

ASYMPTOTIC EXPANSIONS FOR A CLASS OF GENERALIZED HOLOMORPHIC EISENSTEIN SERIES: APPLICATIONS TO WEIERSTRASS' ELLIPTIC FUNCTION AND RAMANUJAN'S FORMULA FOR $\zeta(2k+1)$

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ABSTRACT. We shall establish complete asymptotic expansions for a class of generalized holomorphic Eisenstein series, when the associated parameter z tends to both 0 and ∞ through the complex upper half-plane $\mathfrak{H}^+$. These expansions are further applied to deduce several variants of classical Euler's and Ramanujan's formula for specific values of the Riemann zeta-function, as well as to show various functional relations for the classical Eisenstein series, and Weierstraß' elliptic and allied functions in terms of generalized Lambert series.

1. INTRODUCTION

Throughout the paper, s denotes a complex variable, z a complex parameter, and a , b , μ and ν real parameters. Let $\mathfrak{H}^\pm$ denote the complex upper and lower half-planes, respectively, where the argument of each branch is chosen as

$$\mathfrak{H}^+ = \{z \in \mathbb{C}^\times \mid 0 < \arg z < \pi\} \quad \text{and} \quad \mathfrak{H}^- = \{z \in \mathbb{C}^\times \mid -\pi < \arg z < 0\}.$$

It is frequently used in the sequel the notation $e(s) = e^{2\pi is}$, and the parameter $\tau = e^{\mp\pi i/2}z$ for $z \in \mathfrak{H}^\pm$, where τ varies within the sector $|\arg \tau| < \pi/2$.

We now define the generalized Eisenstein series $F_{\mathbb{Z}^2}^\pm(s; a, b; \mu, \nu; z)$ by

$$(1.1) \quad F_{\mathbb{Z}^2}^\pm(s; a, b; \mu, \nu; z) = \sum'_{m,n=-\infty}^{\infty} \frac{e((a+m)\mu + (b+n)\nu)}{\{a+m+(b+n)z\}^s} \quad (\operatorname{Re} s > 2),$$

where the primed summation symbols hereafter indicate that the possibly emerging singular terms such as $1/0^s$ are to be omitted, and the branch of each summand is chosen such that $\arg\{(a+m)+(b+n)z\}$ falls within the range $]-\pi, \pi]$ in $F_{\mathbb{Z}^2}^+$, and within $[-\pi, \pi[$ in $F_{\mathbb{Z}^2}^-$. The main object of this paper is the arithmetical mean of $F_{\mathbb{Z}^2}^\pm$ defined by

$$(1.2) \quad F_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z) = \frac{1}{2} \{F_{\mathbb{Z}^2}^+(s; a, b; \mu, \nu; z) + F_{\mathbb{Z}^2}^-(s; a, b; \mu, \nu; z)\},$$

for which we shall show that complete asymptotic expansions exist when both $\tau \rightarrow \infty$ (Theorem 1) and $\tau \rightarrow 0$ (Theorems 2 and 3) through the sector $|\arg \tau| < \pi/2$; the combination of Theorems 1–3 can further be applied to obtain several variants of the celebrated

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formulae of Euler and of Ramanujan for specific values of the Riemann zeta-function, as well as to deduce various functional relations for the classical Eisenstein series and for Weierstraß' elliptic and allied functions. One can see that a hidden (but crucial) rôle is played by the connection formula (2.23) below for Kummer's confluent hypergeometric functions in producing various functional relations for zeta-functions, Eisenstein series and elliptic functions mentioned above.

We give here a brief overview of the research related to holomorphic and non-holomorphic Eisenstein series of complex variables.

Lewittes [17] first obtained a transformation formula for

$$(1.3) \quad F(s; z) = F_{\mathbb{Z}^2}(s; 0, 0; 0, 0; z)$$

(with the notation in (1.0)), which was applied to show a modular relation connecting $F(2; z)$ with $F(2; -1/z)$; this transformation formula can be viewed as a prototype of our Theorem 1 below. He further established in [18] a transformation formula for a more general $F_{\mathbb{Z}^2}(s; a, b; 0, 0; z)$, which was extensively applied to study its modular relations when the modular group $SL_2(\mathbb{Z})$ acts on the associated parameter $z \in \mathfrak{H}^+$. A subsequent research was made by Berndt [1], who especially treated in this respect a class of generalized Dedekind eta-functions and Dedekind sums. Let $\zeta(s)$ denote the Riemann zeta-function. Berndt [2] then made a further research into this direction in connection with Euler's and Ramanujan's formulae for specific values of $\zeta(s)$.

On the other hand, Matsumoto [22] more recently derived complete asymptotic expansions for $F(s; z)$ when both $z \rightarrow 0$ and $z \rightarrow \infty$ through $\mathfrak{H}^+$; the latter can be viewed as a prototype of our Theorem 2 below. A transformation formula for a two variable analogue of (1.1) was obtained by Lim [21], while the first author [10] derived complete asymptotic expansions for a generalized *non-holomorphic* Eisenstein series of the form

$$\psi_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z) = \sum_{m, n=-\infty}^{\infty} \frac{e((a+m)\mu + (b+n)\nu)}{|a+m+(b+n)z|^{2s}} \quad (\operatorname{Re} s > 1)$$

both as $z \rightarrow 0$ and as $z \rightarrow \infty$ through the sector $\mathfrak{H}^+$. It has very recently been shown by the authors [15] that complete asymptotic expansions exist for a two variable analogue of $F(s; z)$, when the associated parameters $\mathbf{z} = (z_1, z_2)$ vary within the sectors $\mathfrak{H}^\pm$ so as that the distance $|z_2 - z_1|$ tends to both 0 and ∞ .

2. MAIN RESULTS

Prior to state our main results, we prepare several necessary notations.

Let $\kappa \in \mathbb{R}$ be a parameter. We then introduce the Lerch zeta-function $\phi(s, c, \kappa)$, together with its companion $\psi(s, c, \kappa)$, defined by

$$(2.1) \quad \phi(s, c, \kappa) = \sum_{k=0}^{\infty} \frac{e(k\kappa)}{(c+k)^s} \quad (\operatorname{Re} s > 1),$$

$$(2.2) \quad \psi(s, c, \kappa) = \sum_{k=0}^{\infty} \frac{e((c+k)\kappa)}{(c+k)^s} = e(c\kappa)\phi(s, c, \kappa),$$

which can be continued to entire functions if $\kappa \in \mathbb{R} \setminus \mathbb{Z}$, while for $\kappa \in \mathbb{Z}$ the former (or for $\kappa = 0$ the latter) reduces to the Hurwitz zeta-function $\zeta(s, c)$, also for $\kappa \in \mathbb{R}$ and $c = 1$ to the exponential zeta-function $\zeta_\kappa(s) = e(\kappa)\psi(s, 1, \kappa) = \psi(s, 1, \kappa)$, and hence to

the Riemann zeta-function $\zeta(s) = \zeta(s, 1) = \zeta_\kappa(s)$ if $\kappa \in \mathbb{Z}$. Note that

$$(2.3) \quad \phi(s, 0, \kappa) = e(\kappa)\phi(s, 1, \kappa) \quad \text{and} \quad \psi(s, 0, \kappa) = \psi(s, 1, \kappa)$$

hold by the convention of primed summation symbols; this implies that $\zeta_\kappa(s) = \phi(s, 0, \kappa) = \psi(s, 0, \kappa)$. The functional equation for $\phi(s, c, \kappa)$ (see, for e.g., [19][20]) with a slight extension asserts as follows.

