

On the existence of ground states for the Pauli-Fierz model with a variable mass

By

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Abstract

The purpose of this paper is to review [9]. The existence of ground states of the Pauli-Fierz model with a variable mass is considered. This paper presents the outline of the proof of it under the infrared regularity condition.

§ 1. Introduction

The Pauli-Fierz model describes a minimal interaction between a low energy electron and a quantized radiation field, where the electron is governed by a Schrödinger operator. The Pauli-Fierz Hamiltonian is the physical quantity corresponding to the energy of the system and is realized as a self-adjoint operator on a certain Hilbert space and its bottom of the spectrum is called the ground state energy. An eigenvector associated with the ground state energy is called a ground state, if it exists.

The existence of ground states of the Pauli-Fierz Hamiltonian is investigated in [1, 2, 4, 8, 10, 12]. In [2, 8], the infrared regularity condition is not assumed. In [4, 8], the existence of ground states is shown for arbitrary values of coupling constants. The uniqueness of the ground state of the Pauli-Fierz Hamiltonian is proven in [11].

The Pauli-Fierz Hamiltonian with a variable mass is considered in this paper. It is derived from the analogy of the Nelson model on a pseudo Riemannian manifold [5, 6, 7]. Under the infrared regularity condition, this Hamiltonian has ground states for all values of a coupling constant when a variable mass decays sufficiently fast.

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§ 2. Definition of the Pauli-Fierz model

§ 2.1. Hilbert space of states

We consider the Hilbert space of states of total system as

$$\mathcal{H} := \mathcal{H}_P \otimes \mathcal{F},$$

where

$$\mathcal{H}_P := L^2(\mathbb{R}^3)$$

describes state space of one electron and $\mathcal{F}$ is the boson Fock space over $L^2(\mathbb{R}^3; \mathbb{C}^2)$ defined by

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n L^2(\mathbb{R}^3; \mathbb{C}^2) \right].$$

Here $\bigotimes_s^n L^2(\mathbb{R}^3; \mathbb{C}^2)$ denotes the n -fold symmetric tensor product of $L^2(\mathbb{R}^3; \mathbb{C}^2)$ with $\bigotimes_s^0 L^2(\mathbb{R}^3; \mathbb{C}^2) = \mathbb{C}$. The inner product on $\mathcal{F}$ is given by

$$(2.1) \quad (\Psi, \Phi)_{\mathcal{F}} = \overline{\Psi^{(0)}} \Phi^{(0)} + \sum_{n=1}^{\infty} \int_{\mathbb{R}^{3n}} \overline{\Psi^{(n)}(k_1, \dots, k_n)} \Phi^{(n)}(k_1, \dots, k_n) dk_1 \cdots dk_n.$$

The Hilbert space $\mathcal{H}$ can be identified with

$$(2.2) \quad \mathcal{H} \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{F} dx \cong L^2(\mathbb{R}^3) \oplus \left[\bigoplus_{n=1}^{\infty} L_{\text{sym}}^2(\mathbb{R}^{3+3n}; \mathbb{C}^2) \right].$$

Here $L_{\text{sym}}^2(\mathbb{R}^{3+3n}; \mathbb{C}^2)$ is the set of $L^2(\mathbb{R}^{3+3n}; \mathbb{C}^2)$ -functions such that

$$f(x, k_1, \dots, k_n) = f(x, k_{\sigma(1)}, \dots, k_{\sigma(n)})$$

for an arbitrary permutation σ .

Let T be a densely defined closable operator on $L^2(\mathbb{R}^3; \mathbb{C}^2)$. Then $\Gamma(T)$ and $d\Gamma(T)$ are defined by

$$(2.3) \quad \Gamma(T) := \bigoplus_{n=0}^{\infty} \otimes^n T, \quad d\Gamma(T) := \bigoplus_{n=0}^{\infty} \otimes^n T^{(n)},$$

where $\otimes^0 T = 1$, $T^{(n)} := \overline{\sum_{k=1}^n 1 \otimes \cdots \otimes 1 \otimes T \otimes 1 \cdots \otimes 1}$ and $T^{(0)} = 0$. The number operator is defined by

$$N := d\Gamma(1).$$

The annihilation operator $a(f)$ and the creation operator $a^\dagger(f)$ smeared by $f \in L^2(\mathbb{R}^3; \mathbb{C}^2)$ on $\mathcal{F}$ are defined by

$$(2.4) \quad D(a^\dagger(f)) = \left\{ \Psi \in \mathcal{F} \mid \sum_{n=1}^{\infty} n \|S_n(f \otimes \Psi^{(n-1)})\|^2 < \infty \right\},$$

