

ON DEFINING EQUATIONS OF THREE VARIANTS OF THE GROTHENDIECK-TEICHMÜLLER GROUP

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0. INTRODUCTION

The Grothendieck-Teichmüller group $\widehat{GT}$ is a subgroup of the automorphism group $\text{Aut}\widehat{F}_2$ of the free pro-finite group of $\widehat{F}_2$ of rank 2. It is parameterized by elements of $\widehat{\mathbf{Z}}^\times \times [\widehat{F}_2, \widehat{F}_2]$ and is defined by three relations (I), (II) and (III) (see §1). It is the pro-finite group version of the pro-algebraic group GT [Dr]. In his study of Galois representation on fundamental groups, Y. Ihara showed that the absolute Galois group $G_{\mathbf{Q}} = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ can be embedded into $\widehat{GT}$ in [Ih94]. It is still open whether $G_{\mathbf{Q}}$ is equal to $\widehat{GT}$ or not. But recently there appeared other several new-type relations satisfied by $G_{\mathbf{Q}}$ in $\widehat{GT}$ (see [Ih00], [LNS], [NT] and also [NT*]). In this article, we show a certain relationship among them. In §1, we shall recall the definition of $\widehat{GT}$ and make a small remark (*Proposition 1.3*) which has been possibly unknown. §2 is a review of definitions of three variants Π ([LNS]), GTK ([Ih00]) and GTA ([Ih00]) of $\widehat{GT}$. In §3, we introduce main results of the author's master's thesis [F], which is on a relationship among defining equations of Π , GTK and GTA , and give a short sketch of its proof.

1. REVIEW OF THE DEFINITION OF $\widehat{GT}$

Let $\widehat{F}_2$ be the pro-finite free group of rank 2 with generators x and y . We define $\widehat{GT}$ to be the set of pairs $(\lambda, f) \in \widehat{\mathbf{Z}}^\times \times \widehat{F}_2'$ (where $\widehat{F}_2'$ means the topological commutator subgroup $[\widehat{F}_2, \widehat{F}_2]$ of $\widehat{F}_2$) satisfying

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the following three relations:

$$\left\{ \begin{array}{ll} \text{(I)} & f(x, y)f(y, x) = 1 \quad ((2\text{-cycle relation})) \\ \text{(II)} & f(z, x)z^m f(y, z)y^m f(x, y)x^m = 1 \\ & \quad \text{for } m = \frac{1}{2}(\lambda - 1), \, xyz = 1 \quad ((3\text{-cycle relation})) \\ \text{(III)} & f(x_{12}, x_{23})f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51}) = 1 \quad \text{in } \hat{P}_5^* \\ & \quad ((5\text{-cycle relation})). \end{array} \right.$$

Here $\hat{P}_5^*$ is the pro-finite pure sphere braid group with 5 strings and $x_{i,j} := x_{i,j}^{(5)}$ ($1 \leq i, j \leq 5$) are its standard generators [Ih94]. For $f \in \widehat{F}_2$ and elements α, β of a pro-finite group H , $f(\alpha, \beta)$ stands for the image of f by the homomorphism $\phi : \widehat{F}_2 \rightarrow H$ defined by $\phi(x) = \alpha$, $\phi(y) = \beta$. An element $\sigma = (\lambda, f) \in \widehat{GT}$ induces an endomorphism of $\widehat{F}_2$ by $\sigma(x) = x^\lambda$, $\sigma(y) = f^{-1}y^\lambda f$, from which we get an embedding $\widehat{GT} \hookrightarrow \text{End}\widehat{F}_2$ and in fact we can regard $\widehat{GT}$ as a sub-monoid of $\text{End}\widehat{F}_2$ ([Dr]).

Definition 1.1 ([Dr]). The *Grothendieck-Teichmüller group* $\widehat{GT}$ is the group of invertible elements of $\widehat{GT}$:

$$\widehat{GT} := \left\{ \sigma = (\lambda, f) \in \text{Aut } \widehat{F}_2 \mid (\lambda, f) \text{ satisfies (I) } \sim \text{(III)}. \right\}.$$

Remark 1.2. The above 5-cycle relation (III) is different from the original relation of $\widehat{GT}$ ([Dr])

$$\begin{aligned} \text{(III)}_{\text{DR}} \quad & f(x_{12}^{(4)}, x_{23}^{(4)} x_{24}^{(4)}) f(x_{13}^{(4)} x_{23}^{(4)}, x_{34}^{(4)}) \\ & = f(x_{23}^{(4)}, x_{34}^{(4)}) f(x_{12}^{(4)} x_{13}^{(4)}, x_{24}^{(4)} x_{34}^{(4)}) f(x_{12}^{(4)}, x_{23}^{(4)}) \quad \text{in } \hat{B}_4 \end{aligned}$$

which appeared in [NT] ([NT*]), where $\hat{B}_n$ ($n \in \mathbb{N}$) stands for the pro-finite braid group with n -strings. But we can show that (I)+(III) is equivalent to (I)+(III)_{DR}.