Proposition 1 ([16, Lemma 3]). *For any $c, \kappa \in [0, 1]$, we have the functional equation*

$$(2.4) \quad \phi(s, c, \kappa) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \{e^{\pi i(1-s)/2} \psi(1-s, \kappa, -c) + e^{-\pi i(1-s)/2} \psi(1-s, 1-\kappa, c)\},$$

which reduces if $\kappa \in \{0, 1\}$ to

$$(2.5) \quad \zeta(s, c) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \{e^{\pi i(1-s)/2} \zeta_{-c}(1-s) + e^{-\pi i(1-s)/2} \zeta_c(1-s)\},$$

while if $c \in \{0, 1\}$ to

$$(2.6) \quad \zeta_\kappa(s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \{e^{\pi i(1-s)/2} \zeta(1-s, \kappa) + e^{-\pi i(1-s)/2} \zeta(1-s, 1-\kappa)\}$$

with the convention in (2.3).

Let $\langle x \rangle = x - [x]$ for any $x \in \mathbb{R}$ denote the fractional part of x . Then the functional equation (2.4) can be extended to the following form with a satisfactory extension of the domain of parameters.

Proposition 2 ([16, Lemma 4]). *For any $c, \kappa \in \mathbb{R}$, we have the functional equation*

$$(2.7) \quad \begin{aligned} \psi(s, \langle c \rangle, \kappa) &= e(c\kappa) \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \{e^{\pi i(1-s)/2} \psi(1-s, \langle \kappa \rangle, -c) \\ &\quad + e^{-\pi i(1-s)/2} \psi(1-s, \langle -\kappa \rangle, c)\}. \end{aligned}$$

Let r be a complex variable, and q a complex (base) parameter with $|q| < 1$. We further introduce the generalized Lambert series $\mathcal{S}_r(c, d; \kappa, \lambda; q)$, defined for any real c, d, κ and λ with $c, d \geq 0$ by

$$(2.8) \quad \mathcal{S}_r(c, d; \kappa, \lambda; q) = e(c'\kappa) \sum_{l=0}^{\infty} \frac{e((d+l)\lambda) q^{c'(d+l)}}{(d+l)^r \{1 - e(\kappa)q^{d+l}\}}$$

upon the convention (used hereafter) for any $c \in [0, +\infty[$ that

$$c' = \begin{cases} c & \text{if } c > 0, \\ 1 & \text{if } c = 0. \end{cases}$$

Further let $\delta(x)$ for $x \in \mathbb{R}$ denote the symbol which equals 1 or 0 according to $x \in \mathbb{Z}$ or otherwise, and $\Gamma(s)$ the gamma function and $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any $n \in \mathbb{Z}$ the shifted factorial.

We proceed to state our first main result.

Theorem 1 ([16, Theorem 1]). *Set*

$$(2.9) \quad \begin{aligned} \mathcal{A}(s, a, \mu) &= \psi(s, \langle -a \rangle, -\mu) \cos(\pi s) + \psi(s, \langle a \rangle, \mu) \\ &= e(a\mu) \frac{(2\pi)^s}{2\Gamma(s)} \{e^{-\pi is/2} \psi(1-s, \langle -\mu \rangle, a) + e^{\pi is/2} \psi(1-s, \langle \mu \rangle, -a)\}, \end{aligned}$$

where the second equality holds by (2.7). Then for any real a, b, μ and ν , and any $z \in \mathfrak{H}^+$, we have the formula

$$(2.10) \quad \begin{aligned} F_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z) &= \delta(b) \mathcal{A}(s, a, \mu) \\ &\quad + e(a\mu) \frac{(2\pi)^s}{\Gamma(s)} \{e^{-\pi i s/2} \mathcal{S}_{1-s}(\langle b \rangle, \langle -\mu \rangle; \nu, a; q) \\ &\quad + e^{\pi i s/2} \mathcal{S}_{1-s}(\langle -b \rangle, \langle \mu \rangle; -\nu, -a; q)\}, \end{aligned}$$

which is valid in the whole s -plane.

Remark. The formula (2.10) can be viewed as a transformation formula, and at the same time as a convergent asymptotic expansion when $\tau \rightarrow \infty$ through the sector $|\arg \tau| < \pi/2$, where the asymptotic series are given by $\mathcal{S}_{1-s}(\langle \pm b \rangle, \langle \mp \mu \rangle; \pm \nu, \pm a; q)$ on the right side, since each term of $\mathcal{S}_r(c, d; \kappa, \lambda; q)$ in (2.8) is of order $O\{e^{-2\pi\tau c'(d'+l)}/(d'+l)^r\}$ when $\tau \rightarrow \infty$ ($l = 0, 1, \dots$).

Let $\widetilde{\mathbb{C}}^\times$ denote the universal covering of the punctured complex plane $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, where the mapping $\widetilde{\mathbb{C}}^\times \ni \widetilde{Y} \mapsto \log \widetilde{Y} = \log |\widetilde{Y}| + i \arg \widetilde{Y} \in \mathbb{C}$ is bijective (with the range of $\arg \widetilde{Y}$ being extended over $\mathbb{R}$). We define for any $X \in \mathbb{C}$ and $\widetilde{Y} \in \widetilde{\mathbb{C}}^\times$ the operation

$$(2.11) \quad \begin{aligned} \widetilde{\mathbb{C}}^\times \ni \widetilde{Y} &\longmapsto \widetilde{Y}^X = \exp(X \log \widetilde{Y}) = \exp\{X(\log |\widetilde{Y}| + i \arg \widetilde{Y})\} \\ &= |\widetilde{Y}|^X \exp(iX \arg \widetilde{Y}) \in \mathbb{C}. \end{aligned}$$

Let $\widetilde{e}(\kappa) \in \widetilde{\mathbb{C}}^\times$ for any $\kappa \in \mathbb{R}$ denote the point defined by $\log \widetilde{e}(\kappa) = 2\pi i \kappa$, and write $\widetilde{e}(0) = \widetilde{1}$. Then $\widetilde{e}(\kappa)^c = e(c\kappa)$ holds for all $c \in \mathbb{R}$ by (2.11).