$$(2.5) \quad (a^\dagger(f)\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), n \geq 1, \quad (a^\dagger(f)\Psi)^{(0)} = 0,$$

$$(2.6) \quad a(f) = (a^\dagger(\bar{f}))^*,$$

where S_n denotes the symmetrization operator of degree n and $D(T)$ the domain of T . $\Omega := (1, 0, 0, \dots) \in \mathcal{F}$ is called the Fock vacuum. Let

$$(2.7) \quad (a(k)\Psi)^{(n)}(k_1, \dots, k_n) := \sqrt{n+1} \Psi^{(n+1)}(k, k_1, \dots, k_n)$$

for $\Psi \in D(N^{1/2})$. Then for almost every k , $a(k)\Psi \in \mathcal{F}$.

§ 2.2. Definition of the Pauli-Fierz model

Let v be a multiplication operator on $L^2(\mathbb{R}^3)$. We introduce assumptions on v .

Assumption 1.

$$(1) \quad \sigma_P(-\Delta + v) \subset (0, \infty);$$

$$(2) \quad v(x) \leq \text{const.} \langle x \rangle^{-\beta} \text{ with } \beta > 3, \text{ where } \langle x \rangle = \sqrt{1 + |x|^2}.$$

Here $\sigma_P(T)$ denotes the set of eigenvalues of T .

Then there exists a unique function $\Psi(k, x)$ such that for $k \neq 0$,

$$(2.8) \quad (-\Delta_x + v(x)) \Psi(k, x) = |k|^2 \Psi(k, x)$$

and $\Psi(k, x)$ satisfies the Lippman-Schwinger equation:

$$(2.9) \quad \Psi(k, x) = e^{ikx} - \frac{1}{4\pi} \int \frac{e^{i|k||x-y|} v(y)}{|x-y|} \Psi(k, y) dy.$$

We will use the regularity properties of $\Psi(k, x)$ below to show the existence of ground states.

Lemma 2.1. *Suppose Assumption 1. Then*

(a)

$$(2.10) \quad |\Psi(k, x) - e^{ikx}| \leq \text{const.} \langle x \rangle^{-1}$$

holds.

(b) $\Psi(k, x)$ is continuously differentiable in x for each fixed k but $k \neq 0$ and

$$(2.11) \quad \frac{\partial}{\partial x_\mu} \Psi(k, x) - ik_\mu e^{ikx} = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \left(\frac{e^{i|k||x-y|}(x_\mu - y_\mu)}{|x-y|^3} - \frac{i|k|e^{i|k||x-y|}(x_\mu - y_\mu)}{|x-y|^2} \right) v(y) \Psi(k, y) dy.$$

In particular, for any compact set D but $0 \notin D$, $\sup_{k \in D, x} \left| \frac{\partial \Psi}{\partial x_\mu}(k, x) \right| < \infty$.

(c) For $k \neq 0$ and $k + h \neq 0$,

$$(2.12) \quad \frac{1}{|h|} |\Psi(k + h, x) - \Psi(k, x)| \leq \text{const.} (1 + |x|),$$

$$(2.13) \quad \frac{1}{|h|} \left| \frac{\partial}{\partial x_\nu} \Psi(k + h, x) - \frac{\partial}{\partial x_\nu} \Psi(k, x) \right| \leq \text{const.} (1 + |k| + |x| + |k| |x|)$$

hold, and $\Psi(k, x)$ and $\frac{\partial}{\partial x_\nu} \Psi(k, x)$ are differentiable in $k \in \mathbb{R}^3 \setminus \{0\}$ for each fixed x .

Let us introduce the dispersion relation and the quantized radiation field with a variable mass v .

Definition 2.2. The dispersion relation with a variable mass is given by

$$(2.14) \quad \hat{\omega} := \sqrt{-\Delta + v}$$

on $L^2(\mathbb{R}^3; \mathbb{C}^2)$, where v is called a variable mass. The free Hamiltonian is defined by the second quantization of $\hat{\omega}$:

$$(2.15) \quad H_f = d\Gamma(\hat{\omega}).$$

Let $m \geq 0$ and $\hat{\omega}_m := \sqrt{-\Delta + v + m^2}$. We set

$$H_f(m) = d\Gamma(\hat{\omega}_m).$$

In order to define the quantized radiation field, we introduce a cutoff functions: $\hat{\varphi}_j^\mu$, $j = 1, 2$, $\mu = 1, 2, 3$.

Assumption 2.

