

ON IMMERSED ORIENTED SURFACES AND THEIR PLANE PROJECTIONS

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1. INTRODUCTION

Throughout the report, all manifolds and maps are differentiable of class C^∞ . Let M be a closed connected surface, f and $g : M \rightarrow \mathbb{R}^3$ immersions. Let $F : M \times [0, 1] \rightarrow \mathbb{R}^3$ be a homotopy between f and g . That is, $F|_{M \times \{0\}} = f$ and $F|_{M \times \{1\}} = g$ hold. We call F a regular homotopy between f and g if a level preserving map $(F \times \text{id}) : M \times [0, 1] \rightarrow \mathbb{R}^3 \times [0, 1]$ defined by $(F \times \text{id})(x, t) = (F(x, t), t)$ is an immersion. James and Thomas [4] proved that the space of immersions $f : M \rightarrow \mathbb{R}^3$ has $2^{\dim H_1(M; \mathbb{Z}_2)}$ connected components. Pinkall [6] associated to any immersion $f : M \rightarrow \mathbb{R}^3$ a $\mathbb{Z}_4$ -valued quadratic form $q_f : H_1(M; \mathbb{Z}_2) \rightarrow \mathbb{Z}_4$ and proved that two immersions f and $g : M \rightarrow \mathbb{R}^3$ are regularly homotopic if and only if $q_f = q_g$ holds.

Let $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be an orthogonal projection and $\tilde{f} : M \rightarrow \mathbb{R}^2$ a stable map. If a stable map $\tilde{f} : M \rightarrow \mathbb{R}^2$ has an immersion $f : M \rightarrow \mathbb{R}^3$ such that $\pi \circ f = \tilde{f}$, we call that $\tilde{f}$ has an immersion lift f and that f is an immersion lift over $\tilde{f}$. Let f and $g : M \rightarrow \mathbb{R}^3$ be immersion lifts over $\tilde{f} : M \rightarrow \mathbb{R}^2$. If there exists a regular homotopy $F : M \times [0, 1] \rightarrow \mathbb{R}^3$ between f and g which satisfies that $\pi \circ (F|_{M \times \{t\}}) = \tilde{f}$ for any $t \in [0, 1]$, then we call that f and g are $\tilde{f}$ -regularly homotopic.

In this report, when M is a closed connected oriented surface, we study $\tilde{f}$ -regular homotopy classes for a fixed stable map $\tilde{f} : M \rightarrow \mathbb{R}^2$ which has an immersion lift.

This report is organized as follows. In Section 2, we give the definition and see properties of a stable map $\tilde{f} : M \rightarrow \mathbb{R}^2$. In Section 3, we restate Haefliger's theorem [3] by putting sings on the apparent contour of $\tilde{f}$. In Section 4, we determine $\tilde{f}$ -regular homotopy classes for a fixed stable map $\tilde{f}$ which has an immersion lift. In Section 5, we give the definitions of generic homotopy and regular homotopy lift. By using the method obtained in the previous section, we introduce a necessary and sufficient condition for existence of regular homotopy lift over the given generic homotopy. As an application, we construct a generic homotopy whose regular homotopy lift corresponds to a sphere eversion.

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2. STABLE MAP

In this section, we give the definition and see properties of a stable map.

Let $\tilde{f} : M \rightarrow \mathbb{R}^2$ be a smooth map of a closed connected surface M into the plane. We denote the set of such maps by $C^\infty(M, \mathbb{R}^2)$, which is equipped with the Whitney C^∞ -topology. A smooth map $\tilde{f}$ is said to be a stable map if in $C^\infty(M, \mathbb{R}^2)$, there exists an open neighborhood U of $\tilde{f}$ such that for any $\tilde{g} \in U$, $\tilde{g}$ is C^∞ right-left equivalent to $\tilde{f}$, i.e., there exist two diffeomorphisms $\Phi : M \rightarrow M$ and $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & M \\ \tilde{f} \downarrow & & \downarrow \tilde{g} \\ \mathbb{R}^2 & \xrightarrow{\varphi} & \mathbb{R}^2 \end{array}$$

is commutative.

For a smooth map $\tilde{f} : M \rightarrow \mathbb{R}^2$, we denote by $S(\tilde{f})$ the set of the points in M where the rank of the differential of $\tilde{f}$ is strictly less than two. We say that $S(\tilde{f}) \subset M$ is a singular set of $\tilde{f}$ and $\tilde{f}(S(\tilde{f})) \subset \mathbb{R}^2$ is an apparent contour of $\tilde{f}$.

