

On Reynolds-Hurkens-Coquand paradox and T -algebras

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Abstract

Recently, Coquand [2] presented a variant on the paradox, as a variation of Reynolds “paradox” [5] in terms of weakly initial T -algebras. We first formalize the notion of the powerful universe U in λHOL such as $\Gamma_U := \{U : \square, \text{in} : T(U) \rightarrow U, \text{out} : U \rightarrow T(U)\}$ with a functor T where $T(U) := \text{Pow}(\text{Pow}(U))$ and $\text{Pow}(U) := U \rightarrow *$. Then rather than a retraction such that $\text{out} \circ \text{in}$ is the identity on $T(U)$, we suppose logical equivalence with kind $*$ as follows:

- $\text{Ret}_1 := \Pi X : T(U). \Pi p : \text{Pow}(U). (\text{out} \circ \text{in } X p) \leftrightarrow X p$,
- $\text{Ret}_2 := \Pi X : T(U). \Pi p : \text{Pow}(U). (T \delta X p) \leftrightarrow X p$,
- $\text{Weak}_1 := \Pi X : T(U). \Pi p : \text{Pow}(U). (\text{out} \circ \text{in } X p) \leftrightarrow T \delta X p$, and
- $\text{Weak}_2 := \Pi X : T(U). \Pi p : \text{Pow}(U). (\text{out} \circ \text{in } X p) \leftrightarrow X (p \circ \delta)$,

where $\delta := (\text{in} \circ \text{out})$. Now, λHOL can prove that $\Gamma_U \vdash \text{Weak}_1 \leftrightarrow \text{Weak}_2$; $\Gamma_U \vdash \text{Ret}_1 \wedge \text{Ret}_2 \rightarrow \text{Weak}$; $\Gamma_U \vdash \text{Ret}_1 \wedge \text{Weak} \rightarrow \text{Ret}_2$; $\Gamma_U \vdash \text{Ret}_2 \wedge \text{Weak} \rightarrow \text{Ret}_1$; $\Gamma_U \vdash \neg \text{Ret}_1$; and $\Gamma_U \vdash \neg \text{Weak}$.

Next, in λU^- , we show a dual one with respect to the weakly initial T -algebra [2], i.e., a weakly final T -coalgebra as another instance of the powerful universe.

1. Introduction

Let λHOL (Church 1940) be a PTS (Pure Type System) with the specification [1], consisting of $*$: $\square$, $\square$: Δ for axioms; and $(*, *)$, $(\square, *)$, $(\square, \square)$ for rules. That is, λHOL is Girard’s system F_ω with the axiom $\square : \Delta$, which is a consistent type system and can introduce variables of kinds, e.g., $X : \square$. It is well-known that in System F, weakly initial T -algebras and final T -coalgebras are impredicatively represented as the categorical data types.

Let λU^- be λHOL together with the rule $(\Delta, \square)$, which can introduce polymorphic kinds like polymorphic types in system F. The use of the rule makes it possible to define powerful universes. Moreover, we work with the rules $(\square, \square)$ and $(\Delta, \square)$ to represent categorical data types, just like system F of $(*, *)$ and $(\square, *)$.

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2. Encoding powerful universe in λHOL

As usual, define logical connectives for types and formulae with the kind $*$. Also define power sets and data types as kinds with the super kind $\square$.

1. Bottom $\perp := \Pi X : *. X$ with kind $*$ by $(\square, *)$.
2. Negation $\neg := \lambda X : *. X \rightarrow \perp$ with kind $(* \rightarrow *)$ by $(\square, *), (\square, \square)$.
3. Conjunction $\wedge := \lambda A, B : *. \Pi X : *. (A \rightarrow B \rightarrow X) \rightarrow X$ with kind $(* \rightarrow * \rightarrow *)$ by $(\square, \square)$.

Write $A \leftrightarrow B$ for $(A \rightarrow B) \wedge (B \rightarrow A)$ with *pair*, *fst*, *snd* where $A, B : *$.

4. For $X : \square$, power set $\text{Pow}(X) := X \rightarrow *$ with super kind $\square$ by $(\square, \square)$.
Write $\text{T}(X)$ for $\text{Pow}(\text{Pow}(X))$ where $X : \square$.

