

ASYMPTOTIC BEHAVIOR OF THE TRANSMISSION EIGENVALUES

GEORGI VODEV

1. DEFINITION OF THE TRANSMISSION EIGENVALUES

Let $\Omega \subset \mathbf{R}^d$, $d \geq 2$, be a bounded, connected domain with a C^∞ smooth boundary $\Gamma = \partial\Omega$. A complex number $\lambda \neq 0$ with $\operatorname{Re} \lambda \geq 0$ will be said to be a transmission eigenvalue if the following problem has a non-trivial solution:

$$\begin{cases} (\nabla c_1(x) \nabla + \lambda^2 n_1(x)) u_1 = 0 & \text{in } \Omega, \\ (\nabla c_2(x) \nabla + \lambda^2 n_2(x)) u_2 = 0 & \text{in } \Omega, \\ u_1 = u_2, \quad c_1 \partial_\nu u_1 = c_2 \partial_\nu u_2 & \text{on } \Gamma, \end{cases} \quad (1)$$

where ν denotes the Euclidean inner unit normal to Γ , $c_j, n_j \in C^\infty(\overline{\Omega})$, $j = 1, 2$ are strictly positive real-valued functions. The transmission eigenvalues can be viewed as the eigenvalues of the non-symmetric operator $\mathcal{A}$ defined by

$$\mathcal{A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{n_1(x)} \nabla c_1(x) \nabla u_1 \\ -\frac{1}{n_2(x)} \nabla c_2(x) \nabla u_2 \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \{(u_1, u_2) \in \mathcal{H} : \nabla c_1(x) \nabla u_1 \in L^2(\Omega), \nabla c_2(x) \nabla u_2 \in L^2(\Omega), \\ u_1 = u_2, \quad c_1 \partial_\nu u_1 = c_2 \partial_\nu u_2 \quad \text{on } \Gamma\}$$

where $\mathcal{H} = H_1 \oplus H_2$, $H_j = L^2(\Omega, n_j(x) dx)$. Then the transmission eigenvalues are the poles of the resolvent $(\mathcal{A} - \lambda^2)^{-1}$ (if it forms a meromorphic family) and the multiplicity of a pole λ_k is defined by

$$\begin{aligned} \operatorname{mult}(\lambda_k) &= \operatorname{rank}(2\pi i)^{-1} \int_{|\lambda - \lambda_k| = \varepsilon} (\lambda^2 - \mathcal{A})^{-1} 2\lambda d\lambda \\ &= \operatorname{tr}(2\pi i)^{-1} \int_{|\lambda - \lambda_k| = \varepsilon} (\lambda^2 - \mathcal{A})^{-1} 2\lambda d\lambda. \end{aligned}$$

Our goal is to study the asymptotic behavior of the counting function $N(r) = \#\{\lambda - \text{trans. eig.} : |\lambda| \leq r\}$, $r > 1$. We will see that it is closely related to the localization of the transmission eigenvalues on the complex plane.

2. THE DIRICHLET-TO-NEUMANN MAP

The Dirichlet-to-Neumann map, $N_j(\lambda) : H^1(\Gamma) \rightarrow L^2(\Gamma)$, associated to the pair (c_j, n_j) is defined by

$$N_j(\lambda)f = \partial_\nu u_j|_\Gamma,$$

where u_j solves the equation

$$\begin{cases} (\nabla c_j(x) \nabla + \lambda^2 n_j(x)) u_j = 0 & \text{in } \Omega, \\ u_j = f & \text{on } \Gamma. \end{cases} \quad (2)$$

Denote by G_j , $j = 1, 2$, the Dirichlet self-adjoint realization of the operator $-n_j^{-1}\nabla c_j \nabla$ on the Hilbert space H_j . It is well-known that $N_j(\lambda)$ is meromorphic with poles the eigenvalues of $\sqrt{G_j}$. Introduce the operator

$$T(\lambda) = c_1 N_1(\lambda) - c_2 N_2(\lambda).$$

We have the following trace formula.

Lemma 1. *Suppose that the inverse $T(\lambda)^{-1}$ exists as a meromorphic function. Then the resolvent of the operator $\mathcal{A}$ is meromorphic, too, and we have the formula*

$$M(\gamma) = M_1(\gamma) + M_2(\gamma) + \text{tr}(2\pi i)^{-1} \int_{\gamma} \frac{dT(\lambda)}{d\lambda} T(\lambda)^{-1} d\lambda \quad (3)$$

where γ is a simple, positively orientied, piecewise smooth, closed curve in the complex plane, which avoids the poles of $T(\lambda)^{-1}$ and the eigenvalues of $\sqrt{G_1}$ and $\sqrt{G_2}$, $M(\gamma)$ is the number of the transmission eigenvalues inside γ , and $M_j(\gamma)$ is the number of the eigenvalues of the operator $\sqrt{G_j}$ inside γ .