The following characterizations of stable maps are well-known (see [2, 8], for example).

Proposition 2.1. *A smooth map $\tilde{f} : M \rightarrow \mathbb{R}^2$ of a closed surface M is a stable map if and only if the following conditions are satisfied.*

- (i) *For every $q \in M$, there exist local coordinates (x, y) and (X, Y) around $q \in M$ and $\tilde{f}(q) \in \mathbb{R}^2$ respectively such that one of the following holds:*
 - (a) $(X \circ \tilde{f}, Y \circ \tilde{f}) = (x, y)$ (q : regular point),
 - (b) $(X \circ \tilde{f}, Y \circ \tilde{f}) = (x, y^2)$ (q : fold point),
 - (c) $(X \circ \tilde{f}, Y \circ \tilde{f}) = (x, y^3 - xy)$ (q : cusp point).
- (ii) *If $q \in M$ is a cusp point, then $\tilde{f}^{-1}(\tilde{f}(q)) \cap S(\tilde{f}) = \{q\}$.*
- (iii) *The map $\tilde{f}|(S(\tilde{f}) \setminus \{\text{cusp points}\})$ is an immersion with normal crossings.*

Note that $S(\tilde{f})$ is a compact 1-dimensional submanifold of M and the number of cusp points is finite. Let $U \subset M$ be a tubular neighborhood of $S(\tilde{f})$. Then the restriction of $\tilde{f}$ on the closure $\text{cl}(M \setminus U)$ is an immersion. By the apparent contour of $\tilde{f}$, $\mathbb{R}^2$ is naturally stratified into 2-, 1- and 0-dimensional strata. The union of 1- and 0-dimensional strata forms $\tilde{f}(S(\tilde{f}))$. On each 1-dimensional stratum, we can define an orientation as follows. We fix the canonical orientation on $\mathbb{R}^2$. Let Ω be a connected component of $\mathbb{R}^2 \setminus \tilde{f}(S(\tilde{f}))$. We associate to Ω a non-negative integer $n_{\tilde{f}}(\Omega)$, which is the number of points in the fiber of $\tilde{f}$ over any point of Ω . Every 1-dimensional stratum is adjacent to exactly two connected components of $\mathbb{R}^2 \setminus \tilde{f}(S(\tilde{f}))$. Since these two components have distinct $n_{\tilde{f}}(\Omega)$ -values, we can orient each 1-dimensional stratum in $\tilde{f}(S(\tilde{f}))$ so that the region with the larger $n_{\tilde{f}}(\Omega)$ -value is on its left. Since $\tilde{f}|(S(\tilde{f}) \setminus \{\text{cusp points}\})$ is an immersion, $S(\tilde{f}) \setminus \{\text{cusp points}\}$ is also oriented.

Suppose that M is an oriented closed surface and $\mathbb{R}^2$ is oriented plane. Let q be a cusp point of a stable map $\tilde{f} : M \rightarrow \mathbb{R}^2$. For a sufficiently small neighborhood U of $f(q)$, the map $\tilde{f}|_V : V \rightarrow U$ has degree ± 1 , where V is the component of $\tilde{f}^{-1}(U)$ containing q . We call q is a positive (resp. negative) cusp if the local degree of $\tilde{f}$ at q equals $+1$ (resp. -1).

3. IMMERSION LIFT

In the following, we assume that M , $\mathbb{R}^3$ and $\mathbb{R}^2$ are oriented. In this case, Haefliger's theorem is restated as follows.

Theorem 3.1 (Haefliger [3]). *A stable map $\tilde{f} : M \rightarrow \mathbb{R}^2$ has an immersion lift if and only if each connected component of $S(\tilde{f})$ has even number of cusp points.*

Let $\tilde{f} : M \rightarrow \mathbb{R}^2$ be a stable map which has an immersion lift. On each connected component of fold points $S(\tilde{f}) \setminus \{\text{cusp points}\}$, we can put a sign $+1$ or -1 which satisfies the following rule.

- Let C and C' be two connected components which adjacent to the same cusp point. Then C and C' have the opposite signs.

If a sign of C is $+1$ (resp. -1), we call C a positive (resp. negative) fold and a sign can be put on each image of fold component. Such a stable map $\tilde{f}$ is called a signed stable map.