It is convenient for describing specific values of $\psi(s, c, \kappa)$ to introduce the sequence of functions $\mathcal{C}_k : \mathbb{C} \times \widetilde{\mathbb{C}}^\times \ni (X, \widetilde{Y}) \mapsto \mathcal{C}_k(X, \widetilde{Y}) \in \mathbb{C}$ ($k = 0, 1, \dots$), defined by the Taylor series expansion (with the variable Z in $\mathbb{C}$)

$$(2.12) \quad \frac{Z \widetilde{Y}^X e^{XZ}}{\widetilde{Y}^1 e^Z - 1} = \sum_{k=0}^{\infty} \frac{\mathcal{C}_k(X, \widetilde{Y})}{k!} Z^k$$

near $Z = 0$ (notice that $\widetilde{Y}^1 = |\widetilde{Y}| \exp(\log \widetilde{Y})$); this in particular implies that

$$(2.13) \quad \mathcal{C}_0(X, \widetilde{Y}) = \begin{cases} \widetilde{Y}^X & \text{if } \widetilde{Y}^1 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathcal{C}_k(X, \widetilde{Y})$ reduces if $\widetilde{Y} = \widetilde{1}$ (and so $\widetilde{Y}^X = 1$) to the usual Bernoulli polynomial $B_k(X)$, and also to the rational function $A_k(Y)$ if $X = 0$, defined by the Taylor series expansion

$$\frac{Z}{Y e^Z - 1} = \sum_{k=0}^{\infty} \frac{A_k(Y)}{k!} Z^k$$

centered at $Z = 0$. Professor Andrzej Schinzel kindly informed me (in a private communication [27]) about the explicit form of $A_k(Y)$ involving Eulerian (not Euler's) numbers (cf. [26, p.215]) in its coefficients. We have further shown in [10] the following properties of $\mathcal{C}_k(X, \widetilde{Y})$.

Proposition 3 ([10, Lemma 3]). *For any integer $k \geq 0$ and any $(X, \tilde{Y}) \in \mathbb{C} \times \widetilde{\mathbb{C}^\times}$, the relations*

$$(2.14) \quad \mathcal{C}_k(1 - X, \tilde{1}/\tilde{Y}) = (-1)^k \mathcal{C}_k(X, \tilde{Y}),$$

$$(2.15) \quad \mathcal{C}_k(0, \tilde{1}/\tilde{Y}) = \begin{cases} (-1)^k \mathcal{C}_k(0, \tilde{Y}) & \text{if } k \neq 1, \\ -\mathcal{C}_1(0, \tilde{Y}) - 1 & \text{if } k = 1 \end{cases}$$

hold, where $\tilde{1}/\tilde{Y} \in \widetilde{\mathbb{C}^\times}$ is the point defined by $|\tilde{1}/\tilde{Y}| = 1/|\tilde{Y}|$ and by $\arg(\tilde{1}/\tilde{Y}) = -\arg \tilde{Y}$.

We proceed to state our second main result.

Theorem 2 ([16, Theorem 2]). *Let a, b, μ and ν be arbitrary real parameter, and $z \in \mathfrak{H}^+$, write $q = e(z) = e^{2\pi iz}$ and $\hat{q} = e(-1/z) = e^{-2\pi i/z}$ for any $z \in \mathfrak{H}^+$, and set*

$$(2.17) \quad \begin{aligned} \mathcal{B}_1(s, a, \mu) &= \sin(\pi s) \psi(s, \langle -a \rangle, -\mu) \\ &= e(a\mu) \frac{(2\pi)^s}{2\Gamma(s)} \left\{ e^{\pi i(1-s)/2} \psi(1-s, \langle -\mu \rangle, a) \right. \\ &\quad \left. + e^{-\pi i(1-s)/2} \psi(1-s, \langle \mu \rangle, -a) \right\}, \end{aligned}$$

$$(2.18) \quad \begin{aligned} \mathcal{B}_2(s, b, \nu) &= e^{\pi is/2} \psi(s, \langle -b \rangle, -\nu) + e^{-\pi is/2} \psi(s, \langle b \rangle, \nu) \\ &= e(b\nu) \frac{(2\pi/\tau)^s}{\Gamma(s)} \psi(1-s, \langle \nu \rangle, -b), \end{aligned}$$

where the second equalities in (2.17) and (2.18) hold by (2.7). Then for any integer $J \geq 0$, in the region $\operatorname{Re} s > 1 - J$, we have the formula

$$(2.19) \quad \begin{aligned} F_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z) &= i\delta(b)\mathcal{B}_1(s, a, \mu) + \delta(a)\mathcal{B}_2(s, b, \nu)\tau^{-s} \\ &\quad + 2\sin(\pi s) \sum_{j=-1}^{J-1} \frac{i^{j+1}(s)_j}{(j+1)!} \psi(s+j, \langle -a \rangle, -\mu) \mathcal{C}_{j+1}(\langle b \rangle, \tilde{e}(\nu)) \tau^j \\ &\quad + R_J(s; a, b; \mu, \nu; z), \end{aligned}$$

where $R_J(s; a, b; \mu, \nu; z)$ is the remainder term satisfying the estimate

$$(2.20) \quad R_J(s; a, b; \mu, \nu; z) = O(|\tau|^J)$$

as $\tau \rightarrow 0$ through the sector $|\arg \tau| \leq \pi/2 - \eta$ with any small $\eta > 0$. Here the constant implied in the O -symbol depends at most on s, a, b, μ, ν, J and η .

We use the symbol $\varepsilon(Z) = \operatorname{sgn}(\arg Z)$ for $|\arg Z| > 0$, and let ${}_1F_1\left(\begin{smallmatrix} \alpha \\ \gamma \end{smallmatrix}; Z\right)$ and $U(\alpha; \gamma; Z)$ denote Kummer's confluent hypergeometric functions of the first and second kind, respectively, defined for any complex α and γ by

$$(2.21) \quad {}_1F_1\left(\begin{smallmatrix} \alpha \\ \gamma \end{smallmatrix}; Z\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k k!} Z^k$$

with $\gamma \neq -k$ ($k = 0, -1, \dots$) and for $|Z| < +\infty$ (cf. [5, 6.1 (1)]), and

$$(2.22) \quad U(\alpha; \gamma; Z) = \frac{1}{\Gamma(\alpha)\{e(\alpha) - 1\}} \int_{\infty}^{(0+)} e^{-Zw} w^{\alpha-1} (1+w)^{\gamma-\alpha-1} dw$$

for $|\arg Z| < \pi/2$, where the latter can be continued to the whole sector $|\arg Z| < 3\pi/2$ by rotating appropriately the path of integration. An application of the connection formula

$$(2.23) \quad {}_1F_1\left(\frac{\alpha}{\gamma}; Z\right) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)} e^{\varepsilon(Z)\pi i \alpha} U(\alpha; \gamma; Z) \\ + \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^{\varepsilon(Z)\pi i(\alpha - \gamma)} e^Z U(\gamma - \alpha; \gamma; e^{-\varepsilon(Z)\pi i} Z),$$

valid in the sectors $0 < |\arg Z| < 3\pi/2$ (cf. [5, 6.7 (7)]), allows us to extract the exponentially small order terms of the form $\mathcal{S}_{1-s}(c, d; \kappa, \lambda; \hat{q})$ with $\hat{q} = e^{-2\pi/\tau}$ as $\tau \rightarrow 0$ from the remainder in (2.19).