5. Composition $g \circ f := \lambda x : X. g(fx)$
 $X, Y, Z : \square, f : (X \rightarrow Y), g : (Y \rightarrow Z) \vdash \lambda x : X. g(fx) : X \rightarrow Z$ by $(\square, \square)$.

6. Contravariant functor $\text{mon}_{X \rightarrow Y}(f) := \lambda q : \text{Pow}(Y). q \circ f$:
 $X, Y : \square, f : X \rightarrow Y \vdash \lambda q : \text{Pow}(Y). q \circ f : \text{Pow}(Y) \rightarrow \text{Pow}(X)$ by $(\square, \square)$.

7. Covariant functor $\text{T}_{X \rightarrow Y}(f) := \text{mon}_{\text{Pow}(Y) \rightarrow \text{Pow}(X)}(\text{mon}_{X \rightarrow Y} f)$
 $X, Y : \square, f : X \rightarrow Y \vdash \lambda F : \text{T}(X). F \circ (\text{mon}_{X \rightarrow Y} f) : \text{T}(X) \rightarrow \text{T}(Y)$ by $(\square, \square)$.

8. Powerful universe $U : \square$ and a well-formed context Γ by $\square : \Delta$ and $(\square, \square)$:
 $\Gamma_U := \{U : \square, \text{in} : (\text{T}(U) \rightarrow U), \text{out} : (U \rightarrow \text{T}(U))\}$
 $\delta := \text{in} \circ \text{out}$

9. Retraction by $(\square, *)$:

$$\begin{aligned} \text{Ret1} &:= (\Pi X : \text{T}(U). \Pi p : \text{Pow}(U). (\text{out} \circ \text{in}) X p \leftrightarrow X p) \\ \Gamma_{\text{ret1}} &:= \Gamma_U \cup \{\text{ret1} : (\Pi X : \text{T}(U). \Pi p : \text{Pow}(U). (\text{out} \circ \text{in}) X p \leftrightarrow X p)\} \\ \alpha_e X p &:= \text{fst}(\text{ret1 } X p) : (\text{out} \circ \text{in}) X p \rightarrow X p \\ \beta_e X p &:= \text{snd}(\text{ret1 } X p) : X p \rightarrow (\text{out} \circ \text{in}) X p \\ \text{Ret2} &:= (\Pi X : \text{T}(U). \Pi p : \text{Pow}(U). \text{T } \delta X p \leftrightarrow X p) \\ \Gamma_{\text{ret2}} &:= \Gamma_U \cup \{\text{ret2} : (\Pi X : \text{T}(U). \Pi p : \text{Pow}(U). \text{T } \delta X p \leftrightarrow X p)\} \end{aligned}$$

10. U is weakly initial (final) by $(\square, *)$:

$$\begin{aligned} \Gamma_U &\vdash \Pi X : \text{T}(U). \Pi p : \text{Pow}(U). (\text{out} \circ \text{in}) X p \leftrightarrow \text{T}(\text{in} \circ \text{out}) X p : * \text{ by } (\square, *). \\ \text{Weak1} &:= \Pi X : \text{T}(U). \Pi p : \text{Pow}(U). (\text{out} \circ \text{in}) X p \leftrightarrow \text{T}(\delta) X p \\ \text{Weak2} &:= (\Pi X : \text{T}(U). \Pi p : \text{Pow}(U). (\text{out} \circ \text{in}) X p \leftrightarrow X (p \circ \delta)) \\ \Gamma_{\text{weak1}} &:= \Gamma_U \cup \{w1 : (\Pi X : \text{T}(U). \Pi p : \text{Pow}(U). (\text{out} \circ \text{in}) X p \leftrightarrow \text{T}(\delta) X p)\} \\ \Gamma_{\text{weak2}} &:= \Gamma_U \cup \{w2 : (\Pi X : \text{T}(U). \Pi p : \text{Pow}(U). (\text{out} \circ \text{in}) X p \leftrightarrow X (p \circ \delta))\} \\ \alpha_w X p &:= \text{fst}(\text{weak2 } X p) : (\text{out} \circ \text{in}) X p \rightarrow X (p \circ \delta) \\ \beta_w X p &:= \text{snd}(\text{weak2 } X p) : X (p \circ \delta) \rightarrow (\text{out} \circ \text{in}) X p \end{aligned}$$

