mathbf{H}_2^0 \ni \tau \mapsto (t_2, \dots, t_5) \in S-D$ by

$$t_i = (-1)^i \sum_{1 \leq \nu_1 < \dots < \nu_i \leq 5} \alpha_{\nu_1} \dots \alpha_{\nu_i} \quad (i = 2, 3, 4, 5). \quad (13)$$

Then, the map F is "lifted" to a map $\hat{F} : \mathbf{H}_2^0 \rightarrow \widehat{S-D}$ and $\hat{F}$ is a global section of the bundle $\widehat{S-D} \rightarrow \mathbf{H}_2^0$. Therefore, the isomorphism (9) is obtained. $\square$

By the theorem, we can take $\mathbf{H}_2^0$ as X and Γ as G in the **Diagram-1**.

5 Results(2) — towards (2-3-2).

5.1 Preliminary.

In the previous section, we used $\hat{F}$ as a global section of the bundle $\widehat{S-D} \rightarrow H_2^0$ to give isomorphism (9). As a result, in our trivialization, we obtain a factor of automorphy $j: \Gamma \times H_2^0 \rightarrow \mathbb{C}^*$. From now on, we obtain j explicitly by clarifying factors of automorphy of t_2, t_3, t_4, t_5 (or $\alpha_1, \dots, \alpha_5$) under the action of Γ , where we regard t_i as functions on H_2^0 by the equality (13). Here we note the following three points.

(5-1-1) Each t_i (or α_i) is a homogeneous polynomial of theta constants.

(5-1-2) Γ is regarded as a subgroup of $Sp(4, \mathbb{Z})$ by the isomorphism (8).

(5-1-3) The transformation formula of theta constants under the action of the full modular group is well-known. (cf. [6])

So we investigate the effects of Γ -action on $\alpha_1, \dots, \alpha_5$.

5.2 Transformation formula of theta constants.

Following to [6], here we give short review of the transformation formula of theta constants defined in (12). It is well-known that, for $M = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in Sp(2g, \mathbb{Z})$, $\tau \in H_g$, $\varepsilon = (\varepsilon' \varepsilon'') \in \mathbb{Z}^{2g}$, the following equality holds:

$$\vartheta_{M \circ \varepsilon}(M \circ \tau) = \kappa(M) \exp(\pi i \phi(M, \varepsilon)) \sqrt{\det(C\tau + D)} \vartheta_{\varepsilon}(\tau), \quad (14)$$

where

$$M \circ \tau := (A\tau + B)(C\tau + D)^{-1}, \quad M \circ \varepsilon := \varepsilon M^{-1} + ((C^t D)_0 (A^t B)_0),$$

$$\phi(M, \varepsilon) := \frac{1}{4} \{-\varepsilon''^t D B^t \varepsilon' + 2\varepsilon''^t C B^t \varepsilon' - \varepsilon''^t C A^t \varepsilon'' + 2(\varepsilon''^t D - \varepsilon''^t C)(A^t B)_0\}$$

where for $g \times g$ matrix $X = (x_{ij})$, we write $X_0 := (x_{11}, x_{22}, \dots, x_{gg})$. In (14), $\kappa(M)$ is a constant, which depends on M and is independent to ε and τ . Moreover, It is well known that $\kappa(M)^8 = 1$ for any $M \in Sp(2g, \mathbb{Z})$. As for $\kappa(M)$, more various properties are known.

5.3 Definition of a group Γ' .

For any two positive integers g and n , the principal congruence subgroup of level n , genus g is defined as follows:

$$\Gamma_g(n) := \{M \in Sp(2g, \mathbb{Z}) | M \equiv I_{2g} \pmod{n}\}.$$

(Note that $\Gamma_g(1) = Sp(2g, \mathbb{Z})$.) It is well known that $\Gamma_2(1)/\Gamma_2(2)$ is isomorphic to the 6-th Symmetric group S_6 . (See, for example, [3].) Usually, the isomorphism is given by the action of $\Gamma_2(1)$ over the following six odd theta characteristics of genus two:

$$\langle 1 \rangle := (0101), \langle 2 \rangle := (0111), \langle 3 \rangle := (1011), \langle 4 \rangle := (1010), \langle 5 \rangle := (1110), \langle 6 \rangle := (1101). \quad (15)$$

Here we give an isomorphism explicitly.