Let $\tilde{f} : M \rightarrow \mathbb{R}^2$ be a signed stable map and U a tubular neighborhood of $S(\tilde{f})$. Since M is oriented, $U \setminus S(\tilde{f})$ is divided into two regions U_+ and U_- where $\tilde{f}|_{U_+}$ (resp. $\tilde{f}|_{U_-}$) is an orientation preserving (resp. reversing) immersion. We construct an immersion lift $f_U : U \rightarrow \mathbb{R}^3$ over $\tilde{f}|_U$ which satisfies the following.

- (1) If C is a positive fold, f is defined as Figure 1(a).
- (2) If C is a negative fold, f is defined as Figure 1(b).
- (3) If q is a positive cusp and negative fold comes in q for the orientation of $S(\tilde{f})$, f is defined as Figure 2(a).
- (4) If q is a positive cusp and positive fold comes in q for the orientation of $S(\tilde{f})$, f is defined as Figure 2(b).
- (5) If q is a negative cusp and negative fold comes in q for the orientation of $S(\tilde{f})$, f is defined as Figure 2(c).
- (6) If q is a negative cusp and positive fold comes in q for the orientation of $S(\tilde{f})$, f is defined as Figure 2(d).

Definition 3.2. Let $\tilde{f} : M \rightarrow \mathbb{R}^2$ be a signed stable map and U a tubular neighborhood of $S(\tilde{f})$. If an immersion $f : M \rightarrow \mathbb{R}^3$ satisfies the above rules (1)–(6) on $f|_U$, we call f an immersion lift over the signed stable map $\tilde{f}$.

4. $\tilde{f}$ -REGULAR HOMOTOPY

In this section, we state that $\tilde{f}$ -regular homotopy classes can be determined.

Theorem 4.1. *If f and $g : M \rightarrow \mathbb{R}^3$ are immersion lifts over the signed stable map $\tilde{f} : M \rightarrow \mathbb{R}^2$, then f and g are $\tilde{f}$ -regularly homotopic.*

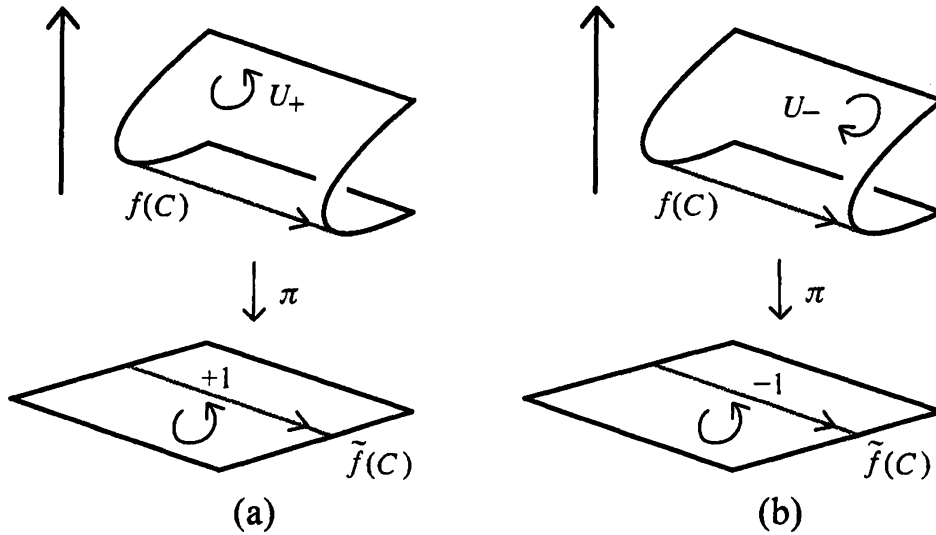


FIGURE 1. Immersion lifts if (a) C is a positive fold, (b) C is a negative fold.

Corollary 4.2. *Let $\tilde{f} : M \rightarrow \mathbb{R}^2$ be a stable map which has an immersion lift. (Note that $\tilde{f}$ is not signed.) The number of $\tilde{f}$ -regular homotopy classes is $2^{\sharp S(\tilde{f})}$, where $\sharp S(\tilde{f})$ is the number of connected components of $S(\tilde{f})$.*

We have a following example which is related to Theorem 4.1.