- (1) The support of $\hat{\varphi}_j^\mu$ is compact;
- (2) $\hat{\varphi}_j^\mu$ is differentiable and the derivative function is bounded;
- (3) (infrared regularity condition)

It holds that

$$(2.16) \quad \int_{\mathbb{R}^3} \frac{|\hat{\varphi}_j^\mu(k)|^{2p}}{|k|^{5p}} dk < \infty \quad \text{for all } 0 < p < 1.$$

Let the test function $\rho_x^\mu = (\rho_x^{\mu,1}, \rho_x^{\mu,2}) \in L^2(\mathbb{R}^3; \mathbb{C}^2) \in L^2(\mathbb{R}^3; \mathbb{C}^2)$ be such that

$$\rho_x^{\mu,j}(y) := (2\pi)^{-3/2} \int \overline{\Psi(k, x)} \Psi(k, y) \hat{\varphi}_j^\mu(k) dk.$$

The quantized radiation field with a variable mass is given by

$$(2.17) \quad A_\mu(x) := \frac{1}{\sqrt{2}} \left(a^\dagger \left(\hat{\omega}^{-1/2} \rho_x^\mu \right) + a \left(\overline{\hat{\omega}^{-1/2} \rho_x^\mu} \right) \right), \quad \mu = 1, 2, 3,$$

for each $x \in \mathbb{R}^3$.

Definition 2.3. Let V be a multiplication operator, and V_+ and V_- the positive part and the negative part of V , respectively. Then the quadratic form q_m^V is defined by

$$(2.18) \quad q_m^V(\Psi, \Phi) = \frac{1}{2} \sum_{\mu=1}^3 \left((p_\mu + \sqrt{\alpha} A_\mu) \Psi, (p_\mu + \sqrt{\alpha} A_\mu) \Phi \right) \\ + \left(H_f^{1/2}(m) \Psi, H_f^{1/2}(m) \Phi \right) + \left(V_+^{1/2} \Psi, V_+^{1/2} \Phi \right) - \left(V_-^{1/2} \Psi, V_-^{1/2} \Phi \right)$$

with the form domain

$$(2.19) \quad Q(q_m^V) = D(|p|) \cap D(H_f^{1/2}(m)) \cap D(|V|^{1/2}).$$

Here α is a coupling constant. When $m = 0$, we denote q^V for q_0^V .

§ 2.3. Generalized Fourier transformation

By [14], under Assumption 1, the generalized Fourier transformation is defined by

$$(2.20) \quad f \mapsto \mathcal{F}f(\cdot) := (2\pi)^{-3/2} \text{l.i.m.} \int f(x) \overline{\Psi(\cdot, x)} dx,$$

which is a unitary transformation on $L^2(\mathbb{R}^3)$. By $1 \otimes \Gamma(\mathcal{F}) : \mathcal{H} \rightarrow \mathcal{H}$, the quadratic form q_m^V is transformed as

$$(2.21) \quad \hat{q}_m^V(\Psi, \Phi) = q_m^V(1 \otimes \Gamma(\mathcal{F}) \Psi, 1 \otimes \Gamma(\mathcal{F}) \Phi) \\ = \frac{1}{2} \sum_{\mu=1}^3 \left((p_\mu + \sqrt{\alpha} \hat{A}_\mu) \Psi, (p_\mu + \sqrt{\alpha} \hat{A}_\mu) \Phi \right) + \left(\hat{H}_f^{1/2}(m) \Psi, \hat{H}_f^{1/2}(m) \Phi \right) \\ + \left(V_+^{1/2} \Psi, V_+^{1/2} \Phi \right) - \left(V_-^{1/2} \Psi, V_-^{1/2} \Phi \right)$$

with the form domain

$$(2.22) \quad Q(\hat{q}_m^V) = D(|p|) \cap D(\hat{H}_f^{1/2}(m)) \cap D(|V|^{1/2}).$$

Here

$$(2.23) \quad \hat{A}_\mu(x) := \frac{1}{\sqrt{2}} \sum_{j=1,2} \left(a^\dagger \left(\frac{\hat{\varphi}_j^\mu \Psi(\cdot, x)}{\sqrt{\omega}} \right) + a \left(\frac{\hat{\varphi}_j^\mu \Psi(\cdot, x)}{\sqrt{\omega}} \right) \right), \quad \omega(k) = |k|,$$

and

$$(2.24) \quad \hat{H}_f(m) := d\Gamma(\omega_m), \quad \omega_m(k) := \sqrt{k^2 + m^2}.$$

We introduce following assumptions on V :

Assumption 3.

(1) V is a measurable function and for almost every $x \in \mathbb{R}^3$, $-\infty < V(x) < \infty$;

(2) For all $\epsilon > 0$, there exists a positive constant C_ϵ such that for $\Psi \in D(|p|)$,

$$(2.25) \quad \|V_-^{1/2} \Psi\|^2 \leq \epsilon \| |p| \Psi \|^2 + C_\epsilon \| \Psi \|^2;$$

(3) $Q(\hat{q}_m^V)$ is dense.

