

On ramifications of Artin-Schreier extensions of surfaces over algebraically closed fields of positive characteristic I

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Abstract

We give an algorithm which computes r , defined by K. Kato in the paper [1], which is an important invariant for Artin-Schreier extensions of surfaces X over fields of positive characteristic. The Swan conductor gives the invariant of ramifications concerning codimension 1 subvarieties of X . This r gives the invariant of ramifications concerning codimension 2 subvarieties of X . The invariant r is important to calculate the Euler Poincaré characteristic of some smooth l -adic sheaf of rank 1 on an open dense subscheme U of X .

Keywords Swan conductor, ramification, surface, Artin-Schreier extension

Research Activity Group Algorithmic Number Theory and Its Applications

1. Introduction

For a perfect field F of characteristic $p > 0$, a smooth surface X over $\text{Spec}(F)$, and an open subscheme U of X such that $D := (X \setminus U)_{\text{red}}$ is a simple normal crossing divisor, we consider an Artin-Schreier extension of X , namely a degree p étale covering of U . Such an extension is characterized by the character $\chi : \pi_1(U) \rightarrow \bar{\mathbb{Q}}_l^\times$ of exact order p , where l is a prime not equal to p . The ramification of χ is understood by the swan conductor $Sw(\chi)$, but it is not enough to understand the ramification of the Artin-Schreier extension. Kato defined an important invariant r depending on a closed point x of X , and χ . We review Kato's definition of r_x in [1, Remark 5.7], and we define r'_x which satisfies $r_x \geq r'_x \geq 0$ and $r_x = r'_x$ for “almost all” extensions. The purpose of this paper is to give an algorithm to compute r'_x in the case $X = \text{Spec}(F[x, y])$. Here F is an algebraically closed field of characteristic $p > 0$.

The Euler Poincaré characteristic of a smooth l -adic sheaf $\mathcal{F}_\chi$ is calculated by r_x , where $\mathcal{F}_\chi$ is the rank 1 sheaf on $U_{\text{ét}}$ corresponding to χ . This is written in Section 4.

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2. Review of the Swan conductor of a discrete valuation ring

2.1 Basic notation

Let F be an algebraically closed field of characteristic $p > 0$. Let X be a smooth surface over $\text{Spec}(F)$. Let D be a simple normal crossing divisor on X . Let $\text{cov}_p : Y \rightarrow X$ be a morphism of schemes of degree p , which is étale over $U := X - D$. Let K be the function field of X , and K' be the function field of Y . By the Artin-Schreier theory, there exists $f \in K$ such that $K' = K(\alpha)$, where α is a solution of the equation.

$$\alpha^p - \alpha = f. \quad (1)$$

2.2 Review of the Swan conductor of an imperfect residue field

Let L be a discrete valuation field, whose residue field F' is a field of characteristic $p > 0$. Let O_L be the integer ring of L . Let v_L be the normalized additive valuation of L . We define the Swan conductor of an Artin-Schreier extension L'/L as follows. We may assume $L' = L(\alpha)$, where $\alpha^p - \alpha = f \in L$. Then we define

$$Sw(L'/L) := \min\{\max\{-v_L(g) \mid g \equiv f \pmod{\mathfrak{B}(L)}\}, 0\}. \quad (2)$$

Here we put $\mathfrak{B}(L) := \{x^p - x \mid x \in L\}$. We also use the notation $Sw(f)$, $Sw_L(f)$ or $Sw(\chi)$ instead of $Sw(L'/L)$, where χ is a non-trivial character of $\text{Gal}(L'/L)$.

Moreover, let X , Y , K , and K' be as in Section 2.1, and D be an irreducible codimension 1 subvariety of X . Then, the normalized additive valuation of K is denoted by v_D . For a non-trivial character χ of $\text{Gal}(K'/K)$, we define $Sw_D(\chi)$ as above, and define $Sw(\chi)$ by

$$Sw(\chi) = \sum_{D \subset X} Sw_D(\chi) \cdot D, \quad (3)$$

where D runs through all irreducible codimension 1 subvarieties of X .

We give a concrete description of $Sw(f)$ next.