Theorem 3 ([16, Theorem 3]). *In the region $\sigma > 1 - J$ with any $J \geq 1$ and in the sectors $0 < |\arg \tau| < \pi/2$, we have the formula*

$$(2.24) \quad R_J(s; a, b; \mu, \nu; z) = e(b\nu) \frac{(2\pi/\tau)^s}{\Gamma(s)} \left\{ \mathcal{S}_{1-s}(\langle a \rangle, \langle \nu \rangle; \mu, -b; \hat{q}) \right. \\ \left. + e^{\varepsilon(\tau)\pi i s} \mathcal{S}_{1-s}(\langle -a \rangle, \langle -\nu \rangle; -\mu, b; \hat{q}) \right\} \\ + (-1)^J e(b\nu) (2\pi/\tau)^s \frac{\sin(\pi s)}{\pi} (s)_J S_J^*(s; a, b; \mu, \nu; z),$$

where the expression

$$(2.25) \quad S_J^*(s; a, b; \mu, \nu; z) = \sum_{m,n=0}^{\infty} \frac{e(-(\langle -a \rangle + m)\mu - (\langle \nu \rangle + n)b)}{(\langle \nu \rangle + n)^{1-s}} \\ \times f_{s,J}(2\pi(\langle -a \rangle + m)(\langle \nu \rangle + n)/\tau) \\ - e^{\varepsilon(\tau)\pi i s} \sum_{m,n=0}^{\infty} \frac{e(-(\langle -a \rangle + m)\mu + (\langle -\nu \rangle + n)b)}{(\langle -\nu \rangle + n)^{1-s}} \\ \times f_{s,J}(2\pi e^{\varepsilon(\tau)\pi i}(\langle -a \rangle + m)(\langle -\nu \rangle + n)/\tau)$$

holds with

$$(2.26) \quad f_{s,J}(Z) = U(s + J; s + J; Z).$$

Furthermore, for any integers J and K with $J \geq 1$ and $K \geq 0$, in the region $\operatorname{Re} s > 1 - J - K$, we have the formula

$$(2.27) \quad S_J^*(s; a, b; \mu, \nu; z) = \frac{(-1)^J e(-b\nu)}{(2\pi)^{s-1}} \sum_{k=0}^{K-1} \frac{i^{J+k+1} (s+J)_k}{(J+k+1)!} \\ \times \psi(s+J+k, \langle -a \rangle, -\mu) \mathcal{C}_{J+k+1}(\langle b \rangle, \tilde{e}(\nu)) \tau^{s+J+k} \\ \times R_{J,K}^*(s; a, b; \mu, \nu; z),$$

valid in the sectors $0 < |\arg \tau| < \pi/2$, where $R_{J,K}^*(s; a, b; \mu, \nu; z)$ is the remainder term satisfying the estimate

$$(2.28) \quad R_{J,K}^*(s; a, b; \mu, \nu; z) = O(|\tau|^{\operatorname{Re} s + J + K})$$

as $\tau \rightarrow 0$ through $\eta \leq |\arg \tau| \leq \pi/2 - \eta$ with any small $\eta > 0$. Here the constant implied in the O -symbol depends at most on s, a, b, μ, ν, J, K and η .

3. VARIANTS OF EULER'S AND RAMANUJAN'S FORMULA FOR $\zeta(s)$

It is in fact possible to deduce from the combination of Theorems 1–3 the celebrated formulae of Euler and Ramanujan for specific values of the Riemann zeta-function as well as their several variants. One can observe that the connection formula (2.23) works as a key ingredient in the background to generate various Ramanujan's type formulae for specific values of zeta-functions.

Theorem 4 ([16, Theorem 4]). *Let $q = e(i\tau) = e^{-2\pi\tau}$ and $\widehat{q} = e(i/\tau) = e^{-2\pi/\tau}$ for any complex τ in the sector $|\arg \tau| < \pi/2$. Then for any real a, b, μ and ν , and any integer $k \neq 0$, we have the formula*

$$\begin{aligned}
 (3.1) \quad & e(a\mu) \{ \delta(b)\psi(k, \langle -\mu \rangle, a) \\
 & + \mathcal{S}_k(\langle b \rangle, \langle -\mu \rangle; \nu, a; q) + (-1)^{k-1} \mathcal{S}_k(\langle -b \rangle, \langle \mu \rangle; -\nu, -a; q) \} \\
 & - (-2\pi)^k \sum_{j=0}^{k+1} \frac{(-i)^j \mathcal{C}_{k+1-j}(\langle b \rangle, \widetilde{e}(\nu)) \mathcal{C}_j(\langle a \rangle, \widetilde{e}(\mu))}{(k+1-j)!j!} \tau^{k-j} \\
 & = e(b\nu)(-i\tau)^{k-1} \{ \delta(a)\psi(k, \langle \nu \rangle, -b) \\
 & + \mathcal{S}_k(\langle a \rangle, \langle \nu \rangle; \mu, -b; \widehat{q}) + (-1)^{k-1} \mathcal{S}_k(\langle -a \rangle, \langle -\nu \rangle; -\mu, b; \widehat{q}) \},
 \end{aligned}$$

whose variant asserts upon replacing $(\tau, q) \mapsto (1/\tau, \widehat{q})$ that

$$\begin{aligned}
 (3.2) \quad & e(b\nu) \{ \delta(a)\psi(k, \langle \nu \rangle, -b) \\
 & + \mathcal{S}_k(\langle a \rangle, \langle \nu \rangle; \mu, -b; q) + (-1)^{k-1} \mathcal{S}_k(\langle -a \rangle, \langle -\nu \rangle; -\mu, b; q) \} \\
 & - (-2\pi)^k \sum_{j=0}^{k+1} \frac{i^j \mathcal{C}_{k+1-j}(\langle a \rangle, \widetilde{e}(\mu)) \mathcal{C}_j(\langle b \rangle, \widetilde{e}(\nu))}{(k+1-j)!j!} \tau^{k-j} \\
 & = e(a\mu)(i\tau)^{k-1} \{ \delta(b)\psi(k, \langle -\mu \rangle, a) \\
 & + \mathcal{S}_k(\langle b \rangle, \langle -\mu \rangle; \nu, a; \widehat{q}) + (-1)^{k-1} \mathcal{S}_k(\langle -b \rangle, \langle \mu \rangle; -\nu, -a; \widehat{q}) \}.
 \end{aligned}$$

The particular case $(\mu, \nu) = (0, 0)$ of Theorem 4 reduces to the following formula for the pairing of $\zeta_a(k)$ and $\zeta_{-b}(k)$.