3. WEYL ASYMPTOTICS FOR THE COUNTING FUNCTION

The following result is proved in [9].

Theorem 1. *Suppose either the condition*

$$c_1(x) \equiv c_2(x) \equiv 1 \quad \text{in } \Omega, \quad n_1(x) \neq n_2(x), \quad \forall x \in \Gamma, \quad (\text{isotropic case}) \quad (4)$$

or the condition

$$c_1(x) \neq c_2(x), \quad \forall x \in \Gamma. \quad (\text{anisotropic case}) \quad (5)$$

Suppose also that the operator $T(\lambda)$ is invertible in a region of the form

$$\left\{ \lambda \in \mathbb{C} : \text{Re } \lambda > 1, |\text{Im } \lambda| \geq C (\text{Re } \lambda)^{1-\kappa} \right\}, \quad C > 0, 0 < \kappa \leq 1, \quad (6)$$

and satisfies there the bound

$$\|T(\lambda)^{-1}\| \leq C_0 |\lambda|^{M_0}, \quad C_0, M_0 > 0.$$

Then we have the asymptotics

$$N(r) = (\tau_1 + \tau_2) r^d + O_{\varepsilon}(r^{d-\kappa+\varepsilon}), \quad \forall 0 < \varepsilon \ll 1, \quad (7)$$

where

$$\tau_j = \frac{\omega_d}{(2\pi)^d} \int_{\Omega} \left(\frac{n_j(x)}{c_j(x)} \right)^{d/2} dx,$$

ω_d being the volume of the unit ball in $\mathbf{R}^d$.

Known results. In the isotropic case when $n_2 \equiv 1$, $n_1(x) > 1$ on Ω , the asymptotic for $N(r)$ with a remainder term $o(r^d)$ is proved by M. Faierman [3] and by L. Robbiano [12].

Idea of the proof. It is inspired by the paper [1] where Weyl type asymptotics have been proved for the counting function of the resonances associated to an exterior transmission problem. We can get an asymptotic for $N(r) - N(r/2)$ by using the trace formula (3), the Weyl asymptotics for the counting functions of the eigenvalues of G_1 and G_2 , and the Theorems of Caratheodory and Jensen. We use in an essential way that $\dim \Gamma = d - 1$.

4. PARABOLIC EIGENVALUE-FREE REGIONS

Thus, the problem of proving Weyl asymptotics for the counting function $N(r)$ is reduced to that one of proving parabolic eigenvalue-free regions. The following result is proved in [14] and concerns the isotropic case.

Theorem 2. *Assume the condition*

$$c_1(x) \equiv c_2(x) \equiv 1 \quad \text{in } \Omega, \quad n_1(x) \neq n_2(x), \quad \forall x \in \Gamma. \quad (8)$$

Then there are no transmission eigenvalues in

$$\left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0, |\operatorname{Im} \lambda| \geq C_\epsilon (\operatorname{Re} \lambda + 1)^{\frac{1}{2} + \epsilon} \right\}, \quad \forall 0 < \epsilon \ll 1.$$

In this case the asymptotic (7) holds with $\kappa = 1/2$.

In the anisotropic case the situation is more interesting and one has to distinguish two subcases. The following result is proved in [14].

Theorem 3. *Assume the condition*

$$(c_1(x) - c_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) < 0, \quad \forall x \in \Gamma. \quad (9)$$

Then there are no transmission eigenvalues in the union of the sets

$$\left\{ \lambda \in \mathbf{C} : 0 \leq \operatorname{Re} \lambda \leq 1, \operatorname{Re} \lambda \geq C_N (|\operatorname{Im} \lambda| + 1)^{-N} \right\}, \quad \forall N \gg 1,$$

and

$$\left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 1, |\operatorname{Im} \lambda| \geq C_\epsilon (\operatorname{Re} \lambda)^{\frac{1}{2} + \epsilon} \right\}, \quad \forall 0 < \epsilon \ll 1.$$

In this case the asymptotic (7) holds with $\kappa = 1/2$.