Proposition 1.3. *The relation (III) implies (I) .*

Proof . Recall that there is a basic projection $p : \hat{P}_5^* \rightarrow \hat{P}_4^*$ by sending $x_{i,j} = x_{i,j}^{(5)} \in \hat{P}_5^*$ to $x_{i,j}^{(4)} \in \hat{P}_4^*$ ($1 \leq i, j \leq 4$) and $x_{i,5} \in \hat{P}_5^*$ to $1 \in \hat{P}_4^*$ ($1 \leq i \leq 5$). It is immediate to see that (III) implies (I) because

$$\begin{aligned} & p(f(x_{12}, x_{23})f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51})) \\ & = f(x_{12}^{(4)}, x_{23}^{(4)})f(x_{23}^{(4)}, x_{34}^{(4)}) = f(x_{12}^{(4)}, x_{23}^{(4)})f(x_{23}^{(4)}, x_{12}^{(4)}) \end{aligned}$$

and $\hat{P}_4^*$ is a free group of rank 2 with generators $x_{12}^{(4)}$ and $x_{23}^{(4)}$. $\square$

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With the $G_{\mathbf{Q}}$ -action on the geometric fundamental group $\pi_1(\mathbf{P}_{\mathbf{Q}}^1 - \{0, 1, \infty\})$ of the projective line minus 3 points, we associate a pro-finite group homomorphism $\varphi : G_{\mathbf{Q}} \rightarrow \text{Aut} \widehat{F}_2$ (recall that $\widehat{F}_2 \simeq \pi_1(\mathbf{P}_{\mathbf{Q}}^1 - \{0, 1, \infty\})$). It follows from Belyi's theorem [Be] that φ is injective. In [Ih90] and [Ih94], it was shown that $\varphi(G_{\mathbf{Q}})$ lies in $\widehat{GT}$. Now to determine whether $G_{\mathbf{Q}}$ is equal to $\widehat{GT}$ or not is a basic open problem, which is also related to a project posed by A. Grothendieck in [Gr]. Recently there appeared other new-type relations which $G_{\mathbf{Q}}$ satisfies in $\widehat{GT}$, for example (I'), (II'), (III'), (IV), (IV'), (V), (A_n) , (K_n) for $n = 1, 2, 3, \dots$ (see [Ih00], [LNS], [NS], [NT] and also [NT*]). Although they do not seem to be deduced from defining relations (I), (II) and (III) of $\widehat{GT}$, it has not been shown yet whether they are really new ones and whether they are enough to characterize $G_{\mathbf{Q}}$ as a subgroup of $\widehat{GT}$.

2. REVIEW OF DEFINITIONS OF Γ , GTK AND GTA

To characterize $G_{\mathbf{Q}}$ in $\widehat{GT}$, three subgroups Γ , GTK and GTA are considered in [Ih00] and [LNS]. The subgroup Γ (resp. GTK) was geometrically constructed by P. Lochak, H. Nakamura, and L. Schneps in [LNS] (resp. by Y. Ihara in [Ih00]). On the other hand, GTA was arithmetically constructed by Y. Ihara in [Ih00]. They all contain $G_{\mathbf{Q}}$, but it has not been known whether they are really proper subgroups of $\widehat{GT}$ and whether they are equal to $G_{\mathbf{Q}}$.

2.1. Γ .