Corollary 4.1 ([16, Corollary 4.1]). *For any real a and b , and any integer $k \neq 0$, we have*

$$\begin{aligned}
 (3.3) \quad & \delta(b)\zeta_a(k) + \mathcal{S}_k(\langle b \rangle, 0; 0, a; q) + (-1)^{k-1} \mathcal{S}_k(\langle -b \rangle, 0; 0, -a; q) \\
 & - (-2\pi)^k \sum_{j=0}^{k+1} \frac{(-i)^j B_{k+1-j}(\langle b \rangle) B_j(\langle a \rangle)}{(k+1-j)!j!} \tau^{k-j} \\
 & = (-i\tau)^{k-1} \{ \delta(a)\zeta_{-b}(k) + \mathcal{S}_k(\langle a \rangle, 0; 0, -b; \widehat{q}) + (-1)^{k-1} \mathcal{S}_k(\langle -a \rangle, 0; 0, b; \widehat{q}) \},
 \end{aligned}$$

whose variant asserts that

$$\begin{aligned}
 (3.4) \quad & \delta(a)\zeta_{-b}(k) + \mathcal{S}_k(\langle a \rangle, 0; 0, -b; q) + (-1)^{k-1} \mathcal{S}_k(\langle -a \rangle, 0; 0, b; q) \\
 & - (-2\pi)^k \sum_{j=0}^{k+1} \frac{i^j B_{k+1-j}(\langle a \rangle) B_j(\langle b \rangle)}{(k+1-j)!j!} \tau^{k-j} \\
 & = (i\tau)^{k-1} \{ \delta(b)\zeta_a(k) + \mathcal{S}_k(\langle b \rangle, 0; 0, a; \widehat{q}) + (-1)^{k-1} \mathcal{S}_k(\langle -b \rangle, 0; 0, -a; \widehat{q}) \}.
 \end{aligned}$$

The particular case $(a, b) = (0, 0)$ of Theorem 4 reduces to the following formula for the pairing of $\zeta(k, \langle -\mu \rangle)$ and $\zeta(k, \langle \nu \rangle)$.

Corollary 4.2 ([16, Corollary 4.2]). *For any real μ and ν , and any integer $k \neq 1$, we have*

$$(3.5) \quad \begin{aligned} & \zeta(k, \langle -\mu \rangle) + \mathcal{S}_k(0, \langle -\mu \rangle; \nu, 0; q) + (-1)^{k-1} \mathcal{S}_k(0, \langle \mu \rangle; -\nu, 0; q) \} \\ & - (-2\pi)^k \sum_{j=0}^{k+1} \frac{(-i)^j A_{k+1-j}(e(\nu)) A_j(e(\mu))}{(k+1-j)! j!} \tau^{k-j} \\ & = (-i\tau)^{k-1} \{ \zeta(k, \langle \nu \rangle) + \mathcal{S}_k(0, \langle \nu \rangle; \mu, 0; \widehat{q}) + (-1)^{k-1} \mathcal{S}_k(0, \langle -\nu \rangle; -\mu, 0; \widehat{q}) \}, \end{aligned}$$

whose variant asserts that

$$(3.6) \quad \begin{aligned} & \zeta(k, \langle \nu \rangle) + \mathcal{S}_k(0, \langle \nu \rangle; \mu, 0; q) + (-1)^{k-1} \mathcal{S}_k(0, \langle -\nu \rangle; -\mu, 0; q) \} \\ & - (-2\pi)^k \sum_{j=0}^{k+1} \frac{i^j A_{k+1-j}(e(\mu)) A_j(e(\nu))}{(k+1-j)! j!} \tau^{k-j} \\ & = (i\tau)^{k-1} \{ \zeta(k, \langle -\mu \rangle) + \mathcal{S}_k(0, \langle -\mu \rangle; \nu, 0; \widehat{q}) + (-1)^{k-1} \mathcal{S}_k(0, \langle \mu \rangle; -\nu, 0; \widehat{q}) \}. \end{aligned}$$

The particular case $(b, \nu) = (0, 0)$ of Theorem 4 reduces to the following formula for the pairing of $\psi(k, \langle -\mu \rangle, a)$ and $\zeta(k)$.

Corollary 4.3 ([16, Corollary 4.3]). *For any real a and μ , and any integer $k \neq 1$, we have*

$$(3.7) \quad \begin{aligned} & e(a\mu) \{ \psi(k, \langle -\mu \rangle, a) + \mathcal{S}_k(0, \langle -\mu \rangle; 0, a; q) + (-1)^{k-1} \mathcal{S}_k(0, \langle \mu \rangle; 0, -a; q) \} \\ & - (-2\pi)^k \sum_{j=0}^{k+1} \frac{(-i)^j B_{k+1-j}(\langle a \rangle, \widetilde{e}(\mu))}{(k+1-j)! j!} \tau^{k-j} \\ & = (-i\tau)^{k-1} \{ \delta(a) \zeta(k) + \mathcal{S}_k(\langle a \rangle, 0; \mu, 0; \widehat{q}) + (-1)^{k-1} \mathcal{S}_k(\langle -a \rangle, 0; \mu, 0; \widehat{q}) \}, \end{aligned}$$

whose variant asserts that

$$(3.8) \quad \begin{aligned} & \delta(a) \zeta(k) + \mathcal{S}_k(\langle a \rangle, 0; \mu, 0; a; q) + (-1)^{k-1} \mathcal{S}_k(\langle -a \rangle, 0, -\mu; 0; q) \} \\ & - (-2\pi)^k \sum_{j=0}^{k+1} \frac{i^j \mathcal{C}_{k+1-j}(\langle a \rangle, \widetilde{e}(\mu)) B_j}{(k+1-j)! j!} \tau^{k-j} \\ & = e(a\mu) (i\tau)^{k-1} \{ \psi(k, \langle -\mu \rangle, a) \\ & + \mathcal{S}_k(0, \langle -\mu \rangle, 0; a, 0; \widehat{q}) + (-1)^{k-1} \mathcal{S}_k(0, \langle \mu \rangle, 0; 0, -a; \widehat{q}) \}. \end{aligned}$$

The particular case $(a, \nu) = (0, 0)$ of Theorem 4 reduces to the following formula for the pairing of $\zeta(k, \langle -\mu \rangle)$ and $\zeta_{-b}(k)$.