Assume the condition

$$(c_1(x) - c_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) > 0, \quad \forall x \in \Gamma. \quad (10)$$

Then there are no transmission eigenvalues in

$$\left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0, |\operatorname{Im} \lambda| \geq C (\operatorname{Re} \lambda + 1)^{\frac{3}{5}} \right\}.$$

In this case the asymptotic (7) holds with $\kappa = 2/5$. Moreover, if in addition to (10) we assume either the condition

$$\frac{n_1(x)}{c_1(x)} \neq \frac{n_2(x)}{c_2(x)}, \quad \forall x \in \Gamma, \quad (11)$$

or the condition

$$\frac{n_1(x)}{c_1(x)} = \frac{n_2(x)}{c_2(x)}, \quad \forall x \in \Gamma, \quad (12)$$

then there are no transmission eigenvalues in

$$\left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0, |\operatorname{Im} \lambda| \geq C_\epsilon (\operatorname{Re} \lambda + 1)^{\frac{1}{2} + \epsilon} \right\}, \quad \forall 0 < \epsilon \ll 1.$$

Remark. One can show that under the condition (9) there are infinitely many transmission eigenvalues in

$$\left\{ \lambda \in \mathbf{C} : 0 \leq \operatorname{Re} \lambda \leq C_N (|\operatorname{Im} \lambda| + 1)^{-N} \right\}$$

and that their counting function, $N^-(r)$, satisfies an asymptotic of the form

$$N^-(r) = \tau_0 r^{d-1} + O(r^{d-2}).$$

Known results. In the isotropic case when $c_1 \equiv c_2 \equiv 1$, $n_2 \equiv 1$, $n_1(x) > 1$ on Ω , it was proved by M. Hitrik, K. Krupchyk, P. Ola and L. Päiväranta [4] that there are no transmission eigenvalues in

$$\left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0, |\operatorname{Im} \lambda| \geq C(\operatorname{Re} \lambda + 1)^{\frac{23}{25}} \right\}.$$

To prove the above theorems we make our problem semi-classical by putting $h = |\operatorname{Re} \lambda^2|^{-1/2}$, $z = h^2 \lambda^2 = \pm 1 + i \operatorname{Im} z$, if $|\operatorname{Re} \lambda^2| \geq |\operatorname{Im} \lambda^2|$, and $h = |\operatorname{Im} \lambda^2|^{-1/2}$, $z = h^2 \lambda^2 = \operatorname{Re} z + i$, if $|\operatorname{Re} \lambda^2| \leq |\operatorname{Im} \lambda^2|$. Clearly, $h \sim |\lambda|^{-1}$. The proof of Theorems 2 and 3 is based on the following semi-classical properties of the Dirichlet-to-Neumann map $N_j(z, h) = -ihN_j(\lambda)$ (see [14]).

Theorem 4. *For every $0 < \epsilon \ll 1$, $0 < h \ll 1$, $|\operatorname{Im} z| \geq h^{1/2-\epsilon}$, the Dirichlet-to-Neumann map $N_j(z, h)$ is an $h - \Psi$ DO of class $OPS_{1/2-\epsilon}^1(\Gamma)$ with a principal symbol*

$$\rho_j(x, \xi) = \sqrt{-r_0(x, \xi) + m_j(x)z} \quad \text{with} \quad \operatorname{Im} \rho_j > 0,$$

where m_j denotes the restriction on Γ of the function n_j/c_j , and r_0 is the principal symbol of the Laplace-Beltrami operator $-\Delta_\Gamma$, Γ being considered as a Riemannian manifold equipped with the Riemannian metric induced by the Euclidean one.

Recall that $a \in S_\delta^k(\Gamma)$, $0 \leq \delta < 1/2$, if $a \in C^\infty(T^*\Gamma)$ satisfies the bounds

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} h^{-\delta(|\alpha|+|\beta|)} \langle \xi \rangle^{k-|\beta|}.$$

It is well-known that for $h - \Psi$ DOs with such symbols there is a very nice calculus (e.g. see [2]).

Thus getting eigenvalue-free regions is reduced to inverting the operator

$$T(z, h) = c_1 N_1(z, h) - c_2 N_2(z, h)$$

with a principal symbol

$$c_1 \rho_1 - c_2 \rho_2 = \frac{\tilde{c}(x)(c_0(x)r_0(x, \xi) - z)}{c_1 \rho_1 + c_2 \rho_2} \quad (13)$$

where $\tilde{c}$ and c_0 are the restrictions on Γ of the functions

$$c_1 n_1 - c_2 n_2 \quad \text{and} \quad \frac{c_1^2 - c_2^2}{c_1 n_1 - c_2 n_2}$$

respectively. In the isotropic case we have $c_0 \equiv 0$ on Γ , while in the anisotropic case we have $c_0(x) \neq 0$, $\forall x \in \Gamma$. Under the condition (9) we have $c_0(x) < 0$, $\forall x \in \Gamma$, while under the condition (10) we have $c_0(x) > 0$, $\forall x \in \Gamma$.

The parametrix of $N_j(z, h)$ is bad when $\operatorname{Re} z = 1$ near the glancing region

$$\Sigma_j = \{(x, \xi) \in T^*\Gamma : r_0(x, \xi) - m_j(x) = 0\}.$$

Therefore, to improve the above results one has to improve the parametrix construction in the glancing region. Indeed, a better parametrix has been constructed in [15] for strictly concave domains valid for $|\operatorname{Im} z| \geq h^{1-\epsilon}$, which led to some improvements in this case.