2.3 A concrete description of the Swan conductor of a discrete valuation field

Replacing f by g such that $g \equiv f \pmod{\mathfrak{B}(L)}$, we may assume that one of the following (i), (ii) or (iii) holds:

- (i) $f \in O_L$.
- (ii) $v_L(f) = -n$ such that $n > 0$ and $(n, p) = 1$.
- (iii) $f = at^{-n}$ where $a \in O_L^\times$, $n > 0$, $p \mid n$, t is a prime element of L , and the residue class of a in F' does not belong to $F'^p := \{x^p \mid x \in F'\}$.

In the case (i), we see $Sw(f) = 0$. In the case (ii) or (iii), we see $Sw(f) = n$.

2.4 The definition of cleanness

Let X , F , D , and Y be as in Section 2.1. The definition of cleanness is written in (3.6) of [1]. We define the cleanness at the closed point $x \in D$ following Kato's paper [1]. We say (X, D) is "Case (I)" at x if D has one irreducible component D'_1 at x , and we say (X, D) is "Case (II)" at x if D has two irreducible components D'_1 and D'_2 at x . From now on, we always assume there exists $t_1 \in O_{X,x}$ such that t_1 is a generator of the defining ideal of D in $O_{X,x}$ in Case (I), and there exist $t_1, t_2 \in O_{X,x}$ such that $t_1 t_2$ is a generator of the defining ideal of D in $O_{X,x}$ in Case (II).

Case (I): We say (1) is clean at $x \in D$ if and only if there exists $g \in K$ such that $g - f \in \mathfrak{B}(K)$ and either of the following holds:

- (I-1) $g \in O_{X,x}$.
- (I-2) $g = u/t_1^n$ ($u \in O_{X,x}^\times$, $n \geq 1$, $\gcd(n, p) = 1$).
- (I-3) $g = t/t_1^n$ ($n \geq 1$, $t \in O_{X,x}$ and (t_1, t) is the maximal ideal).

Case (II): We say (1) is clean at $x \in D$ if and only if either of the following holds:

- (II-1) $g \in O_{X,x}$.
- (II-2) $g = u/t_1^a t_2^b$, ($u \in O_{X,x}^\times$, $a, b \geq 1$ and $\gcd(a, b, p) = 1$).

Here $t_1 t_2 \in O_{X,x}$ is a generator of the defining ideal of D in $O_{X,x}$.

3. Review of the definition of r

Let X , F , D , $x \in D$, and f be as in Section 2.4. Then $r_{(X,D)}(x, f)$ is inductively defined as follows. If (1) is clean at x , $r_{(X,D)}(x, f) := 0$. Otherwise,

$$r_{(X,D)}(x, f) := \mu + \sum_{x' \in pr^{-1}(x)} r_{(X', pr^{-1}(D))}(x', f), \quad (4)$$

where X' is the blow up of X at x , and $pr : X' \rightarrow X$ is the natural map. Here we put $\mu := e(e-1)$ (resp. $\mu := e^2$) in Case (I) (resp. in Case (II)) and

$$e := Sw_{D'_1}(f) - Sw_{pr^{-1}(x)}(f) \text{ in Case (I),}$$

$$e := Sw_{D'_1}(f) + Sw_{D'_2}(f) - Sw_{pr^{-1}(x)}(f) \text{ in Case (II).}$$

It is known $r_{(X,D)}(x, f)$ is well defined (by finite steps) and finite.

3.1 The definition of r' (Special case)

Throughout this section, we always assume $X = \text{Spec}(F[x, y])$, $D := D_1$ or D_2 , and $pt = (0, 0)$, where

$$D_1 := \{(x, y) \in F^2 \mid x = 0\},$$

$$D_2 := \{(x, y) \in F^2 \mid x = 0 \text{ or } y = 0\}.$$

Let X' be the blow-up of X at $pt = (0, 0)$. Then X' have two open coverings U_1 and U_2 , where $U_1 = \text{Spec}(F[y, w])$ and $U_2 = \text{Spec}(F[x, s])$. In $U_1 \cap U_2$, $x = yw$ and $y = xs$ are satisfied.

Now $r'_{(X,D)}(pt, f)$ is inductively defined as follows. It is known that r' is well defined (by finite steps) and finite.