Corollary 4.4 ([16, Corollary 4.4]). *For any real b and μ , and any integers $k \neq 1$, we have*

$$(3.9) \quad \begin{aligned} & \delta(b) \zeta(k, \langle -\mu \rangle) + \mathcal{S}_k(b, \langle -\mu \rangle; 0, 0; q) + (-1)^{k-1} \mathcal{S}_k(\langle -b \rangle, \langle \mu \rangle; 0, 0; q) \\ & - (-2\pi)^k \sum_{j=0}^{k+1} \frac{(-i)^j B_{k+1-j}(\langle b \rangle) A_j(e(\mu))}{(k+1-j)! j!} \tau^{k-j} \\ & = e(b\nu) (-i\tau)^{k-1} \{ \zeta_{-b}(k) + \mathcal{S}_k(0, 0; \mu, -b; \widehat{q}) + (-1)^{k-1} \mathcal{S}_k(0, 0; -\mu, b; \widehat{q}) \}, \end{aligned}$$

whose variant asserts that

$$\begin{aligned}
 (3.10) \quad & \zeta_{-b}(k) + \mathcal{S}_k(0, 0; \mu, -b; q) + (-1)^{k-1} \mathcal{S}_k(0, 0; -\mu, b; q) \\
 & - (-2\pi)^k \sum_{j=0}^{k+1} \frac{i^j A_{k+1-j}(e(\mu)) B_j(\langle b \rangle)}{(k+1-j)! j!} \tau^{k-j} \\
 & = (i\tau)^{k-1} \{ \delta(b) \zeta(k, \langle -\mu \rangle) \\
 & + \mathcal{S}_k(\langle b \rangle, \langle -\mu \rangle; 0, 0; \widehat{q}) + (-1)^{k-1} \mathcal{S}_k(\langle -b \rangle, \langle \mu \rangle; 0, 0; \widehat{q}) \}.
 \end{aligned}$$

The simplest case $(a, b, \mu, \nu) = (0, 0, 0, 0)$ of Theorem 4 reduces to the celebrated formulae of Euler and Ramanujan, respectively, for specific values of $\zeta(s)$.

Corollary 4.5 ([16, Corollary 4.5]). *We have the the following formulae:*

i) for any integer $k \geq 1$,

$$(3.11) \quad \zeta(2k) = \frac{(-1)^{k+1} (2\pi)^{2k}}{2(2k)!} B_{2k};$$

ii) for any integer $k \neq 0$,

$$\begin{aligned}
 (3.12) \quad & \zeta(2k+1) + 2\mathcal{S}_{2k+1}(0, 0; 0, 0; q) + (2\pi)^{2k+1} \sum_{j=0}^{k+1} \frac{(-1)^j B_{2k+2-2j} B_{2j}}{(2k+2-2j)! (2j)!} \tau^{2k+1-2j} \\
 & = (i\tau)^{2k} \{ \zeta(2k+1) + 2\mathcal{S}_{2k+1}(0, 0; 0, 0; \widehat{q}) \}.
 \end{aligned}$$

4. CLASSICAL EISENSTEIN SERIES

We present in this section several applications of Theorems 1–3 to the classical Eisenstein series. Let $E_{2k}(z)$ denote the classical holomorphic Eisenstein series defined for $k \geq 1$ by

$$(4.1) \quad E_{2k}(z) = 1 - \frac{4k}{B_{2k}} \sum_{l=1}^{\infty} \frac{l^{2k-1} q^l}{1 - q^l}$$

with $q = e(z)$ (cf. [4, Chap.4, 4.5 (4.5.1)]). Theorem 1 in fact shows that

$$(4.2) \quad E_{2k}(z) = \frac{(-1)^{k-1} (2k)!}{(2\pi)^{2k} B_{2k}} F_{\mathbb{Z}^2}(2k; 0, 0; 0, 0; z).$$

for any integer $k \geq 1$. We shall treat in what follows the cases $k = 1$ and $k \geq 2$ separately.

Consider first the case $k = 1$. The combination of Theorems 2 and 3 reduces in this case to

$$F_{\mathbb{Z}^2}(2; 0, 0; 0, 0; z) = \frac{\pi^2}{3z^2} + \frac{2\pi i}{z} - \frac{8\pi^2}{z^2} \mathcal{S}_{-1}(0, 0; 0, 0; \widehat{q}),$$

while Theorem 1 applied with $-1/z$ instead of z implies that

$$F_{\mathbb{Z}^2}\left(2; 0, 0; 0, 0; -\frac{1}{z}\right) = \frac{\pi^2}{3} - 8\pi^2 \mathcal{S}_{-1}(0, 0; 0, 0; \widehat{q}),$$

and hence the relation between $F_{\mathbb{Z}^2}(2; 0, 0; 0, 0; z)$ and $F_{\mathbb{Z}^2}(2; 0, 0; 0, 0; -1/z)$ asserts

$$(4.3) \quad F_{\mathbb{Z}^2}(2; 0, 0; 0, 0; z) = \frac{2\pi i}{z} + \frac{1}{z^2} F_{\mathbb{Z}^2}\left(2; 0, 0; 0, 0; -\frac{1}{z}\right),$$

which gives the following transformation formula (cf. [30, Chap.2, 2.4 (2.58)]):

Corollary 4.6 ([16, Corollary 4.6]). *For any $z \in \mathfrak{H}^+$, we have*

$$(4.4) \quad E_2\left(-\frac{1}{z}\right) = \frac{6z}{\pi i} + z^2 E_2(z).$$

One can see that the procedure of derivation above gives a zeta-function theoretic or asymptotic methodological proof of the modular relation for $E_2(z)$.