Definition 2.1 ([LNS]). The *new Grothendieck-Teichmüller group* Γ is the subset defined as follows:

$$\Gamma := \left\{ \sigma = (\lambda, f) \in \widehat{GT} \mid (\lambda, f) \text{ satisfies (III')} \text{ and (IV) below.} \right\}.$$

$$\begin{cases} \text{(III')} & g(x_{45}, x_{51})f(x_{12}, x_{23})f(x_{34}, x_{45}) = f(\sigma_1\sigma_3, \sigma_2^2) \quad \text{in } \hat{B}_5 \\ \text{(IV)} & f(\sigma_1, \sigma_2^4) = \sigma_2^{-8\Psi_2^{(0)}(\sigma)} f(\sigma_1^2, \sigma_2^2) \sigma_1^{-4\Psi_2^{(0)}(\sigma)} (\sigma_2\sigma_1)^{6\Psi_2^{(0)}(\sigma)} \text{ in } \hat{B}_3. \end{cases}$$

Here $g(x, y) \in \widehat{F}_2$ is the auxiliary parameter (depending on $\sigma \in \widehat{GT}$) satisfying $f(x, y) = g(y, x)^{-1}g(x, y)$ which was introduced in [LS]. For the definition of $\Psi_2^{(0)}(\sigma)^\dagger$, see §§2.2. In [LNS] and [NS], it was shown that actually Γ forms a subgroup of $\widehat{GT}$. Note that (III') implies (III). These two relations, (III') and (IV), describe the condition for elements

[†]The 1-cocycle $\rho_2(\sigma)$ and $\rho_3(\sigma)$ (see [LNS], [LS], [NS], [NT] and also [NT*]) are equal to $-\Psi_2^{(0)}(\sigma)$ and $-\Psi_3^{(0)}(\sigma)$ respectively.

of $\widehat{GT}$ to act (as $G_{\mathbf{Q}}$ does) on all types of pro-finite Teichmüller modular groups in a certain consistent way (for more details, see [LNS]).

2.2. GTK . For a natural number n , let H_n be the index n normal subgroup of $\widehat{F}_2$ which is freely generated by $n + 1$ elements $x^n, y, x^{-1}yx, \dots, x^{-(n-1)}yx^{n-1}$. For $\sigma = (\lambda, f) \in \widehat{GT}$, f belongs to H_n , since $H_n \supset [\widehat{F}_2, \widehat{F}_2]$. In [Ih99], Ihara constructed the extended Kummer 1-cocycle $\Psi_n^{(0)}(\sigma)$ for $\sigma \in \widehat{GT}$, which is the image of f by the continuous group homomorphism $H_n \rightarrow \widehat{\mathbf{Z}}$ defined by $x^n \mapsto 0, y \mapsto 1, x^{-j}yx^j \mapsto 0$ ($1 \leq j < n$). We remark that especially for $\sigma \in G_{\mathbf{Q}}$, $\Psi_n^{(0)}(\sigma)$ is the Kummer 1-cocycle which is characterized by $\sigma(\sqrt[n]{n}) = \sqrt[n]{n} \zeta_k^{-\Psi_n^{(0)}(\sigma)}$ for $k \in \mathbf{N}$, where $\zeta_k = \exp(\frac{2\pi i}{k})$. Suppose that $\varphi_n : H_n \rightarrow \widehat{F}_2$ is the continuous group homomorphism defined by $x^n \mapsto x, y \mapsto y, x^{-j}yx^j \mapsto 1$ ($1 \leq j < n$).

Definition 2.2 ([Ih00]). The *Grothendieck-Teichmüller-Kummer group* GTK is the subset defined as follows:

$$GTK_n := \left\{ \sigma = (\lambda, f) \in \widehat{GT} \mid (\lambda, f) \text{ satisfies } (K_n) \text{ below.} \right\}$$

$$GTK := \bigcap_{n \in \mathbf{N}} GTK_n$$

$$(K_n) \quad \varphi_n(f) = y^{\Psi_n^{(0)}(\sigma)} f \quad .$$

It follows immediately from [Ih00] *Proposition 1* that GTK_n and GTK actually form subgroups of $\widehat{GT}$. Relation (K_n) describes the condition for elements of $\widehat{GT}$ to act (as $G_{\mathbf{Q}}$ does) on H_n and $\widehat{F}_2$ consistently with two algebraic morphisms, the Kummer covering $\mathbf{P}_{\mathbf{Q}}^1 - \{0, \mu_n, \infty\} \rightarrow \mathbf{P}_{\mathbf{Q}}^1 - \{0, 1, \infty\}$ defined by $t \mapsto t^n$ and the natural inclusion $\mathbf{P}_{\mathbf{Q}}^1 - \{0, \mu_n, \infty\} \hookrightarrow \mathbf{P}_{\mathbf{Q}}^1 - \{0, 1, \infty\}$ defined by $t \mapsto t$ (for more details, see [Ih00]).