If (1) is clean at x , $r'_{(X,D)}(pt, f) := 0$. Otherwise,

$$r'_{(X,D)}(pt, f) := \mu + \sum_{i=1,2} r'_{(U_i, pr^{-1}(D) \cap U_i)}(pt_i, f).$$

Here $pt_1 = (0, 0) \in U_1$ in the coordinate (x, w) , and $pt_2 = (0, 0) \in U_2$ in the coordinate (y, s) . Note that $U_i \cong X$ as schemes for $i = 1, 2$, and

$$pr^{-1}(D_i) \cap U_j \cong D_1, \quad i = 1, j = 2,$$

$$pr^{-1}(D_i) \cap U_j \cong D_2, \quad \text{otherwise.}$$

Note that we only treat the case $X = \text{Spec}(F[x, y])$, but we can easily generalize the definition of r' for any smooth surface X .

3.2 An algorithm to compute r' (I)

For simplicity, we may assume that $X = \text{Spec}(F[x, y])$, $D := D_1$ or D_2 , and $pt = (0, 0)$, and $f = c_0 y^m / x^n + c_1 y^{m+b_1} / x^{n+a_1} + \dots + c_k y^{m+b_k} / x^{n+a_k}$ for some natural number k and $c_i \in F^\times$ ($i = 0, 1, 2, \dots, k$). Throughout this paper, we assume $a_0 = b_0 = 0$. We may assume $(a_i, b_i) \neq (a_j, b_j)$ for $0 \leq i \neq j \leq k$. From now on, we always assume $p \nmid \gcd(n + a_i, m + b_i)$ for any i .

Theorem 1 *Let X , D , pt , and f be as above. Then $r'_{(X,D_i)}(pt, f)$ depends only on $n, m, a_1, b_1, \dots, a_k, b_k$.*

Proof The function f is written as

$$f = \sum_{i=0}^k \frac{c_i y^{m-n+b_i-a_i}}{w^{n+a_i}} \quad \text{in Case (I),}$$

$$f = \sum_{i=0}^k \frac{c_i s^{m+b_i}}{x^{n-m+a_i-b_i}} \quad \text{in Case (II).}$$

In this blow-up, e is written as

$$e = \max_{0 \leq i \leq k} \{n + a_i, 0\} + \max_{0 \leq i \leq k} \{-m - b_i, 0\} - \max_{0 \leq i \leq k} \{n - m + a_i - b_i, 0\}.$$

From the definition of r' , we see

$$r'_D(X, pt, f) = r'_{D_2}(X', pt_1, f) + r'_D(X', pt_2, f) + \mu. \quad (5)$$

It is easy to see that μ depends only on $n, m, a_1, b_1, \dots, a_k, b_k$. By the induction (explained in detail in the next section), we see r' depends only on $n, m, a_1, b_1, \dots, a_k, b_k$. This completes the proof of the theorem. It allows us to write $r'_{D_i}(n, m; a_1, b_1, \dots, a_k, b_k)$ or $r'_{D_i}(n, m; (a_i, b_i)_{1 \leq i \leq k})$ instead of $r'_{(X,D_i)}(pt, f)$. (QED)

4. Some mathematical properties of r' and an algorithm to compute r'

In this section, we give a theorem and an algorithm to compute r' . The difference between r and r' is described in Section 4.3.

4.1 Some concrete example for r'

Theorem 2 *In Case (I), $r'_{D_1}(n, m) = \max\{(m-1), 0\}n$. In Case (II), if $nm > 0$, then $r'_{D_2}(n, m) = nm$, and if $nm \leq 0$, then $r'_{D_2}(n, m) = 0$.*

Proof We prove this theorem by induction on $|n| + |m|$. At first we treat case (I). As in the proof of Theorem 1, we obtain $r'_{D_1}(n, m) = r'_{D_2}(n, m-n) + r'_{D_1}(n-m, m) + e(e-1)$. If $n > m \geq 1$, the right hand side of this equality is $(n-m)(m-1) + m(m-1) = n(m-1)$ by the induction

hypothesis, and if $m \geq n \geq 1$ and the right hand side of this equality is $n(m-n) + n(n-1) = n(m-1)$ by the induction hypothesis. Next, we treat case (II). By the above subsection, we obtain $r'_{D_2}(n, m) = r'_{D_2}(n, m-n) + r'_{D_2}(n-m, m) + e^2$. We see that the right hand side of this equality is $(n-m)m + m^2 = nm$ if $n > m \geq 1$, and if $m \geq n \geq 1$, that the right hand side of this equality is $n(m-n) + n^2 = nm$. The proof for $mn \leq 0$ is similar. (QED)

4.2 An algorithm to compute r' (II)

The following theorem is essential to compute r' .