We next treat the case $k \geq 2$. The combination of Theorems 2 and 3 in this case reduces to

$$F_{\mathbb{Z}^2}(2k; 0, 0; 0, 0; z) = \frac{(-1)^k 2(2\pi/z)^{2k}}{(2k-1)!} \left\{ -\frac{B_{2k}}{4k} + \mathcal{S}_{1-2k}(0, 0; 0, 0; \widehat{q}) \right\},$$

while Theorem 1 applied with $-1/z$ instead of z implies that

$$F_{\mathbb{Z}^2}\left(2k; 0, 0; 0, 0; -\frac{1}{z}\right) = \frac{(-1)^k 2(2\pi)^{2k}}{(2k-1)!} \left\{ -\frac{B_{2k}}{4k} + \mathcal{S}_{1-2k}(0, 0; 0, 0; \widehat{q}) \right\},$$

and hence the relation between $F_{\mathbb{Z}^2}(2; 0, 0; 0, 0; z)$ and $F_{\mathbb{Z}^2}(2; 0, 0; 0, 0; -1/z)$ asserts

$$(4.5) \quad F_{\mathbb{Z}^2}(2k; 0, 0; 0, 0; z) = \frac{1}{z^{2k}} F_{\mathbb{Z}^2}\left(2k; 0, 0; 0, 0; -\frac{1}{z}\right),$$

which gives the transformation formula:

Corollary 4.7 ([16, Corollary 4.7]). *For any $z \in \mathfrak{H}^+$, we have*

$$(4.6) \quad E_{2k}\left(-\frac{1}{z}\right) = z^{2k} E_{2k}(z) \quad (k \geq 2).$$

It is known for $k \geq 2$ that the double series expression

$$E_{2k}(z) = \frac{1}{2} \sum_{\substack{c,d=-\infty \\ (c,d)=1}}^{\infty} \frac{1}{(cz+d)^{2k}}$$

is valid (cf. [4, Chap.4, 4.5 (4.5.1)]). One can therefore see that the procedure of derivation above successfully(?) gives a stupidly lengthy proof(!) of the modular relation for $E_{2k}(z)$ with $k \geq 2$.

5. WEIERSTRASS' ELLIPTIC AND ALLIED FUNCTIONS

We present in this section several applications of Theorems 1–3 to Weierstraß' elliptic and allied functions. Let $\omega = (\omega_1, \omega_2) \in \mathbb{C}^2$ be a fundamental parallelogram with $\text{Im}(\omega_2/\omega_1) > 0$. Set $\omega_2/\omega_1 = z$, and choose the branch with $\arg z \in]0, \pi[$. Weierstraß' elliptic function with the periods $\omega = (\omega_1, \omega_2)$ is defined by

$$(5.1) \quad \wp(w \mid \omega) = \frac{1}{w^2} + \sum_{\substack{m,n=-\infty \\ (m,n) \neq (0,0)}}^{\infty} \left\{ \frac{1}{(w - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right\}$$

(cf. [6, 13.12 (4)]), while (allied) Weierstraß' zeta and sigma functions by

$$(5.2) \quad \zeta(w \mid \omega) = \frac{1}{w} + \sum_{\substack{m,n=-\infty \\ (m,n) \neq (0,0)}}^{\infty} \left\{ \frac{1}{w - m\omega_1 - n\omega_2} + \frac{1}{m\omega_1 + n\omega_2} + \frac{w}{(m\omega_1 + n\omega_2)^2} \right\},$$

$$(5.3) \quad \sigma(w \mid \omega) = w \prod_{\substack{m,n=-\infty \\ (m,n) \neq (0,0)}}^{\infty} \left(1 - \frac{w}{m\omega_1 + n\omega_2}\right) \exp \left\{ \frac{w}{m\omega_1 + n\omega_2} + \frac{1}{2} \left(\frac{w}{m\omega_1 + n\omega_2} \right)^2 \right\},$$

respectively (cf. [6, 13.12 (6) and (11)]). It suffices in fact to study the elliptic and allied functions defined with the normalized periods $\mathbf{z} = (1, z)$, in view of the relations

$$\wp(cw \mid c\omega) = c^{-2} \wp(w \mid \omega), \quad \zeta(cw \mid c\omega) = c^{-1} \zeta(w \mid \omega), \quad \sigma(cw \mid c\omega) = c\sigma(w \mid \omega)$$

for any $c \in \mathbb{C}^\times$. One can then see that the limiting relation

$$(5.4) \quad \wp(w \mid \mathbf{z}) = \lim_{\substack{s \rightarrow 2 \\ \operatorname{Re} s > 2}} \{F_{\mathbb{Z}^2}(s; a, b; 0, 0; z) - F_{\mathbb{Z}^2}(s; 0, 0; 0, 0; z)\}$$

is valid for any $w = a + bz \in \mathbb{C}$ with $(a, b) \in \mathbb{R}^2$, since the limiting point $s = 2$ is located in the boundary of the region where the defining series in (1.1) converges absolutely. Theorem 1 can therefore be applied on the right side of (5.4) to show the following expression of $\wp(w \mid \mathbf{z})$.

Corollary 4.8 ([16, Corollary 4.8]). *For any $w = a + bz \in \mathbb{C}$ with $(a, b) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$, we have*

$$(5.5) \quad \wp(w \mid \mathbf{z}) = -\frac{\pi^2}{3} E_2(z) + \frac{\delta(b)\pi^2}{\sin^2 \pi a} - 4\pi^2 \{ \mathcal{S}_{-1}(\langle b \rangle, 0; 0, a; q) + \mathcal{S}_{-1}(\langle -b \rangle, 0; 0, -a; q) \}.$$

Combining Theorems 2, 3 and Corollary 4.8, we obtain the period change formula for $\wp(w \mid \mathbf{z})$ in the form

$$(5.6) \quad \wp(w \mid \mathbf{z}) = \frac{1}{z^2} \wp\left(\frac{w}{z} \mid \widehat{\mathbf{z}}\right),$$

where $\widehat{\mathbf{z}} = (1, -1/z)$ are the dual periods (cf. [30, Chap.2, 2.4]). We next write the (base) parameter corresponding to the half period as $p = e(z/2) = e^{-\pi\tau}$ (i.e. $q = p^2$), and then define the Weierstrassian invariants by

$$(5.7) \quad e_1(\mathbf{z}) = \wp\left(\frac{1}{2} \mid \mathbf{z}\right), \quad e_2(\mathbf{z}) = \wp\left(\frac{z}{2} \mid \mathbf{z}\right), \quad e_3(\mathbf{z}) = \wp\left(\frac{1+z}{2} \mid \mathbf{z}\right).$$