For any $M \in \Gamma_2(1)$, the following map:

$$\varepsilon \bmod (2\mathbb{Z})^4 \mapsto M \circ \varepsilon \bmod (2\mathbb{Z})^4 \quad (16)$$

gives rise to a permutation of the six elements in (15). Hence for any $M \in \Gamma_2(1)$ and for any $i \in \{1, \dots, 6\}$, there is a $M(i) \in \{1, \dots, 6\}$ such that

$$\langle M(i) \rangle \equiv M \circ \langle i \rangle \bmod (2\mathbb{Z})^4.$$

Then the map $i \mapsto M(i)$ is a permutation of $\{1, \dots, 6\}$. Therefore we have a group homomorphism $\Gamma_2(1) \rightarrow S_6$. It can be easily seen that the homomorphism is surjective, and that its kernel is $\Gamma_2(2)$. So we obtain an isomorphism $\Gamma_2(1)/\Gamma_2(2) \cong S_6$.

Here we treat the following subgroup Γ' :

Definition 5.1 $\Gamma' := \{M \in \Gamma_2(1) | M \circ (1101) \equiv (1101) \bmod (2\mathbb{Z})^4\}$.

This subgroup Γ' has the following property.

$$\Gamma_2(2) \subset \Gamma' \subset \Gamma_2(1) \quad \text{and} \quad \Gamma'/\Gamma_2(2) \cong S_5.$$

The following lemma brings the above subgroup Γ' to our notice.

Lemma 5.2 (A'Campo [1]) $\Gamma' = \Gamma$.

5.4 Factors of automorphy of α_i .

We denote $\chi(M) := \kappa(M)^2 \exp[2\pi i \phi(M, (1101))]$ for any $M \in \Gamma_2(1)$. We can easily obtain the following obvious lemma by using transformation formula of theta constants.

Lemma 5.3 $\Gamma' \ni M \mapsto \chi(M) \in \mathbb{C}^*$ is a group homomorphism.

Moreover, the transformation formula (14) implies the following lemma.

Lemma 5.4 ([4]) For any $i \in \{1, 2, 3, 4, 5\}$, for any $M = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in \Gamma$, and for any $\tau \in \mathbb{H}_2$, the following equality holds.

$$\alpha_{M(i)}(M \circ \tau) = \chi(M) \det(C\tau + D)^3 \alpha_i(\tau).$$

By the lemma, consequently, we obtain the following theorem.

Theorem 5.5 ([4]) Under the trivialization of $\widehat{S-D}$ by $\hat{F}$, we conclude that

$$j(M, \tau)^2 = \chi(M) \det(C\tau + D)^3 \quad (\forall M = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in \Gamma, \tau \in \mathbb{H}_2).$$

6 Comparison with the ring of Siegel modular forms.

In the frame given in section 2, if we take $X = \mathbb{H}_2$, $G = \Gamma_2(1)$ and $j\left(\begin{pmatrix} D & C \\ B & A \end{pmatrix}, \tau\right) := \det(C\tau + D)$, then the ring of (G, j) -automorphic forms on X is the ring of ordinary Siegel modular forms of genus two. Igusa showed (in [2], [3]) that the ring is

$$\mathbb{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}, \chi_{35}] \quad (\text{indices denote weights}) \quad (17)$$

where $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ are algebraically independent over $\mathbb{C}$, and $\chi_{35}^2 \in \mathbb{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}]$. On the other hand, the ring which we pay attention to is

$$\mathbb{C}[t_2, t_3, t_4, t_5] \quad (\text{indices denote half of weights})$$

where t_2, t_3, t_4, t_5 are algebraically independent over $\mathbb{C}$. Therefore, though we have not yet found conditions which determine the ring $\mathbb{C}[t_2, t_3, t_4, t_5]$ in the ring of (Γ, j) -automorphic forms, we can already see that algebraic structures of our ring $\mathbb{C}[t_2, t_3, t_4, t_5]$ is different from that of the ring (17).

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