Lemma 1 *Let $X, Y, Z : \square, f : X \rightarrow Y, g : Z \rightarrow X, h : W \rightarrow Z, p : \text{Pow}(Y), P : \text{T}(X)$. Then we have the following elementary facts under $=_\beta$.*

1. $(f \circ g) \circ h =_\beta f \circ (g \circ h)$
2. $(\text{mon } g) \circ (\text{mon } f) =_\beta \text{mon}(f \circ g)$
3. $(\text{T } f) \circ (\text{T } g) =_\beta \text{T}(f \circ g)$
4. $\text{mon } f \circ p =_\beta p \circ f$
5. $\text{T } f \circ P =_\beta P \circ (\text{mon } f)$

We write simply $\Gamma \vdash A$ if $\Gamma \vdash M : A$ for some M .

2.1. Retraction and Γ_{ret} in λHOL

Definition 1 1. $X_0 := \lambda p : (\text{Pow } U). \Pi u : U. p u \rightarrow \neg(\text{out } u p)$ with $\text{T}(U)$.

2. $q_0 := \lambda u : U. \Pi p : (\text{Pow } U). p u \rightarrow \neg(\text{out } u p)$ with $\text{Pow}(U)$.
3. $x_0 := \text{in } X_0$ with U .

Under Γ_{ret1} , λHOL proves the following proposition.

Proposition 1 1. $\Gamma_{\text{ret1}} \vdash l_1 : X_0 q_0$ for some l_1 .

2. $\Gamma_{\text{ret1}} \vdash l_2 : q_0 x_0$ for some l_2 .

Proof. (1) $l_1 := \lambda u : U. \lambda h : q_0 u. h q_0 h$ proves $X_0 q_0$, i.e., $\Pi u : U. q_0 u \rightarrow \neg(\text{out } u q_0)$.

(2) $l_2 := \lambda p : \text{Pow}(U). \lambda h : p x_0. \lambda h_1 : X_0 p. (\alpha_e X_0 p) h_1 x_0 h h_1$ proves $q_0 x_0$, i.e., $\Pi p : \text{Pow}(U). p x_0 \rightarrow \neg(\text{out } x_0 p)$.

Proposition 2 1. λHOL proves that $\Gamma_{\text{ret1}} \vdash \perp$, and then $\Gamma_U \vdash \neg \text{Ret1}$ and $U : \square \vdash \Pi \text{in} : (\text{T}U \rightarrow U). \Pi \text{out} : (U \rightarrow \text{T}U). \neg \text{Ret1}$.

2. Let $F := (\alpha_e X_0 q_0) \circ (\beta_e X_0 q_0) : X_0 q_0 \rightarrow X_0 q_0$. λHOL never enjoy the reduction such that $F l \rightarrow_\beta l$.

Proof. (1) λHOL proves that $\Gamma_{\text{ret1}} \vdash l_1 x_0 l_2 (\beta_e X_0 q_0 l_1) : \perp$, and hence $\Gamma_U \vdash \lambda \text{ret1} : \text{Ret1}. l_1 x_0 l_2 (\beta_e X_0 q_0 l_1) : \neg \text{Ret1}$, and then $\Pi \text{in} : (\text{T}U \rightarrow U). \Pi \text{out} : (U \rightarrow \text{T}U). \neg \text{Ret1}$ by $(\square, *)$.

(2) Towards a contradiction, suppose that $\alpha_e X_0 q_0 (\beta_e X_0 q_0 l) \rightarrow_\beta l$, i.e., type isomorphism $(\text{out} \circ \text{in}) X p \simeq X p$. Then the head reduction gives

$$\begin{aligned}
 & l_1 x_0 l_2 (\beta_e X_0 q_0 l_1) \\
 & \rightarrow_\beta^* l_2 q_0 l_2 (\beta_e X_0 q_0 l_1) \\
 & \rightarrow_\beta^* \alpha_e X_0 q_0 (\beta_e X_0 q_0 l_1) x_0 l_2 (\beta_e X_0 q_0 l_1) \\
 & \rightarrow_\beta^* l_1 x_0 l_2 (\beta_e X_0 q_0 l_1) \rightarrow_\beta^* \dots,
 \end{aligned}$$

leading to a contradiction to the fact that λHOL is strongly normalizing.