2.3. GTA . By the $\widehat{F}_2^{ab} (:= \widehat{F}_2' / \widehat{F}_2')$ -action on $(\widehat{F}_2')^{ab} := \widehat{F}_2' / [\widehat{F}_2', \widehat{F}_2']$ induced from the conjugation $n \mapsto fnf^{-1}$ for $n \in \widehat{F}_2'$ and $f \in \widehat{F}_2$, we can regard $(\widehat{F}_2')^{ab}$ as a free $A_2 (:= \widehat{\mathbf{Z}}[[\widehat{F}_2^{ab}]])$ -module of rank 1, generated by the class $\overline{[x, y]} \in (\widehat{F}_2')^{ab}$ of $[x, y] := xyx^{-1}y^{-1} \in \widehat{F}_2'$ (for more details, see [Ih99]). Thus the action of $\sigma = (\lambda, f) \in \widehat{GT}$ on $(\widehat{F}_2')^{ab}$ induced from that on $\widehat{F}_2$ is expressed $\sigma(\overline{[x, y]}) = B'_\sigma \cdot \overline{[x, y]}$, where $B'_\sigma \in A_2^\times$. The *adelic beta function* $B_\sigma \in A_2^\times$ was defined by $B'_\sigma = \frac{x^\lambda - 1}{x - 1} \frac{y^\lambda - 1}{y - 1} B_\sigma$ in [A] (for $\sigma \in G_{\mathbf{Q}}$) and [Ih99] (for $\sigma \in \widehat{GT}$). By the

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embedding constructed by G. W. Anderson in [A], B_σ can be regarded as a function on $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ valued in $\widehat{\mathbb{Z}} \otimes \mathbb{Q}^{ab}$, where $\mathbb{Q}^{ab}$ stands for the maximal abelian extension field over $\mathbb{Q}$. In [Ih99] *Proposition 1.6.1*, it was shown that the adelic beta function has much analogy with the classical beta function. Especially it is remarkable that the adelic beta function $B_\sigma(s_1, s_2)$ ($\sigma \in \widehat{GT}$, $(s_1, s_2) \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$) can be split into the product of the *adelic gamma function* Γ_σ , $B_\sigma(s_1, s_2) = \frac{\Gamma_\sigma(s_1)\Gamma_\sigma(s_2)}{\Gamma_\sigma(s_1+s_2)}$, where Γ_σ is a function on $\mathbb{Q}/\mathbb{Z}$ valued in the product $\prod_{p:\text{prime}} \mathbb{W}(\overline{\mathbb{F}_p})$ of

the Witt vector ring $\mathbb{W}(\overline{\mathbb{F}_p})$ of $\overline{\mathbb{F}_p}$. In [A](i) *Corollary 8.6.3*, Anderson showed the n -th multiplication formula of the adelic gamma function Γ_σ for elements σ of $G_{\mathbb{Q}}$ as an analogy of Gauß' n -th multiplication formula of the classical gamma function:

$$(A_n) \quad \prod_{nc=0} \frac{\Gamma_\sigma(s+c)}{\Gamma_\sigma(c)} \cdot \frac{1}{\Gamma_\sigma(ns)} = 1 \otimes \exp[2\pi i \cdot n\Psi_n^{(0)}(\sigma)s]$$

by using Deligne's theory of absolute Hodge cycles. Ihara guessed that this arithmetic relation (A_n) could be a key condition to distinguish $G_{\mathbb{Q}}$ from $\widehat{GT}$ and considered the following new subgroup of $\widehat{GT}$ containing $G_{\mathbb{Q}}$.

Definition 2.3 ([Ih00]). The *Grothendieck-Teichmüller-Anderson group* GTA is the subset defined as follows:

$$\begin{aligned} GTA_n &:= \left\{ \sigma = (\lambda, f) \in \widehat{GT} \mid \sigma \text{ satisfies } (A_n) \right\} \\ GTA &:= \bigcap_{n \in \mathbb{N}} GTA_n. \end{aligned}$$

It can be checked directly from definitions of $\Psi_n^{(0)}(\sigma)$ and $\Gamma_\sigma(s)$ that GTA and GTA_n actually form subgroups of $\widehat{GT}$.

The relationship among the above three subgroups $\mathbb{F}$, GTK and GTA has not been fully understood yet. But we remark that it was shown in [Ih99] that the relation (K_n) implies $(D\log A_n)$ which stands for the logarithmic derivative of the equation (A_n) .