Proposition 2.4. *Suppose Assumptions 1, 2 and 3. Then there exists the unique self-adjoint operator $\hat{H}_m^V$ such that $Q(\hat{q}_m^V) = D(|\hat{H}_m^V|^{1/2})$ and for all Ψ and $\Phi \in Q(\hat{q}_m^V)$,*

$$\hat{q}_m^V(\Psi, \Phi) - E^V(m)(\Psi, \Phi) = \left((\hat{H}_m^V - E^V(m))^{1/2} \Psi, (\hat{H}_m^V - E^V(m))^{1/2} \Phi \right).$$

Here we denote the ground state energy of $\hat{q}_m^V$ by

$$(2.26) \quad E^V(m) := \inf_{\Psi \in Q(\hat{q}_m^V), \|\Psi\|=1} \hat{q}_m^V(\Psi, \Psi).$$

Formally, the Pauli-Fierz Hamiltonian H_m^V is given by

$$(2.27) \quad H_m^V := \frac{1}{2} \sum_{\mu, \nu} (p_\mu + \sqrt{\alpha} A_\mu) a_{\mu\nu} (p_\nu + \sqrt{\alpha} A_\nu) + H_f(m) + V.$$

Here $\{a_{\mu, \nu}\}_{\mu, \nu=1,2,3} = \{a_{\mu, \nu}(x)\}_{\mu, \nu=1,2,3}$ is positive definite. We consider only the case of $a_{\mu\nu}(x) = \delta_{\mu, \nu}$ for simplicity.

§ 3. Binding condition

We introduce functions ϕ_R and $\tilde{\phi}_R$ below. Let $\phi \in C^\infty(\mathbb{R}^3)$ be such that for all $x \in \mathbb{R}^3$, $0 \leq \phi(x) \leq 1$ and

$$\phi(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2. \end{cases}$$

Let $\tilde{\phi} \in C^\infty(\mathbb{R}^3)$ be such that for all $x \in \mathbb{R}^3$, $0 \leq \tilde{\phi}(x) \leq 1$ and

$$\phi(x)^2 + \tilde{\phi}(x)^2 = 1.$$

We set for $R > 0$,

$$(3.1) \quad \phi_R(x) := \phi(x/R), \quad \tilde{\phi}_R(x) := \tilde{\phi}(x/R).$$

Let

$$(3.2) \quad E^V(R, m) = \inf_{\|\tilde{\phi}_R \Psi\|=1, \Psi \in D(\hat{H}_m^V)} (\tilde{\phi}_R \Psi, \hat{H}_m^V \tilde{\phi}_R \Psi).$$

$\lim_{R \rightarrow \infty} E^V(R, m) - E^V(m)$ formally describes ionization energy by definition, it is expected that positive ionization energy yields ground state.

Assumption 4 (Binding condition).

$$(3.3) \quad E^V(m) < \lim_{R \rightarrow \infty} E^V(R, m).$$

§ 4. Massive case

The existence of ground states in the case of $m > 0$ is considered in this section.

Theorem 4.1. *Let $m > 0$. Suppose Assumptions 1-4. Then ground states of $\hat{H}_m^V$ exist for all values of a coupling constant.*

Outline of Proof. Let $\{\Psi^j\}_j \subset Q(\hat{q}_m^V)$ be a sequence such that weakly converges to 0. It suffices to show that

$$(4.1) \quad \liminf_{j \rightarrow \infty} \hat{q}_m^V(\Psi^j, \Psi^j) > E^V(m).$$

We can suppose that $\sup_j \hat{q}_m^V(\Psi^j, \Psi^j) < \infty$. Let ϕ_R and $\tilde{\phi}_R$ be in (3.1).

$$(4.2) \quad \begin{aligned} \hat{q}_m^V(\Psi^j, \Psi^j) &= \hat{q}_m^V(\Psi_R^j, \Psi_R^j) + \hat{q}_m^V(\tilde{\Psi}_R^j, \tilde{\Psi}_R^j) \\ &\quad - \frac{1}{2} \|(|\nabla \phi_R| \otimes 1) \Psi^j\|^2 - \frac{1}{2} \|(|\nabla \tilde{\phi}_R| \otimes 1) \Psi^j\|^2. \end{aligned}$$

holds. Here $\Psi_R^j = \phi_R \Psi^j$ and $\tilde{\Psi}_R^j = \tilde{\phi}_R \Psi^j$. Let j_1 and j_2 be nonnegative, smooth functions on $\mathbb{R}^3$ such that

$$(4.3) \quad j_1(k) = \begin{cases} 1 & \text{if } |k| < 1, \\ 0 & \text{if } |k| > 2 \end{cases} \quad \text{and} \quad j_1(k)^2 + j_2(k)^2 = 1.$$