It should be remarked that we had $F l_1 x_0 l_2 (\beta_e X_0 q_0 l_1) =_\beta l_1 x_0 l_2 (\beta_e X_0 q_0 l_1)$ in the above so that the proof of the retract gave a certain fixed point of $F : X_0 q_0 \rightarrow X_0 q_0$. This simple example shows that an existence of type isomorphism $(\text{out} \circ \text{in}) X_0 p_0 \simeq X_0 p_0$ leads to a fixed point of functions with $X_0 q_0 \rightarrow X_0 q_0$.

2.2. Weak initiality (cofinality) and Γ_{weak} in λHOL

Under Γ_{weak_i} , λHOL proves the following proposition. Simply write Γ_{weak} for Γ_{weak_i} with $i = 1, 2$

Proposition 3 1. $\Gamma_{weak1} \vdash Weak2$, and $\Gamma_{weak2} \vdash Weak1$.

2. $\Gamma_{weak}, X : \mathbf{T}(U), p : \mathbf{Pow}(U) \vdash L_1 : \neg(out \circ in X p) \rightarrow \neg(X(p \circ \delta))$ for some L_1 .

3. $\Gamma_{weak}, X : \mathbf{T}(U), p : \mathbf{Pow}(U) \vdash L_2 : \neg(X(p \circ \delta)) \rightarrow \neg(out \circ in X p)$ for some L_2 .

Proof. (1) $\mathbf{T}(in \circ out) X p =_{\beta} X \circ \mathbf{mon}(in \circ out) p =_{\beta} X(\mathbf{mon}(in \circ out) p) =_{\beta} X(p \circ \delta)$.

(2) From the assumption *Weak*, $L_1 := \lambda z : \neg(out \circ in X p). \lambda b : X(p \circ \delta). z(\beta_w X p b)$.

(3) $L_2 := \lambda z : \neg(X(p \circ \delta)). \lambda a : (out \circ in X p). z(\alpha_w X p a)$.

Definition 2 1. $X_0 := \lambda p : (\mathbf{Pow} U). \Pi u : U. p u \rightarrow \neg(out u p)$ with $\mathbf{T}(U)$.

2. $p_0 := \lambda u : U. \Pi p : (\mathbf{Pow} U). p(\delta u) \rightarrow \neg(out u p)$ with $\mathbf{Pow}(U)$.

3. $x_0 := in X_0$ with U .

Under Γ_{weak} , λHOL proves the following proposition.

Proposition 4 1. $\Gamma_{weak} \vdash stableX_0 : (\Pi p : (\mathbf{Pow} U). X_0 p \rightarrow X_0(\mathbf{mon} \delta p))$

2. $\Gamma_{weak} \vdash stablep_0 : (\Pi z : U. p_0 z \rightarrow p_0(\delta z))$

3. $\Gamma_{weak} \vdash lem1 : (\Pi p : (\mathbf{Pow} U). p x_0 \rightarrow \neg(X_0 p))$

4. $\Gamma_{weak} \vdash lem2 : (p_0 x_0) = (\Pi p : (\mathbf{Pow} U). p(\delta x_0) \rightarrow \neg(out x_0 p))$

5. $\Gamma_{weak} \vdash lem3 : (X_0 p_0) = (\Pi u : U. p_0 u \rightarrow \neg(out u p_0))$

Proof. We show here the cases of 1 and 5. Leave it to the reader to prove the remaining cases.