3. MAIN RESULTS

Theorem 3.1. *Relations (I), (II), (IV) and (K_2) imply (A_{2^n}) for $n = 1, 2, 3, \dots$.*

Sketch of the proof. At first, we introduce a new parameter f_+ on the rank 3 free group $\widehat{F}_3$ with generators W , X and Y . Recall that the index 2 subgroup H_2 (§§2.2) of $\widehat{F}_2$ generated by xyx^{-1} , x^2 and y can be

identified with $\widehat{F_3}$ by sending xyx^{-1} , x^2 , y into W , X , Y respectively. Since $y^{-2\Psi_2^{(0)}(\sigma)}f$ lies on H_2 , we can associate an element $f_+(W, X, Y)$ of $\widehat{F_3}$ such that $y^{-2\Psi_2^{(0)}(\sigma)}f = f_+(xyx^{-1}, x^2, y)$. It is immediate that the equation (K₂) is equivalent to

$$(1) \quad y^{\Psi_2^{(0)}(\sigma)}f_+(xyx^{-1}, x^2, y) = f_+(1, x, y) .$$

It can be checked by calculation that (IV) is also re-expressed in terms of $f_+ \in \widehat{F_3}$ as follows:

$$(2) \quad f_+(W, X, Y)f_+(X^{-1}W^{-1}Y^{-1}, Y, X) = 1 .$$

From equations (I), (II) and (IV), we can also deduce the following equation

$$(3) \quad Y^m f_+(W, X, Y) X^m f_+(Z, W, X) W^m f_+(Y, Z, W) \\ Z^m f_+(X, Y, Z) = 1 \quad \text{for} \quad WXYZ = 1$$

On Anderson's duplication formula (A₂), we can show the following equivalence by using pro-finite free differential calculus developed in the appendix of [Ih99].

$$(A_2) \quad \frac{\Gamma_\sigma(s)\Gamma_\sigma(s+\frac{1}{2})}{\Gamma_\sigma(\frac{1}{2})\Gamma_\sigma(2s)} = 1 \otimes \exp[2\pi i \cdot \Psi_2^{(0)}(\sigma)s] \\ \iff B_\sigma(x, x) = x^{-2\Psi_2^{(0)}(\sigma)}B_\sigma(x, -1) \\ \iff \frac{x^\lambda + 1}{x^m(x+1)}B_\sigma(x, \frac{1}{x^2}) = x^{-2\Psi_2^{(0)}(\sigma)}B_\sigma(-1, x) \\ \iff \frac{x^\lambda + 1}{x^m(x+1)} \left\{ 1 - (x-1) \frac{df}{dx}(x, \frac{1}{x^2}) \right\} = x^{-2\Psi_2^{(0)}(\sigma)} \left\{ 1 + 2 \frac{df}{dx}(-1, x) \right\} \\ \iff \frac{x^\lambda + 1}{x^m(x+1)} \left[1 - (x-1) \left\{ x^{-2\Psi_2^{(0)}(\sigma)} \frac{df_+}{dW}(\frac{1}{x^2}, x^2, \frac{1}{x^2}) \cdot (1 - \frac{1}{x^2}) \right. \right. \\ \left. \left. + \frac{df_+}{dX}(\frac{1}{x^2}, x^2, \frac{1}{x^2}) \cdot 2x \right\} \right] \\ - x^{-2\Psi_2^{(0)}(\sigma)} \left[1 + 2x^{2\Psi_2^{(0)}(\sigma)} \left\{ (1-x) \frac{df_+}{dW}(x, 1, x) - 2 \frac{df_+}{dX}(x, 1, x) \right\} \right] = 0$$

By (1) and Anderson's identity ([Ih99] *Theorem A.1*),

$$\iff \frac{x^\lambda + 1}{x^m(x+1)} \left\{ 1 - x^{-2\Psi_2^{(0)}(\sigma)} \cdot (x-1) \frac{df_+}{dX}(1, x, \frac{1}{x^2}) \right\} \\ - x^{-2\Psi_2^{(0)}(\sigma)} \left[1 + 2x^{2\Psi_2^{(0)}(\sigma)} \left\{ (1-x) \frac{df_+}{dW}(x, 1, x) - 2 \frac{df_+}{dX}(x, 1, x) \right\} \right] = 0$$

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By (2) and Anderson's identity,

$$\begin{aligned} \iff & \frac{x^\lambda + 1}{x^m(x+1)} x^{-2\Psi_2^{(0)}(\sigma)} \left\{ 1 - (x^2 - 1) \frac{df_+}{dW} \left(1, x, \frac{1}{x^2} \right) \right\} \\ & - x^{-2\Psi_2^{(0)}(\sigma)} \left\{ 1 - 2 \left(\frac{x-1}{x} \right) \frac{df_+}{dW} \left(\frac{1}{x^2}, x, 1 \right) \right\} = 0. \end{aligned}$$

By direct calculation, it is possible to deduce the validity of the above last equation from (2), (3) and Anderson's identity.