We set $\hat{j}_{l,P} = j_l(-i\nabla_k/P)$, $l = 1, 2$, and

$$(4.4) \quad \hat{j}_P \Psi = \hat{j}_{1,P} \Psi \oplus \hat{j}_{2,P} \Psi,$$

for $\Psi \in L^2(\mathbb{R}^3; \mathbb{C}^2)$. Let us define the isometric operator from $\mathcal{F}$ to $\mathcal{F} \otimes \mathcal{F}$ by

$$(4.5) \quad \begin{aligned} & d\tilde{\Gamma}(\hat{j}_P) a^\dagger(h_1) \cdots a^\dagger(h_n) \Omega \\ &= a^\dagger(\hat{j}_{1,P} h_1) \cdots a^\dagger(\hat{j}_{1,P} h_n) \Omega \oplus a^\dagger(\hat{j}_{2,P} h_1) \cdots a^\dagger(\hat{j}_{2,P} h_n) \Omega. \end{aligned}$$

By the localization argument (see [8]), it holds that

$$(4.6) \quad \liminf_{j \rightarrow \infty} \hat{q}_m^V(\Psi_R^j, \Psi_R^j) \geq (E^V(m) + m) \liminf_{j \rightarrow \infty} \|\Psi_R^j\|^2 + o_R(P^0)$$

and

$$(4.7) \quad \hat{q}_m^V(\tilde{\Psi}_R^j, \tilde{\Psi}_R^j) \geq E_{R,m}^V \|\tilde{\Psi}_R^j\|^2 + o(R^0).$$

Here $o_R(P^0)$ goes to zero as $P \rightarrow \infty$ for each fixed $R > 0$. By (4.2), (4.6) and (4.7), we can see that

$$(4.8) \quad \liminf_{j \rightarrow \infty} \hat{q}_m^V(\Psi^j, \Psi^j) \geq E^V(m) + \min\{m, E^V(R, m) - E^V(m)\}.$$

By the binding condition, we obtain (4.1). $\square$

§ 5. The case of $m = 0$

Throughout in this section, we suppose Assumptions 1, 2, 3 and Assumption 4 with $m = 0$. Φ_m denotes the normalized ground state of $\hat{H}_m^V$. Similarly to the case of $v = 0$, the following lemma holds.

Lemma 5.1. *Let $\{m_j\}_{j=1}^\infty$ be a sequence converging to 0. Then*

$$\lim_{j \rightarrow \infty} E^V(m_j) = E^V(0)$$

and for sufficiently small $0 < m$, the binding condition holds.

The pull through formula below leads to a photon number bound (Lemma 5.3 and Corollary 5.4) and a photon derivative bound (Lemma 5.6).

Lemma 5.2 (Pull through formula). *Let $f \in D(\omega_m)$. Then $a(f)\Phi_m \in Q(\hat{q}_m^V)$ and for all $\eta \in Q(\hat{q}_m^V)$,*

$$(5.1) \quad \begin{aligned} & \hat{q}_m^V(\eta, a(f)\Phi_m) - E^V(m)(\eta, a(f)\Phi_m) \\ &= -\sqrt{\alpha}(\eta, (\bar{f}, \bar{G}) \cdot (p + \sqrt{\alpha}\hat{A})\Phi_m) + \frac{i\sqrt{\alpha}}{2}(\eta, (\bar{f}, \nabla_x \cdot \bar{G})\Phi_m) - (\eta, a(\omega_m f)\Phi_m). \end{aligned}$$

holds. Here

$$G_j^\mu(k, x) := \frac{\hat{\varphi}_j^\mu(k) \Psi(k, x)}{\sqrt{2\omega(k)}}.$$

Lemma 5.3. *Let $\theta = (\theta_1, \theta_2) \in L^\infty(\mathbb{R}^3; \mathbb{R}^2)$. Then*

$$(5.2) \quad \|d\Gamma(\theta^2)^{1/2} \Phi_m\|^2 \leq C\alpha \sum_{\mu, j} \int \frac{\hat{\varphi}_j^\mu(k)^2 \theta_j(k)^2}{\omega(k) \omega_m(k)^2} dk,$$

where C is a constant independent of α and sufficiently small m .