(1) *stableX₀* : $(\Pi p : (\mathbf{Pow} U). X_0 p \rightarrow X_0(\mathbf{mon} \delta p))$

$$\begin{array}{c}
 \frac{[p : \mathbf{Pow}(U) \vdash h_p : X_0 p]}{h_p : \Pi u : U. p u \rightarrow \neg(out u p)} =_{\beta} \\
 \frac{L_1 : \neg(out(\delta u) p) \rightarrow \neg(out u(p \circ \delta)) \quad \frac{h_p(\delta u) : p(\delta u) \rightarrow \neg(out(\delta u) p)}{Prop3} u := (\delta u)}{L_1 \circ (h_p(\delta u)) : p(\delta u) \rightarrow \neg(out u(p \circ \delta))} \\
 \frac{L_1 \circ (h_p(\delta u)) : p(\delta u) \rightarrow \neg(out u(p \circ \delta))}{L_1 \circ (h_p(\delta u)) : (\mathbf{mon} \delta p) u \rightarrow \neg(out u(\mathbf{mon} \delta p))} =_{\beta} \\
 \frac{\lambda u : U. L_1 \circ (h_p(\delta u)) : \Pi u : U. (\mathbf{mon} \delta p) u \rightarrow \neg(out u(\mathbf{mon} \delta p))}{\lambda u : U. \lambda x. L_1(h_p(\delta u)) : X_0(\mathbf{mon} \delta p)} (\square, *) \\
 \frac{\lambda h_p : X_0 p. \lambda u : U. L_1 \circ (h_p(\delta u)) : X_0 p \rightarrow X_0(\mathbf{mon} \delta p)}{\lambda p : \mathbf{Pow}(U). \lambda h_p : X_0 p. \lambda u : U. L_1 \circ (h_p(\delta u)) : \Pi p : \mathbf{Pow}(U). X_0 p \rightarrow X_0(\mathbf{mon} \delta p)} (*, *)
 \end{array}$$

(5) *lem3* : $(X_0 p_0) = (\Pi u : U. p_0 u \rightarrow \neg(out u p_0))$

$$\begin{array}{c}
\frac{\frac{stablep_0 : \Pi u : U. p_0 u \rightarrow p_0(\delta u)}{[h_u : p_0 u] \quad \frac{stablep_0 u : p_0 u \rightarrow p_0(\delta u)}{stablep_0 u h_u : p_0(\delta u)}} \quad \frac{[h_z : p_0 u]}{h_z : \Pi p : \mathbf{Pow}(U). p(\delta u) \rightarrow \neg(out u p)} \quad =_{\beta} \\
\frac{stablep_0 u h_u : p_0(\delta u)}{h_z p_0 : p_0(\delta u) \rightarrow \neg(out u p_0)} \quad p := p_0 \\
\frac{h_u p_0 (stablep_0 u h_u) : \neg(out u p_0)}{\lambda h_u : p_0 u. h_u p_0 (stablep_0 u h_u) : p_0 u \rightarrow \neg(out u p_0)} (*, *) \\
\frac{\lambda u : U. \lambda h_u : p_0 u. h_u p_0 (stablep_0 u h_u) : \Pi u : U. p_0 u \rightarrow \neg(out u p_0)}{(\square, *)}
\end{array}$$

Let $loop := lem1 p_0 lem2 lem3$.

Proposition 5 1. $\lambda\mathbf{HOL}$ proves that $\Gamma_{weak} \vdash loop : \perp$.

2. $\lambda\mathbf{HOL}$ proves that $\Gamma_U \vdash \neg Weak$ and then $U : \square \vdash \Pi in : \mathbf{T}U \rightarrow U. \Pi out : U \rightarrow \mathbf{T}U. \neg Weak$.

3. Encoding of sets and algebras in $\lambda\mathbf{U}^-$

We define instances of the powerful universe, i.e., a weakly initial T -algebra $\mathbf{A}$ as in [2] and in addition here, a weakly final T -coalgebra $\mathbf{B}$, as follows.

- For $A, B : \square$, product $A \times B := \Pi X : \square. (A \rightarrow B \rightarrow X) \rightarrow X$ with $\square$.
 $\langle M, N \rangle := \lambda X : \square. \lambda z : (A \rightarrow B \rightarrow X). z M N$ with $A \times B$
 $\pi_1 M := M A (\lambda x : A. \lambda y : B. x)$ with A where $M : A \times B$
 $\pi_2 M := M B (\lambda x : A. \lambda y : B. y)$ with B where $M : A \times B$
- For $A, B : \square$, coproduct $A + B := \Pi X : \square. (A \rightarrow X) \rightarrow (B \rightarrow X) \rightarrow X$ with $\square$.
- For $G(X) : \square$ with $X : \square$,
 existential quantifier $\Sigma X : \square. G(X) := \Pi Y : \square. (\Pi X : \square. G(X) \rightarrow Y) \rightarrow Y$ with $\square$.
 $[B, M] := \lambda Y : \square. \lambda z : (\Pi X : \square. G(X) \rightarrow Y). z B M$ with $\Sigma X : \square. G(X)$.
 $(\mathbf{let} [X, z] = M \mathbf{in} N) := MC(\lambda X : \square. \lambda z : G(X). N)$ with $C : \square$,
 where $N : C$ and $M : (\Sigma X : \square. G(X))$.