Example 4.3. Let T^2 be an oriented torus and l and m longitude and meridian of T^2 , respectively. Let $\tilde{f}$ and $\tilde{g} : T^2 \rightarrow \mathbb{R}^2$ be signed stable maps which satisfy the following properties. They do not have cusp points, $\tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g}))$, both signs are the same, $\tilde{f}|l = \tilde{g}|l$ and $\tilde{g}|m$ are plane curves whose rotation numbers equal 2 (or -2), $\tilde{f}|m$ is a simple closed plane curve. See Figure 3. By the theorem of Pinkall [6], immersion lifts f and $g : T^2 \rightarrow \mathbb{R}^3$ over $\tilde{f}$ and $\tilde{g}$ respectively are not regularly homotopic.

Theorem 3.1 and Example 4.3 mean that if $M \neq S^2$, an apparent contour with sign does not determine a regular homotopy class. We need information of immersion $\tilde{f}|(M \setminus S(\tilde{f}))$.

5. REGULAR HOMOTOPY LIFT OVER A GENERIC HOMOTOPY

Let $\tilde{f}$ and $\tilde{g} : M \rightarrow \mathbb{R}^2$ be stable maps and $\tilde{F} : M \times [0, 1] \rightarrow \mathbb{R}^2$ a homotopy between $\tilde{f}$ and $\tilde{g}$. If $\tilde{F}$ satisfies the following conditions, we call $\tilde{F}$ a generic homotopy between $\tilde{f}$ and $\tilde{g}$ (see [5]).

- (1) There is a finite set of parameter values $0 < t_1 < \dots < t_n < 1$ (possibly empty) in $(0, 1)$.
- (2) For any $t \in (0, 1) \setminus \{t_1, \dots, t_n\}$, $\tilde{F}|M \times \{t\} : M \times \{t\} \rightarrow \mathbb{R}^2$ is a stable map.
- (3) For each t_i and a sufficiently small positive value ε , the moves of apparent contours of $\tilde{F}|M \times \{t\}$ ($t \in (t_i - \varepsilon, t_i + \varepsilon)$) are classified into lips (type L), beaks (type B), swallowtail (type S), cusp-fold (type C), self-tangency (type K) or triple point (type T).