Outline of proof of Lemma 5.3. Inserting $\eta = a(f)\Phi_m$ into (5.2), we have

$$(5.3) \quad (a(f)\Phi_m, a(\omega_m f)\Phi_m) \leq -\sqrt{\alpha} \left(a(f)\Phi_m, (\bar{f}, \bar{G}) \cdot (p + \sqrt{\alpha}\hat{A})\Phi_m \right) \\ + \frac{i\sqrt{\alpha}}{2} (a(f)\Phi_m, (\bar{f}, \nabla_x \cdot \bar{G})\Phi_m).$$

Let $f := \omega_m \theta g_i$. Here $\{g_i\}_{i=1}^\infty$ is a complete orthonormal system such that each $g_i \in D(\omega_m^{1/2})$. Note that

$$(5.4) \quad \sum_{i=1}^\infty \left(a(\omega_m^{-1/2} \theta g_i) \Phi_m, a(\omega_m^{1/2} \theta g_i) \Phi_m \right) \\ = \sum_{j=1,2} \int_{\mathbb{R}^3} \theta_j(k)^2 \|a_j(k)\Phi_m\|^2 dk = \|d\Gamma(\theta^2)^{1/2} \Phi_m\|^2.$$

Then by (5.3) and (5.4),

$$(5.5) \quad \|d\Gamma(\theta^2)^{1/2} \Phi_m\|^2 \\ \leq 2\alpha \int_{\mathbb{R}^3} \omega_m(k)^{-2} \|\theta(k)G(k) \cdot (p + \sqrt{\alpha}\hat{A})\Phi_m\|^2 dk \\ + \frac{\alpha}{2} \int_{\mathbb{R}^3} \omega_m(k)^{-2} \|\theta(k)\nabla_x \cdot G(k)\Phi_m\|^2 dk.$$

can be estimated. Since $\Psi(k, x)$ and $\hat{\varphi}(k)\frac{\partial}{\partial x_\mu}\Psi(k, x)$ are bounded in k and x , we can see that the lemma follows. $\square$

From Lemma 5.3, we can see that following facts hold.

Corollary 5.4. *It holds that*

- (1) $\sup_{m < m_0} \|N^{1/2}\Phi_m\| < \infty,$
- (2) $\text{supp } \Phi_m^{(n)}(x, \cdot) \subset \Pi_{k=1}^n [\cup_{j,\mu} \text{supp } \hat{\varphi}_j^\mu].$

We can show the spatial exponentially decay of Φ_m for many external potentials. See [13].

Assumption 5.

(1) For sufficiently large $|x|$, $V(x) > \text{const.}|x|^{2n}$.

(2) $\liminf_{|x| \rightarrow \infty} V(x) > \inf \sigma(H_P)$ and for all $t > 0$, $e^{-tH_P} : L^2 \rightarrow L^\infty$ with

$$\|e^{-tH_P} f\|_{L^\infty(\mathbb{R}^3)} \leq \text{const.} \|f\|_{L^2(\mathbb{R}^3)},$$

where $H_P = -\frac{1}{2}\Delta + V$.

Theorem 5.5. Suppose Assumption 5. Then for some c and $m_0 > 0$,

$$(5.6) \quad \sup_{0 < m < m_0} \|\exp(c|x|)\Phi_m\| < \infty.$$

holds.

Outline of Proof. Since $\Phi_m = e^{tE} e^{-t\hat{H}_m^V} \Phi_m$, by the functional integral representation of $e^{-t\hat{H}_m^V}$, we can see that for all $t \geq 0$,

$$(5.7) \quad \|\Phi_m(x)\|_{\mathcal{F}} \leq C e^{tE^V(m)} \mathbb{E}^x \left[e^{-\int_0^t V(B_s) ds} \right]$$

holds. Here $(B_t)_{t \geq 0}$ denotes Brownian motion starting from x . C is a constant independent of x and m .

$$(5.8) \quad e^{t(x)E^V(m)} \mathbb{E}^x \left[e^{-\int_0^{t(x)} V(B_s) ds} \right] \leq C_1 \exp(-C_2|x|^{n+1})$$

and

$$(5.9) \quad e^{t'(x)E^V(m)} \mathbb{E}^x \left[e^{-\int_0^{t'(x)} V(B_s) ds} \right] \leq C'_1 \exp(-C'_2|x|)$$

hold. Here $t(x) = |x|^{1-n}$, $t'(x) = \beta|x|$. (5.8) and (5.9) are called Carmona's estimate [3]. By (5.7), (5.8) and (5.9), the theorem can be proven.

Lemma 5.6. Suppose Assumption 5. Let $1 \leq p < 2$. Then

(a) $\Phi_m^{(n)} \in H^1(\mathbb{R}^{3+3n})$ for all $n \geq 0$;

(b) $\{\|\Phi_m^{(n)}\|_{W^{1,p}(\Omega)}\}_{0 < m \leq m_0}$ is bounded, where m_0 is sufficiently small number and Ω is any measurable and bounded set in $\mathbb{R}^{3+3n}$.