Theorem 3 *If $a_j \geq a_i$, and $b_j \leq b_i$ ($i \neq j$), then we have*

$$r'_D(n, m; (a_{i'}, b_{i'})_{1 \leq i' \leq k}) = r'_D(n, m; (a_{i'}, b_{i'})_{1 \leq i' (\neq i) \leq k}).$$

Proof We assume that $f \in y^{m'}/x^{n'}O_{X,pt}^\times + y^{m'+b'_1}/x^{n'+a'_1}O_{X,pt}^\times + \cdots + y^{m'+a'_l}/x^{n'+a'_l}O_{X,pt}^\times$ for some natural number l . We see that $r'_{(X,D)}(pt, f) = r'(n', m'; a'_1, b'_1, \dots, a'_l, b'_l)$ by the equality (5) and that $O_{X,pt}^\times \subset O_{U_i,pt_i}^\times$ for $i = 1, 2$. Note that if $a_j \geq a_i$, and $b_j \leq b_i$, then we have

$$\begin{aligned} c_j y^{m'+b'_j}/x^{n'+a'_j}O_{X,pt}^\times + c_i y^{m'+b'_i}/x^{n'+a'_i}O_{X,pt}^\times \\ \in y^{m'+b'_j}/x^{n'+a'_j}O_{X,pt}^\times. \end{aligned}$$

This completes the proof of the theorem. (QED)

We denote

$$\begin{aligned} J_a &:= \left\{ j \mid b_j < \max_{j+1 \leq i \leq k} \{b_i\} \right\}, \\ J_b &:= \left\{ j \mid a_j > \max_{0 \leq i \leq j-1} \{a_i\} \right\}. \end{aligned}$$

Corollary 4 *If $0 < a_1 < \cdots < a_k$, then*

$$\begin{aligned} r'_{D_i}(n, m; (a_j, b_j)_{1 \leq j \leq k}) \\ = r'_{D_i}(n, m, (a_j, b_j)_{j \in J_a}), \quad \text{and if } b_{j_1} \leq 0, \\ r'_{D_i}(n, m; (a_j, b_j)_{1 \leq j \leq k}) \\ = r'_{D_i}(n - a_{j_1}, m - b_{j_1}; (a_j - a_{j_1}, b_j - b_{j_1})_{j(\neq j_1) \in J_a}). \end{aligned}$$

Here we put $j_1 = \min J_a$. If $0 < b_1 < \cdots < b_k$, then

$$r'_{D_i}(n, m; (a_j, b_j)_{1 \leq j \leq k}) = r'_{D_i}(n, m, (a_j, b_j)_{j \in J_b}).$$

Similar statements hold for r . From Theorem 3, we have to compute r' when $0 < a_1 < \cdots < a_k$ and $0 < b_1 < \cdots < b_k$.

We call k the length of $r'(n, m; a_1, b_1, \dots, a_k, b_k)$, and we call $\#J_a$ (or $\#J_a - 1$ in the case $b_{j_1} \leq 0$) the essential length of $r'(n, m; a_1, b_1, \dots, a_k, b_k)$. We call $a_1 + b_1 + \cdots + a_k + b_k$ the depth of r' , and we call

$$\text{depth}_{ess}(r') := \sum_{j \in J_a} (a_j + b_j)$$

the essential depth of r' if $b_{j_1} > 0$. We call

$$\text{depth}_{ess}(r') := \sum_{b_{j_1} \neq j \in J_a} (a_j - a_{j_1} + b_j - b_{j_1})$$

the essential depth of r' if $b_{j_1} \leq 0$. We describe an algorithm to compute $r'_{(X,D)}(n, m, a_1, b_1, \dots, a_k, b_k)$. It is calculated by the induction on the pair of length and essential depth, where the order of the pair is determined by the dictionary order.

(Step 1) By Theorem 3, we may assume that $0 < a_1 < \cdots < a_k$ and $0 < b_1 < \cdots < b_k$.