Since we can prove that conditions (A_n) and (A_m) implies (A_{n+m}) for $n, m \in \mathbb{N}$, it is easy to see that one relation (A_2) implies infinite ones (A_{2^n}) for $n = 1, 2, 3, \dots$. Thus we get the claim. $\square$

Therefore, we see that arithmetic relations (A_{2^n}) for $n = 1, 2, 3, \dots$ follows from geometric relations, (I), (II), (IV) and (K_2) .

Corollary 3.2. $GTK \cap \Gamma \subseteq GTA_{2^\infty}$, where $GTA_{2^\infty} := \bigcap_{n \in \mathbb{N}} GTA_{2^n}$.

Proof . It follows immediately from *Theorem 3.1*. $\square$

Thus we get a relationship among the arithmetic subgroup GTA_{2^∞} and the geometric subgroups GTK and Γ .

REFERENCES

- [A] Anderson, G. W.; (i) The hyperadelic gamma function: a précis. Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986), 1–19, Adv. Stud. Pure Math., 12, North-Holland, Amsterdam, 1987.
(ii) The hyperadelic gamma function. Invent. Math. 95 (1989), no. 1, 63–131.
- [Be] Belyĭ, G. V.; Galois extensions of a maximal cyclotomic field. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 2, 267–276, 479.
- [Dr] Drinfel'd, V. G.; On quasitriangular quasi-Hopf algebras and a group closely connected with $\text{Gal}(\overline{Q}/Q)$. ; Leningrad Math. J. 2 (1991), no. 4, 829–860
- [F] Furusho, H.; Geometric and arithmetic subgroups of the Grothendieck-Teichmüller group. (in Japanese), Master's thesis, RIMS, Kyoto University, March 2000, in preparation to translate into English.
- [Gr] Grothendieck, A.; Esquisse d'un programme. Geometric Galois actions, 1, 5–48, Edited by Leila Schneps and Pierre Lochak. London Mathematical Society Lecture Note Series, 242. Cambridge University Press, Cambridge, 1997
- [Ih90] Ihara, Y.; Braids, Galois groups, and some arithmetic functions. Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 99–120, Math. Soc. Japan, Tokyo, 1991.
- [Ih91] ——— Automorphisms of pure sphere braid groups and Galois representations. The Grothendieck Festschrift, Vol. II, 353–373, Progr. Math., 87, Birkhäuser Boston, Boston, MA, 1990.

*Here, we use (I) and (II).

- [Ih94] ——— On the embedding of $\text{Gal}(\overline{Q}/Q)$ into $\widehat{GT}$. London Math. Soc. Lecture Note Ser., 200, The Grothendieck theory of dessins d'enfants (Luminy, 1993), 289–321, Cambridge Univ. Press, Cambridge, 1994.
- [Ih99] ——— On beta and gamma functions associated with the Grothendieck-Teichmüller groups. Aspects of Galois theory, 144–179, London Math. Soc. Lecture Note Ser., 256, Cambridge Univ. Press, Cambridge, 1999.
- [Ih00] ——— On beta and gamma functions associated with the Grothendieck-Teichmüller group II. J. Reine Angew. Math. 527 (2000), 1–11.
- [LNS] Lochak, P., Nakamura, H., Schneps, L. ; On a new version of the Grothendieck-Teichmüller group. C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), no. 1, 11–16.
- [LS] Lochak, P., Schneps, L. ; A cohomological interpretation of the Grothendieck-Teichmüller group. With an appendix by C. Scheiderer. Invent. Math. 127 (1997), no. 3, 571–600.
- [NS] Nakamura, H., Schneps, L. ; On a subgroup of the Grothendieck-Teichmüller group acting on the tower of profinite Teichmüller modular groups. Invent. Math. 141 (2000), no. 3, 503–560.
- [NT] Nakamura, H., Tsunogai, H. ; Harmonic and equianharmonic equations in the Grothendieck-Teichmüller group. Forum Math. (to appear).
- [NT*] Nakamura, H., Tsunogai, H. ; Harmonic equations in the Grothendieck-Teichmüller group. (in this volume).

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