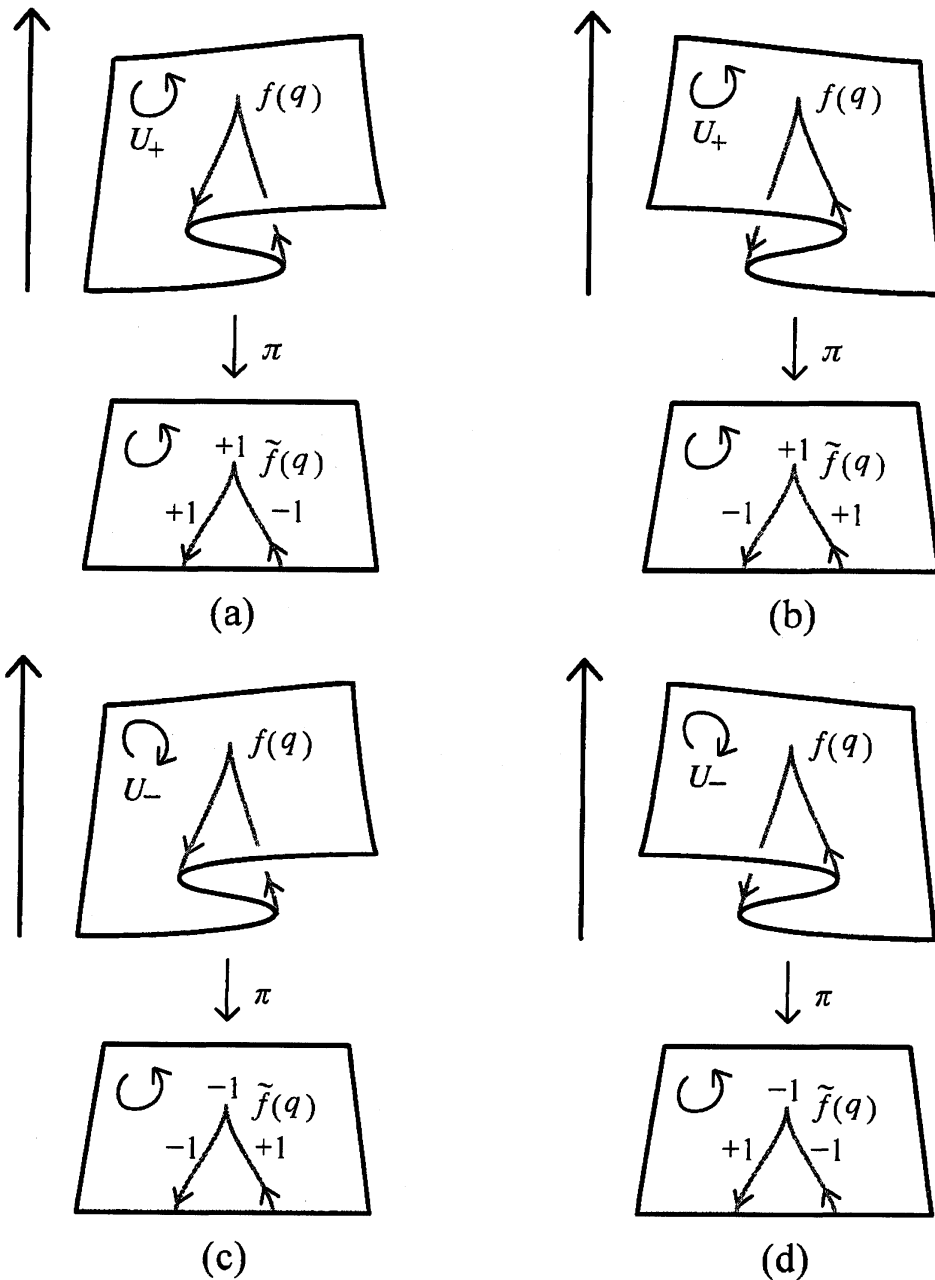


FIGURE 2. Immersion lifts if (a) q is a positive cusp and negative fold comes in, (b) q is a positive cusp and positive fold comes in, (c) q is a negative cusp and negative fold comes in, (d) q is a negative cusp and positive fold comes in.

We call each t_i a bifurcation point on a generic homotopy $\tilde{F}$.

Let $\tilde{F} : M \times [0, 1] \rightarrow \mathbb{R}^2$ be a generic homotopy between signed stable maps $\tilde{f}$ and $\tilde{g}$ and let f and g immersion lifts over $\tilde{f}$ and $\tilde{g}$, respectively. If there exists a regular homotopy $F : M \times [0, 1] \rightarrow \mathbb{R}^3$ between f and g such that $\pi \circ F = \tilde{F}$, we call F a regular homotopy lift over $\tilde{F}$.

Theorem 5.1. *Let $\tilde{f}$ and $\tilde{g} : M \rightarrow \mathbb{R}^2$ be signed stable maps. If there exists a generic homotopy $\tilde{F} : M \times [0, 1] \rightarrow \mathbb{R}^2$ between $\tilde{f}$ and $\tilde{g}$ which preserves sign*

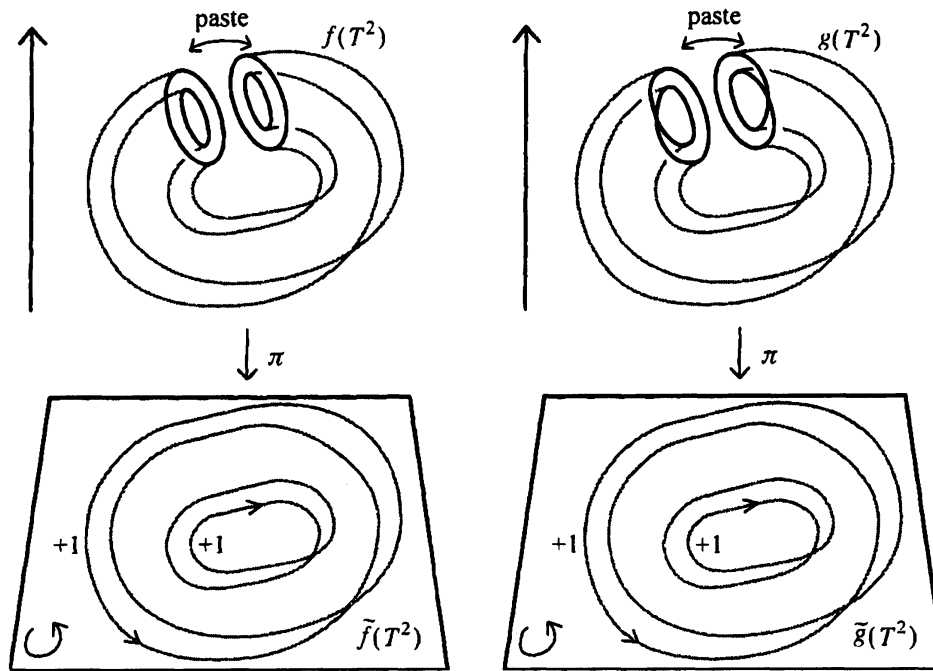


FIGURE 3. Two stable maps $\tilde{f}$ and $\tilde{g} : T^2 \rightarrow \mathbb{R}^2$ which satisfy that $\tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g}))$ and both apparent contours have positive signs. But their immersion lifts f and $g : T^2 \rightarrow \mathbb{R}^3$ are not regularly homotopic.