5. OPTIMAL EIGENVALUE-FREE REGIONS

We can improve the above eigenvalue-free regions if $\Sigma_1 \cap \Sigma_2 = \emptyset$. More precisely, we have the following (see [16]).

Theorem 5. *Assume either the condition (8) or the condition (9). Then there are no transmission eigenvalues in*

$$\{\lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 1, |\operatorname{Im} \lambda| \geq C > 0\}. \quad (14)$$

In this case the asymptotic (7) holds with $\kappa = 1$.

The eigenvalue-free region (14) has been previously proved in [10] in the case of a ball and constant coefficients. It is shown by Leung and Colton [6] that in the isotropic case when Ω is a ball and the refraction indices n_1 and n_2 constants, the eigenvalue-free region (14) is optimal. In the anisotropic case we also have the following (see [16]).

Theorem 6. *Assume the conditions (10) and (11). Then there are no transmission eigenvalues in*

$$\{\lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0, |\operatorname{Im} \lambda| \geq C \log(\operatorname{Re} \lambda + 2)\}, \quad C > 0. \quad (15)$$

In this case the asymptotic (7) holds with $\kappa = 1$.

Define the cut-off function $\chi_j^0 \in C_0^\infty(T^*\Gamma)$ by

$$\chi_j^0(x, \xi) = \phi((r_0(x, \xi) - m_j(x))\delta^{-2})$$

where $0 < \delta \ll 1$ is a small parameter independent of h and z , and $\phi \in C_0^\infty(\mathbf{R})$, $0 \leq \phi \leq 1$, $\phi(t) = 1$ for $|t| \leq 1$, $\phi(t) = 0$ for $|t| \geq 2$, is also independent of h and z . Theorems 5 and 6 follow from the following (see [16]).

Theorem 7. *Let $\operatorname{Re} z = 1$ and let $0 < \epsilon < 1$ be arbitrary. Then, for every $0 < \delta \ll 1$ there are constants $C_\delta > 1$ and $0 < h_0(\epsilon, \delta) \ll 1$ such that we have*

$$\|N_j(z, h) - \operatorname{Op}_h(\rho_j(1 - \chi_j^0) + hb_j)\|_{L^2(\Gamma) \rightarrow H_h^1(\Gamma)} \leq C\delta \quad (16)$$

for $C_\delta h \leq |\operatorname{Im} z| \leq h^\epsilon$, $0 < h \leq h_0(\epsilon, \delta)$, where $C > 0$ is a constant independent of h , z and δ , and $b_j \in S_0^0(\Gamma)$ is independent of h , z and the function n_j .

Here $H_h^1(\Gamma)$ denotes the Sobolev space equipped with the semi-classical norm.

6. THE DEGENERATE ISOTROPIC CASE

We will consider the case when

$$c_1(x) \equiv c_2(x) \equiv 1 \quad \text{in } \Omega, \quad n_1(x) = n_2(x), \quad \forall x \in \Gamma.$$

We have the following (see [17]).

Theorem 8. *Assume that there is an integer $j \geq 1$ such that*

$$\partial_\nu^s(n_1(x) - n_2(x)) = 0, \quad \forall x \in \Gamma, \quad 0 \leq s \leq j-1, \quad (17)$$

and

$$\partial_\nu^j(n_1(x) - n_2(x)) \neq 0, \quad \forall x \in \Gamma. \quad (18)$$

Then there are no transmission eigenvalues in

$$\left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0, |\operatorname{Im} \lambda| \geq C(\operatorname{Re} \lambda + 1)^{1-\kappa_j} \right\},$$

where $\kappa_j = 2(3j+2)^{-1}$. *In this case the asymptotic (7) holds with $\kappa = \kappa_j$.*

It has been previously proved by Lakshtanov and Vainberg [5] that under the conditions (17) and (18) there are no transmission eigenvalues in $|\arg \lambda| \geq \epsilon$, $|\lambda| \geq C_\epsilon \gg 1$, $\forall 0 < \epsilon \ll 1$.

7. OPEN PROBLEMS

Conjecture 1. *For an arbitrary domain Ω , the counting function of the transmission eigenvalues satisfies the Weyl asymptotics*

$$N(r) = (\tau_1 + \tau_2)r^d + O(r^{d-1}). \quad (19)$$

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UNIVERSITÉ DE NANTES, LABORATOIRE DE MATHÉMATIQUES JEAN LERAY, 2 RUE DE LA HOUSSINIÈRE, BP 92208, 44322 NANTES CEDEX 03, FRANCE
E-mail address: Georgi.Vodev@univ-nantes.fr