Note that $(\mathbf{let} [X, z] = [B, M] \mathbf{in} N) \rightarrow_{\beta} N[X := B, z := M]$.

Now the context Γ_U in $\lambda\mathbf{HOL}$ can be established by the following algebras in $\lambda\mathbf{U}^-$, so that the condition *Weak* holds in $\lambda\mathbf{U}^-$ under the weakly initial and cofinal algebras, respectively.

3.1. Weakly initial algebra

We introduce the powerful universe $\mathbf{A}$ with **fold**, **intro**, and **match**. Then the following Proposition 6 can be established [2].

- Powerful universe $\mathbf{A} := \Pi X : \square. (\mathbf{T}(X) \rightarrow X) \rightarrow X$ with super kind $\square$.
 Let $\mathbf{fold}_X^{\mathbf{A}} := \lambda f. \lambda a^{\mathbf{A}}. a X f$ with $(\mathbf{T}(X) \rightarrow X) \rightarrow (\mathbf{A} \rightarrow X)$.
 Let $\mathbf{intro} := \lambda u : \mathbf{T}(\mathbf{A}). \lambda X : \square. \lambda f. f(\mathbf{T} \mathbf{A} X (\mathbf{fold}_X^{\mathbf{A}} f) u)$ with $\mathbf{T}(\mathbf{A}) \rightarrow \mathbf{A}$.
 Let $\mathbf{match} := \mathbf{fold}_{\mathbf{T}(\mathbf{A})}^{\mathbf{A}} (\mathbf{T}(\mathbf{T} \mathbf{A}) \mathbf{intro})$ with $\mathbf{A} \rightarrow \mathbf{T}(\mathbf{A})$, i.e., $\mathbf{match}(a) = a(\mathbf{T} \mathbf{A})(\mathbf{T} \mathbf{intro})$.
- Proposition 6** $\langle \mathbf{A}, \mathbf{intro} \rangle$ is weakly initial: for any $f : \mathbf{T}(X) \rightarrow X$, the following is commutative.

$$\begin{aligned}
& ((\mathbf{fold}_X^{\mathbf{A}} f) \circ \mathbf{intro}) =_{\beta} (f \circ (\mathbf{T}(\mathbf{fold}_X^{\mathbf{A}} f))) \\
& (\mathbf{match} \circ \mathbf{intro}) =_{\beta} ((\mathbf{T} \mathbf{intro}) \circ (\mathbf{T} \mathbf{match}))
\end{aligned}$$

3.2. Weakly final co-algebra

We here define the powerful universe $\mathbf{B}$ together with `unfold`, `in`, and `out` as well. Then the following Proposition 7 introduces a weakly final T -coalgebra.

1. Powerful universe $\mathbf{B} := \Sigma X : \square. (X \rightarrow T(X)) \times X$ with super kind $\square$.

Let $\text{unfold}_X^{\mathbf{B}} := \lambda f. \lambda z : X. [X, \langle f, z \rangle]$ with $(X \rightarrow T(X)) \rightarrow (X \rightarrow \mathbf{B})$.

Let $\text{out} := \lambda b : \mathbf{B}. (\text{let } [X, \beta] = b \text{ in } T(\text{unfold}_X(\pi_1 \beta)) (\pi_1 \beta (\pi_2 \beta)))$
with $\mathbf{B} \rightarrow T(\mathbf{B})$.

Let $\text{in} := \text{unfold}_{T\mathbf{B}}^{\mathbf{B}}(T(T\mathbf{B}) \text{out})$ with $T\mathbf{B} \rightarrow \mathbf{B}$, i.e., $\text{in}(u) = [T\mathbf{B}, \langle T \text{out}, u \rangle]$.