convention as depicted in Figures 4 and 5, then $\tilde{F}$ has a regular homotopy lift $F : M \times [0, 1] \rightarrow \mathbb{R}^3$.

As an application of Theorem 5.1, we have the following example.

Example 5.2. If $\tilde{f}$ and $\tilde{g} : S^2 \rightarrow \mathbb{R}^2$ are signed stable maps such that $\tilde{f}(S^2) = \tilde{g}(S^2) = D^2$, $\tilde{f}(S(\tilde{f})) = \tilde{g}(S(\tilde{g}))$ is a simple closed curve and the sign of $S(\tilde{f})$ (resp. $S(\tilde{g})$) is $+1$ (resp. -1). Then there is a generic homotopy $\tilde{F} : S^2 \times [0, 1] \rightarrow \mathbb{R}^2$ between $\tilde{f}$ and $\tilde{g}$ which has a regular homotopy lift $F : S^2 \times [0, 1] \rightarrow \mathbb{R}^3$. See Figure 6. By the definitions of $\tilde{f}$, $\tilde{g}$, the regular homotopy lift F over $\tilde{F}$ corresponds to an eversion of the embedded sphere.

Our eversion in Example 5.2 is almost same as the eversion given by Francis [1]. But in his picture, self intersections of immersed spheres were not drawn. Professor Mikami Hirasawa and the author draw a regular homotopy over the generic homotopy of Figure 6, precisely. So, we can follow how self intersections move during our sphere eversion. Our eversion will appear in their preparing paper.

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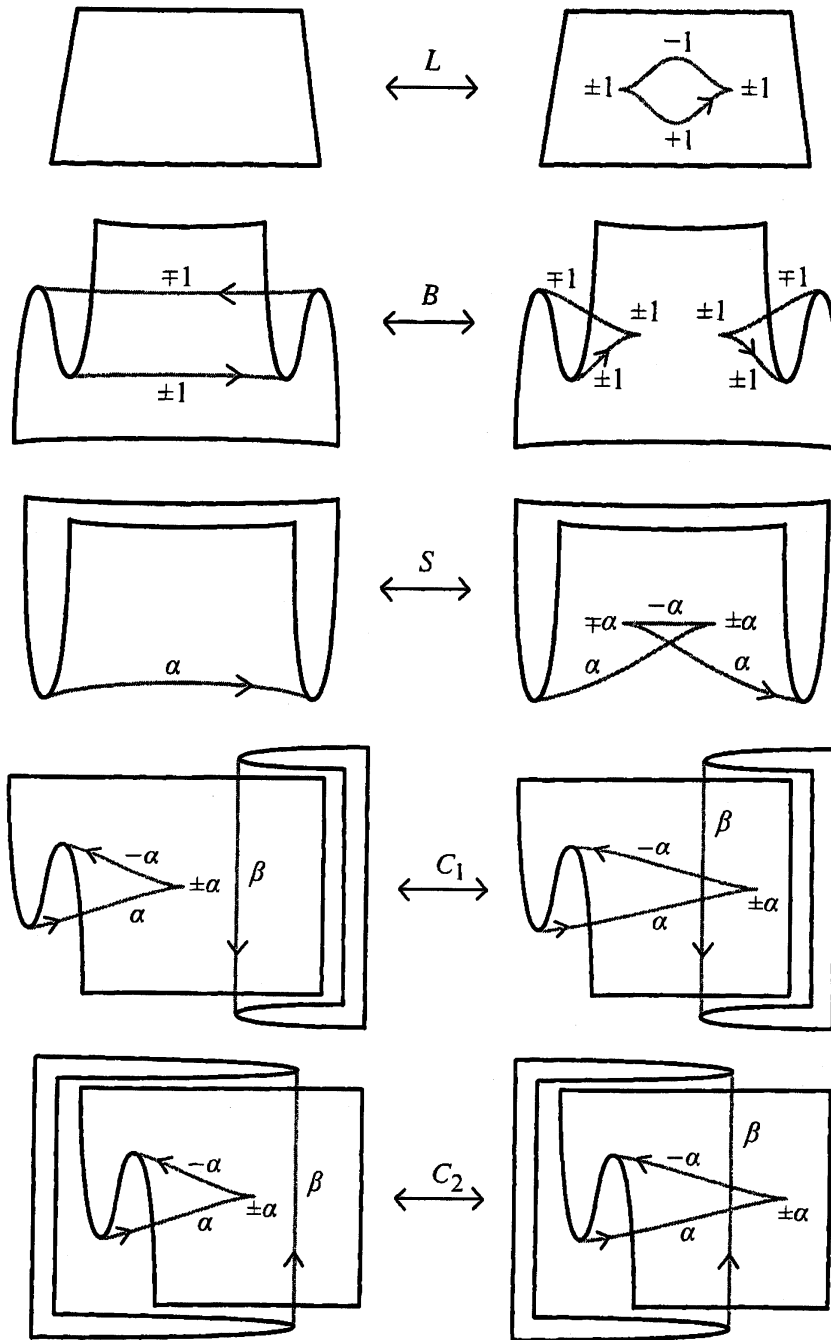