(Step 2) Using the definition of r' , to compute $r'(n, m; a_1, b_1, \dots, a_k, b_k)$, we need to calculate the values of

- (1) $r'(n-m, m; a_1-b_1, b_1, \dots, a_k-b_k, b_k)$,
- (2) $r'(n, m-n; a_1, b_1-a_1, \dots, a_k, b_k-a_k)$.

(Step 3) Note that the essential lengths of (1) and (2) are less than or equal to k .

(Case 3-1) If the essential lengths of (1) and (2) are less than k , then use the induction on k .

(Case 3-2) If the essential length of (1) or (2) is k , then all $a_i - b_i$ ($i = 1, 2, \dots, k$) have the same sign. Then we use the induction on the essential depth of r' .

4.3 Difference between r and r'

Let X be a smooth surface, and X' be the blow-up of X at the closed point $x \in X$. Note that $pr^{-1}(x) \simeq \mathbb{P}_F^1$, where $pr : X' \rightarrow X$ is the projection. We describe when r and r' differ. For simplicity let $X = \text{Spec}(F[x, y])$, $D := D_1$ or D_2 , $pt = (0, 0)$ and

$$f \in \frac{1}{x^{n_1}y^{n_2}} \left(\sum_{w=0}^a d_w x^w y^{a-w} + \mathfrak{m}^{a+1} \right) \quad (6)$$

for some non-negative integers n_1, n_2 . Here $\mathfrak{m} = (x, y)$ is the maximal ideal of $O_{X,pt}$.

Theorem 5 *Let assumptions be as above. Then $(X', pr^{-1}(D), f)$ is not clean at the point of $pr^{-1}(pt)$ other than $(0, 1)$ in Case (I), (other than $(0, 1)$ and $(1, 0)$ in Case (II)) if and only if $h(x, y) := \sum_{w=0}^a d_w x^w y^{a-w}$ has a double root at that point. More precisely, the roots of $h(x, y)$ correspond one-to-one with the points of $\mathbb{P}_F^1$. The double roots of h correspond one-to-one with the non-clean points of $\mathbb{P}_F^1$.*

Proof This is merely the definition of cleanness. (QED)

But this difficulty is easily solved by the coordinate transformation (the translation).

Theorem 6 *For simplicity, let X and f be as in Section 3.1. We assume $D = D_2$. Let $X_s \rightarrow X_{s-1} \rightarrow \cdots \rightarrow X_0 = X$ be a successive blow up of X at the closed point of Case (II). If X_s is a clean model, then $r_{(X,D)}(pt, f) = r'_{X,D}(pt, f)$.*

Proof This theorem is a consequence of Theorem 5. (QED)

4.4 Application

Let X, F, D, Y , and χ be as in Section 2. Let $\mathcal{F}_\chi$ be the smooth sheaf on $U_{\acute{e}t}$ corresponding to χ . Let $\chi_c(U, \mathcal{F})$ be the (compact support étale cohomological) Euler number of the sheaf $\mathcal{F}$ on $U_{\acute{e}t}$. By the calculation of r , we understood explicitly the formula (5.7.1) of [1]. More precisely, we obtain the following equality (c.f. [2]).

Theorem 7 (Saito [2]) *Let X be a smooth proper surface over $\mathrm{Spec}(F)$. Then*

$$\chi_c(U, \mathcal{F}_X) - \chi_c(U) = -\deg c_{\mathcal{F}_X}. \quad (7)$$

Here $c_{\mathcal{F}_X} \in CH_0(X)$ is defined by

$$c_{\mathcal{F}_X} := \sum_{x \in X} r_D(x, \chi) \cdot x - \left(Sw(\chi), Sw(\chi) + K_X^{\log} \right). \quad (8)$$

Note that $K_X^{\log}$ is the log canonical divisor of (X, D) , and that $(,)$ is the intersection pairing. As is well known, $\chi_c(U, \mathcal{F})$ is defined by

$$\chi_c(U, \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i H_c^i(U_{\acute{e}t}, \mathcal{F}). \quad (9)$$

We simply write $\chi_c(U)$ for $\chi_c(U, \mathbb{Q}_l)$.

4.5 Further topics

Further discussion is given in [3]. In that paper, we treat the general smooth surface X , and give a mathematical description of r .

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