2. **Proposition 7** $\langle \mathbf{B}, \text{out} \rangle$ is weakly final: for any $f : X \rightarrow T(X)$, the following is commutative.

$$(\text{out} \circ (\text{unfold}_X^{\mathbf{B}} f)) =_{\beta} ((T(\text{unfold}_X^{\mathbf{B}} f)) \circ f)$$

$$(\text{out} \circ \text{in}) =_{\beta} ((T \text{in}) \circ (T \text{out}))$$

3.3. Computational behaviour of the proof $\vdash \perp$ in λU^-

Now for both algebras, we obtain the same computational behaviour of the proof, which do not produce a term that reduces to itself. In the proof Proposition4, the use of Prop3 can be simply replaced with $=_{\beta}$ by removing L_1, L_2 , and then reset `loop` := `lem1 p0 lem2 lem3`. Then `loop` has type $\perp$, and there exists an infinite reduction path with head redexes of `lem1`, 2, 3 [3], which means that redexes (cut-formulae) cannot be removed as follows.

$$\begin{aligned} \text{loop} &\rightarrow_{\beta}^+ \text{lem3 } x_0 \text{ lem2 } (\text{stableX}_0 p_0 \text{ lem3}) \\ &\rightarrow_{\beta}^+ \text{lem2 } p_0 (\text{stablep}_0 x_0 \text{ lem2}) (\text{stableX}_0 p_0 \text{ lem3}) \\ &\rightarrow_{\beta}^+ \text{lem1 } (p_0 \circ \delta) (\text{stablep}_0 x_0 \text{ lem2}) (\text{stableX}_0 p_0 \text{ lem3}) \\ &\rightarrow_{\beta}^+ \text{lem1 } (p_0 \circ \delta \circ \delta) (\text{stablep}_0 (\delta x_0) (\text{stablep}_0 x_0 \text{ lem2})) \\ &\quad (\text{stableX}_0 (p_0 \circ \delta) (\text{stableX}_0 p_0 \text{ lem3})) \\ &\rightarrow_{\beta}^+ \dots \end{aligned}$$

4. Concluding Remarks

Independently of the concrete structure of the powerful universe, λHOL can prove $\perp$ under the assumption Γ_{weak} or Γ_{ret} , so that λHOL derives $\Gamma_U \vdash \neg \text{Weak}$; $\Gamma_U \vdash \neg \text{Ret1}$; $\Gamma_U \vdash \text{Ret1} \wedge \text{Ret2} \rightarrow \text{Weak}$; $\Gamma_U \vdash \text{Ret1} \wedge \text{Weak} \rightarrow \text{Ret2}$; $\Gamma_U \vdash \text{Ret2} \wedge \text{Weak} \rightarrow \text{Ret1}$.

λU^- introduces not only the weakly initial T -algebra [2], but also the dual one, i.e., a weakly final T -coalgebra as instances of the powerful universe.

It should be remarked that natural transformations can be used to characterize functions on final coalgebras having a unique fixed point [6], and this property should be applied to our final T -coalgebra.

References

- [1] H. Barendregt: Lambda Calculi with Types, Handbook of Logic in Computer Science Vol. 2, Oxford University Press, 1992.
- [2] Th. Coquand: A variation of Reynolds-Hurkens Paradox, In V. Capretta, R. Krebbers, F. Wiedijk, editors, *Logics and Type Systems in Theory and Practice*, LNCS vol. 14560, pp. 111-117, 2024.
- [3] Th. Coquand: Private communication, June 14, 2024.

- [4] A.J.C. Hurkens: A simplification of Girard's paradox, In Mariangiola Dezani-Ciancaglini and Gordon Plotkin, editors, *Typed Lambda Calculi and Applications*, LNCS Vol. 902, pp. 266–278, 1995.
- [5] J.C.Reynodls and G.D.Plotkin: On functors expressible in the polymorphic typed lambda calculus. *Inf. Comput.* **105**, pp. 1–29, 1993.
- [6] D. Pavlovi'c: Gurded induction on final coalgebras, In B. Jacobs, L. Moss, H. Reichel, J.J.M.M. Rutten, editors, *Proc. 1st Intnat. Workshop on Coalgebraic Methods in Computer Science*, 1998.