FIGURE 4. Bifurcations of type L , B , S and C which have regular homotopy lifts. Here, $\alpha = \pm 1$ and $\beta = \pm 1$ and α and β vary independently.

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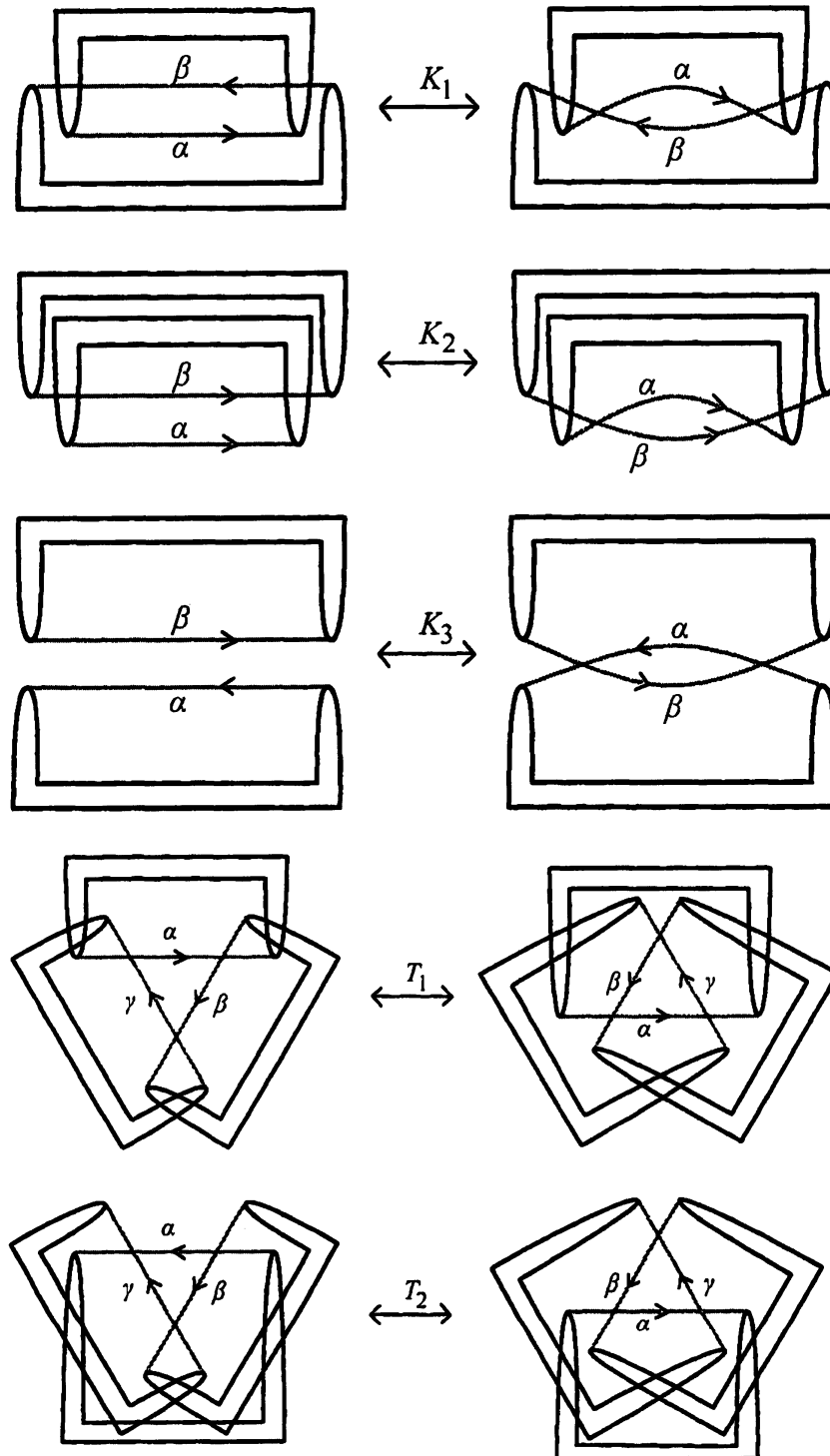