Here $W^{1,p}(\Omega)$ is the Sobolev space.

Outline of proof. Let $f = \omega_m^{-1/2} \theta g_i$. By the pull through formula with $f(x)$ replaced by $f(x+h) - f(x)$, similarly to the proof of Lemma 5.3, we can see that for

almost every k and sufficiently small h ,

$$(5.10) \quad \begin{aligned} & \| |h|^{-1}(a(k+h) - a(k)) \Phi_m \|^2 \\ & \leq \frac{\text{const.}}{\omega_m(k)^2} \left(\sum_{\mu=1}^3 (\| |h|^{-1}(\delta_h G_\mu)(k) \Phi_m \|^2 + \| |h|^{-1}(\nabla_x \delta_h G_\mu)(k) \Phi_m \|^2) \right. \\ & \quad \left. + \| |h|^{-1}(\nabla_x \cdot \delta_h G)(k) \Phi_m \|^2 + \sum_{j,\mu} \frac{\hat{\varphi}_j^\mu(k+h)^2}{\omega_m(k+h)^2 \omega(k+h)} \frac{|\omega(k+h) - \omega(k)|^2}{|h|^2} \right). \end{aligned}$$

Here $(\delta_h G_\mu)(k) = G_\mu(k+h, x) - G_\mu(k, x)$. By Lemma 2.1 (c) and Assumption 5,

$$(5.11) \quad \begin{aligned} & \| |h|^{-1}(a(k+h) - a(k)) \Phi_m \|^2 \\ & \leq C \omega_m(k)^{-2} \sum_{\nu,j} \left((1 + |k|^{-3}) \hat{\varphi}_j^\nu(k)^2 + |k|^{-1} \sum_{\lambda} |\partial_\lambda \hat{\varphi}_j^\nu(k)|^2 \right) \end{aligned}$$

holds for almost every k and sufficiently small $|h|$. Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. Thus by Alaoglu theorem, for almost every k , there exists the sequence $\{h_l(k)\}_{l=1}^\infty$ depending on k so that

$$\lim_{l \rightarrow \infty} h_l(k) = 0$$

and $|h_l(k)|^{-1}(a(k - |h_l(k)|e_\mu) - a(k))\Phi_m$ weakly converges to some vector $v_\mu(k)$:

$$v_\mu(k) := \text{w-} \lim_{l \rightarrow \infty} |h_l(k)|^{-1}(a(k - |h_l(k)|e_\mu) - a(k))\Phi_m.$$

It can be proven that $v_\mu^{(n)}(k)(x, k_1, \dots, k_n)$ is the weak derivative $\Phi_m^{(n+1)}(x, k, k_1, \dots, k_n)$ with respect to k_μ . Thus by (5.11), (a) and (b) are proven directly. $\square$

Theorem 5.7. *Let $m = 0$. Suppose Assumption 5. Then ground states of $\hat{H}^V$ exist for all values of a coupling constant.*

By Lemmas 5.3, 5.6 and Theorem 5.5, Theorem 5.7 can be proven similarly to [8, Theorem 2.1].

§ 6. Remarks on infrared cutoffs

We assumed the infrared regularity condition, but in the case of $v = 0$, we can show the existence of ground states of $\hat{H}$ without the infrared regularity condition. In the case of $v \neq 0$,

$$(6.1) \quad \Psi(k, x) - e^{ikx} = \sum_{n=1}^{\infty} \left(\frac{1}{4\pi} \right)^n \int_{\mathbb{R}^3} \frac{e^{i|k| \sum_{j=1}^n |y_j - y_{j-1}|} \Pi_{j=1}^n v(y_j)}{\Pi_{j=1}^n |y_j - y_{j-1}|} dy_1 \cdots dy_n$$

and

$$\begin{aligned} & \nabla_x \Psi(k, x) - ik e^{ikx} \\ &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \left(\frac{e^{i|k||x-y|}(x-y)}{|x-y|^3} - \frac{i|k|e^{i|k||x-y|}(x-y)}{|x-y|} \right) v(y) \Psi(k, y) dy \end{aligned}$$

hold. Here $y_0 := x$. The right hand side of (6.1) is not $O(|k|)$, ($k \rightarrow 0$). This is the reason that we assumed the infrared regularity condition. To see this, let us consider the case of $v = 0$. Set $v = 0$ and $\hat{\varphi}_j^\mu(k) = \chi_\Lambda(k) e_j^\mu(k)$, $j = 1, 2$, $\mu = 1, 2, 3$, where χ_Λ is the characteristic function of the set $\{k \mid |k| < \Lambda\}$ and $e_1(k)$ and $e_2(k)$, $k \in \mathbb{R}^3 \setminus \{0\}$ are polarization vectors given by

$$(6.2) \quad e_1(k) := \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}} \quad \text{and} \quad e_2(k) := \frac{k \times e_1(k)}{|k|}.$$