Then Corollary 4.8 in fact implies the following Lambert series expressions for Weierstrassian invariants (cf. [30, Chap.4, 4.2 (4.46)–(4.48)])

$$(5.8) \quad \begin{aligned} e_1(\mathbf{z}) &= 4\pi^2 \left\{ \frac{1}{6} + 4 \sum_{l=1}^{\infty} \frac{(2l-1)p^{4l-2}}{1-p^{4l-2}} \right\}, \\ e_2(\mathbf{z}) &= 4\pi^2 \left\{ -\frac{1}{12} - 2 \sum_{l=1}^{\infty} \frac{(2l-1)p^{2l-2}}{1-p^{2l-2}} \right\}, \\ e_3(\mathbf{z}) &= 4\pi^2 \left\{ -\frac{1}{12} + 2 \sum_{l=1}^{\infty} \frac{(2l-1)p^{2l-1}}{1+p^{2l-1}} \right\}, \end{aligned}$$

which further yield a significant relation (cf. [30, Chap.4, 4.2 (4.49)]):

$$(5.9) \quad e_1(\mathbf{z}) + e_2(\mathbf{z}) + e_3(\mathbf{z}) = 0.$$

Furthermore, combining Theorems 2, 3 and Corollary 4.8, we obtain the period change formulae for Weierstrassian invariants:

$$(5.10) \quad e_j(\mathbf{z}) = e_j(\widehat{\mathbf{z}}) \quad (j = 1, 2, 3).$$

We next consider Weierstraß' zeta function. It is misleading to validate that $\zeta(w \mid z)$ is defined to be the limit

$$\lim_{\substack{s \rightarrow 1 \\ \operatorname{Re} s > 1}} \{F_{\mathbb{Z}^2}(s; a, b; 0, 0; z) - F_{\mathbb{Z}^2}(s; 0, 0; 0, 0; z) + swF_{\mathbb{Z}^2}(s+1; 0, 0; 0, 0; z)\},$$

since the limiting point $s = 1$ is located in the exterior of the region where the defining series in (1.1) converges absolutely. We rather take another route for defining Weierstraß' zeta function in terms of $\wp(w \mid z)$, which asserts

$$(5.11) \quad \zeta(w \mid z) = \frac{1}{w} - \int_0^w \left\{ \wp(u \mid z) - \frac{1}{u^2} \right\} du$$

(cf. [6, 13.12 (7)]). The expression in (5.5) can therefore be integrated to show the following formula for $\zeta(w \mid z)$.

Corollary 4.9 ([16, Corollary 4.9]). *For any $w = a + bz \in \mathbb{C}$ with $(a, b) \in]-1, 1[^2 \setminus \{(0, 0)\}$, we have*

$$(5.12) \quad \begin{aligned} \zeta(w \mid z) = & \frac{\pi^2}{3} E_2(z)w + \delta(b)\pi \cot \pi a - (\operatorname{sgn} b)\pi i \\ & - 2\pi i \{ \mathcal{S}_0(\langle b \rangle, 0; 0, a; q) - \mathcal{S}_0(\langle -b \rangle, 0; 0, -a; q) \}. \end{aligned}$$

Weierstraß' eta invariants are defined by

$$(5.13) \quad \eta_1(z) = \zeta\left(\frac{1}{2} \mid z\right), \quad \eta_2(z) = \zeta\left(\frac{z}{2} \mid z\right), \quad \eta_3(z) = \zeta\left(-\frac{1+z}{2} \mid z\right).$$

Corollary 4.9 therefore gives the evaluations

$$(5.14) \quad \begin{aligned} \eta_1(z) &= \frac{\pi^2}{6} E_2(z), \\ \eta_2(z) &= \frac{\pi^2}{6} E_2(z)z - \pi i, \\ \eta_3(z) &= -\frac{\pi^2}{6} E_2(z)(1+z) + \pi i, \end{aligned}$$

which imply the classical Legendre relations (cf. [6, 13.12 (10)])

$$(5.15) \quad \begin{aligned} \eta_1(z) \cdot \frac{z}{2} - \eta_2(z) \cdot \frac{1}{2} &= \frac{\pi i}{2}, \\ \eta_2(z) \cdot \left(-\frac{1+z}{2}\right) - \eta_3(z) \cdot \frac{z}{2} &= \frac{\pi i}{2}, \\ \eta_3(z) \cdot \frac{1}{2} - \eta_1(z) \cdot \left(-\frac{1+z}{2}\right) &= \frac{\pi i}{2}. \end{aligned}$$

We finally consider Weierstraß' sigma function. It is misleading again to validate that $\log \sigma(w \mid z)$ is defined to be the limit

$$\begin{aligned} \lim_{\substack{s \rightarrow 0 \\ \operatorname{Re} s > 0}} \left\{ -\frac{\partial}{\partial s} F_{\mathbb{Z}^2}(s; a, b; 0, 0; z) + \frac{\partial}{\partial s} F_{\mathbb{Z}^2}(s; 0, 0; 0, 0; z) - wF_{\mathbb{Z}^2}(s+1; 0, 0; 0, 0; z) \right. \\ \left. + \frac{1}{2} w^2 F_{\mathbb{Z}^2}(s+2; 0, 0; 0, 0; z) \right\}, \end{aligned}$$

since the limiting point $s = 0$ is again located in the exterior of the region where the defining series in (1.1) converges absolutely. We rather take another route for defining Weierstraß' sigma function in terms of $\zeta(w | z)$, which asserts

$$(5.16) \quad \log \sigma(w | z) = \log w + \int_0^w \left\{ \zeta(u | z) - \frac{1}{u} \right\} du$$

(cf. [6, 13.12 (12)]). We use the customary notation $(z; q) = \prod_{l=0}^{\infty} (1 - zq^l)$ for any $z \in \mathbb{C}$ in the sequel. Then the expression in (5.12) can therefore be integrated to show the following formula for $\log \sigma(w | z)$.

Corollary 4.10 ([16, Corollary 4.10]). *For any $w = a + bz \in \mathbb{C}$ with $(a, b) \in]-1, 1[^2 \setminus \{(0, 0)\}$, we have*

$$(5.17) \quad \begin{aligned} \log \sigma(w | z) = & \frac{\pi^2}{6} E_2(z) w^2 + (\operatorname{sgn} b) \pi i \left(\frac{1}{2} - w \right) + \delta(b) \log(2 \sin \pi a) \\ & - \mathcal{S}_1(\langle b \rangle, 0; 0, a; q) - \mathcal{S}_1(\langle -b \rangle, 0; 0, -a; q) \\ & + 2\mathcal{S}_1(0, 0; 0, 0; q) - \log 2\pi, \end{aligned}$$

whose exponential form asserts

$$(5.18) \quad \begin{aligned} \sigma(w | z) = & \exp \left\{ \frac{\pi^2}{6} E_2(z) w^2 + (\operatorname{sgn} b) \pi i \left(\frac{1}{2} - w \right) \right\} (2 \sin \pi a)^{\delta(b)} \\ & \times \frac{(e(a)q^{\langle b \rangle'}; q)_{\infty} (e(-a)q^{\langle -b \rangle'}; q)_{\infty}}{2\pi(q; q)_{\infty}^2}, \end{aligned}$$

where (only) the many valued term $\log(2 \sin \pi a)$ on the right side of (5.17) becomes one valued after its exponentiation on the right side of (5.18).

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