FIGURE 5. Bifurcations of type K and T which have regular homotopy lifts. Here, $\alpha = \pm 1, \beta = \pm 1$ and $\gamma = \pm 1$ and α, β and γ vary independently.

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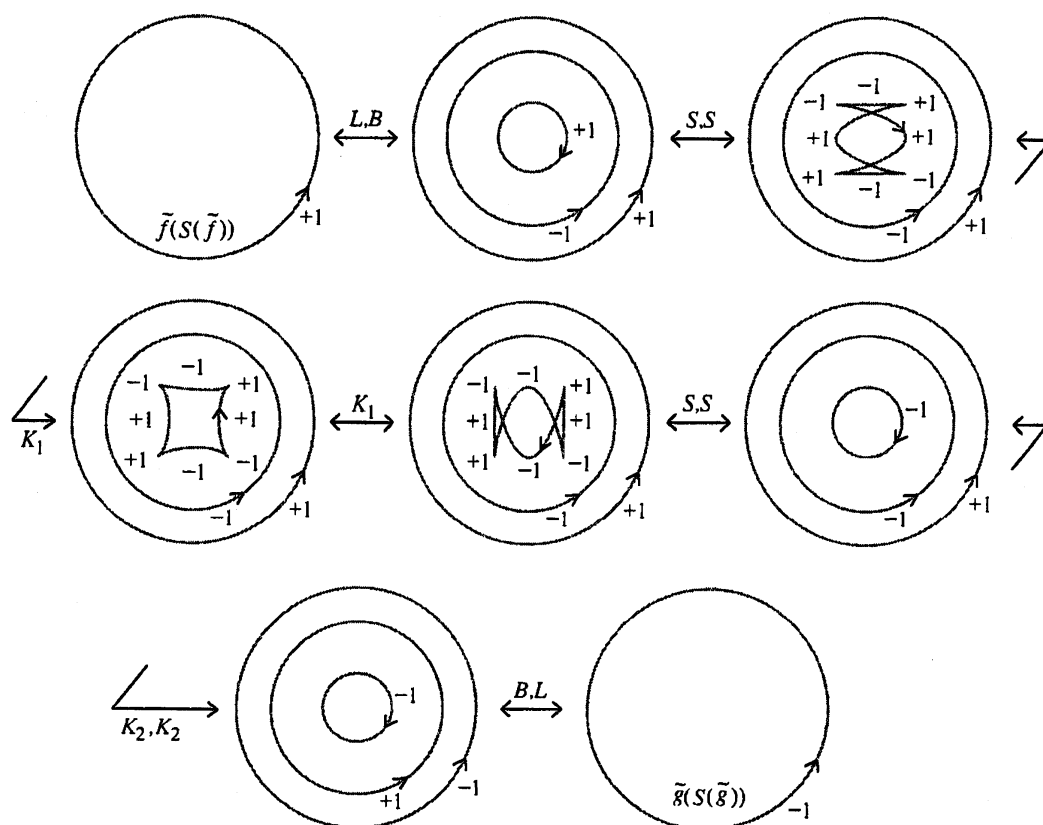


FIGURE 6. A sequence of apparent contours of a generic homotopy between $\tilde{f}$ and $\tilde{g} : S^2 \rightarrow \mathbb{R}^2$ which has a regular homotopy lift. This regular homotopy corresponds to a sphere eversion.