Note that the infrared regularity condition is not assumed in this case. Define the unitary operator U as

$$(6.3) \quad U := \exp[i\sqrt{\alpha}x \cdot \hat{A}(0)].$$

Put

$$(6.4) \quad \hat{q}_m^V(\Psi, \Phi) := \hat{q}_m^V(U\Psi, U\Phi)$$

and

$$(6.5) \quad \tilde{\hat{A}}(x) := \hat{A}(x) - \hat{A}(0).$$

Then

$$\begin{aligned} (6.6) \quad & \tilde{q}_m^V(\Psi, a(f)\tilde{\Phi}_m) - E^V(m)(\Psi, a(f)\tilde{\Phi}_m) \\ &= -\sqrt{\alpha}(\Psi, (\bar{f}, \tilde{G})(p + \sqrt{\alpha}\tilde{A})\tilde{\Phi}_m) - (\Psi, a(\omega_m f)\tilde{\Phi}_m) + i(\Psi, (\bar{f}, \omega_m w)\tilde{\Phi}_m) \end{aligned}$$

follows. Here $\tilde{\Phi}_m = U\Phi_m$, $w_j := \frac{\chi_\Lambda(k)e_j(k) \cdot x}{\sqrt{\omega(k)}}$ and $\tilde{G}_j^\mu := \frac{\chi_\Lambda(k)e_j^\mu(k)(e^{ikx}-1)}{\sqrt{2\omega(k)}}$. Similarly to the proof of Theorem 5.3, we have

$$(6.7) \quad \|a(k)\tilde{\Phi}_m\|^2 \leq \text{const.} \omega(k)^{-2} \left\{ \|\tilde{G}\tilde{\Phi}_m\|^2 + \|\nabla_x \tilde{G}\tilde{\Phi}_m\|^2 + \omega(k)^2 \|w\tilde{\Phi}_m\|^2 \right\} \chi_\Lambda(k).$$

Since $|e^{ikx} - 1| \leq |k||x|$ and $|\nabla_x e^{ikx}| = |k|$, by the exponential decay of $\tilde{\Phi}_m$, it holds that

$$(6.8) \quad \|d\Gamma(\theta^2)^{1/2} \tilde{\Phi}_m\|^2 \leq C\alpha \sum_j \int \frac{\chi_\Lambda(k) \theta_j(k)^2}{\omega(k)} dk.$$

Here C is a constant independent of α and m for sufficiently small m . Also by (6.6), for almost every k and sufficiently small h ,

$$(6.9) \quad \begin{aligned} & \| (a(k+h) - a(k)) \tilde{\Phi}_m \|^2 \\ & \leq \frac{\text{const.}}{\omega(k)^2} \left\{ \|\delta_h \tilde{G} \tilde{\Phi}_m\|^2 + \|\nabla_x \delta_h \tilde{G} \tilde{\Phi}_m\|^2 + \omega(k)^2 \|\delta_h(\omega w) \tilde{\Phi}_m\|^2 \right. \\ & \quad \left. + \frac{1}{|k+h|} \| |x| \tilde{\Phi}_m \|^2 |\omega(k+h) - \omega(k)| \chi_\Lambda(k+h) \right\} \end{aligned}$$

can be proven. Since $|e^{ikx} - 1| \leq |k||x|$ and $|\nabla_x e^{ikx}| = |k|$, by Assumption 5, we can see that

$$(6.10) \quad \begin{aligned} & |h|^{-1} \| (a(k+h) - a(k)) \tilde{\Phi}_m \|^2 \\ & \leq \text{const.} \left\{ \frac{1}{|k|(k_1^2 + k_2^2)} + \frac{1}{|k-h|((k_1-h_1)^2 + (k_2-h_2)^2)} \right\} \end{aligned}$$

holds. This inequality implies that $\{\|\tilde{\Phi}_m^{(n)}\|_{W^{1,p}(\Omega)}\}_{0 < m \leq m_0}$ with $1 \leq p < 2$ is bounded, where Ω is a bounded set and m_0 a sufficiently small number. Therefore in this case, the existence of ground states can be proven without the infrared regularity condition. Key inequalities are

$$(6.11) \quad |e^{ikx} - 1| \leq |k||x|$$

and

$$(6.12) \quad |\nabla_x e^{ikx}| = |k|.$$

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