

FIXED POINT THEOREMS FOR MAPPINGS DETERMINED BY SEVERAL PARAMETERS IN METRIC SPACES

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1. FIXED POINT THEOREMS IN METRIC SPACES: TYPE 1 [5]

Let (X, d) be a metric space. A mapping T from X into itself is said to be widely more generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ such that

$$\begin{aligned} & \alpha d(Tx, Ty)^2 + \beta d(x, Ty)^2 + \gamma d(Tx, y)^2 + \delta d(x, y)^2 \\ & \quad + \varepsilon d(x, Tx)^2 + \zeta d(y, Ty)^2 \\ & \leq 0 \end{aligned}$$

for any $x, y \in X$. Such a mapping is called an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely more generalized hybrid mapping.

Lemma 1.1. *Let (X, d) be a metric space, let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely more generalized hybrid mapping from X into itself, and $\lambda \in [0, 1]$. Then T is an $(\alpha, (1 - \lambda)\beta + \lambda\gamma, \lambda\beta + (1 - \lambda)\gamma, \delta, (1 - \lambda)\varepsilon + \lambda\zeta, \lambda\varepsilon + (1 - \lambda)\zeta)$ -widely more generalized hybrid mapping from X into itself.*

Lemma 1.2. *Let X be a metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely more generalized hybrid mapping from X into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

- (1) $\alpha + (1 - \lambda)\varepsilon + \lambda\zeta + 2 \min\{\lambda\beta + (1 - \lambda)\gamma, 0\} \geq 0;$
- (2) $\alpha + \delta + \varepsilon + \zeta + 4 \min\{\lambda\beta + (1 - \lambda)\gamma, 0\} > 0.$

Then $\{T^n x \mid n \in \mathbb{N} \cup \{0\}\}$ is a Cauchy sequence for any $x \in X$.

By Lemma lemma:Cauchy-m we obtain directly the following theorem.

Theorem 1.1. *Let X be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely more generalized hybrid mapping from X into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

- (1) $\alpha + (1 - \lambda)\varepsilon + \lambda\zeta + 2 \min\{\lambda\beta + (1 - \lambda)\gamma, 0\} \geq 0;$
- (2) $\alpha + \delta + \varepsilon + \zeta + 4 \min\{\lambda\beta + (1 - \lambda)\gamma, 0\} > 0.$

Then for any $x \in X$ there exists $\lim_{n \rightarrow \infty} T^n x$.

By Theorem 1.1, we obtain the following theorem.

Theorem 1.2. *Let X be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely more generalized hybrid mapping from X into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

- (1) $\alpha + (1 - \lambda)\varepsilon + \lambda\zeta + 2 \min\{\lambda\beta + (1 - \lambda)\gamma, 0\} \geq 0;$
- (2) $\alpha + \delta + \varepsilon + \zeta + 4 \min\{\lambda\beta + (1 - \lambda)\gamma, 0\} > 0;$
- (3) $\alpha + (1 - \lambda)(\beta + \zeta) + \lambda(\gamma + \varepsilon) > 0.$

Then T has a fixed point u , where $u = \lim_{n \rightarrow \infty} T^n x$ for any $x \in X$. Additionally, if $\alpha + \beta + \gamma + \delta > 0$, then T has a unique fixed point.

Proof. By Theorem 1.1 there exists $u \in X$ such that $u = \lim_{n \rightarrow \infty} T^n x$. Replacing x and y by $T^n x$ and u , respectively, we obtain

$$\begin{aligned} & \alpha d(T^{n+1}x, Tu)^2 + ((1-\lambda)\beta + \lambda\gamma)d(T^n x, Tu)^2 \\ & + (\lambda\beta + (1-\lambda)\gamma)d(T^{n+1}x, u)^2 + \delta d(T^n x, u)^2 \\ & + ((1-\lambda)\varepsilon + \lambda\zeta)d(T^n x, T^{n+1}x)^2 + (\lambda\varepsilon + (1-\lambda)\zeta)d(u, Tu)^2 \\ & \leq 0. \end{aligned}$$

Since $u = \lim_{n \rightarrow \infty} T^n x$, we obtain

$$(\alpha + (1-\lambda)(\beta + \zeta) + \lambda(\gamma + \varepsilon))d(u, Tu)^2 \leq 0.$$

Since $\alpha + (1-\lambda)(\beta + \zeta) + \lambda(\gamma + \varepsilon) > 0$, we obtain $d(u, Tu)^2 \leq 0$ and hence u is a fixed point of T .

Furthermore, suppose that $\alpha + \beta + \gamma + \delta > 0$ holds. Let u and v be fixed points of T . Then we obtain

$$\begin{aligned} & \alpha d(Tu, Tv)^2 + \beta d(u, Tv)^2 + \gamma d(Tu, v)^2 + \delta d(u, v)^2 \\ & + \varepsilon d(u, Tu)^2 + \zeta d(v, Tv)^2 \\ & = (\alpha + \beta + \gamma + \delta)d(u, v)^2 \leq 0. \end{aligned}$$

Since $\alpha + \beta + \gamma + \delta > 0$, we obtain $d(u, v)^2 \leq 0$ and hence $u = v$. $\square$

2. FIXED POINT THEOREMS IN METRIC SPACES: TYPE 2 [7]

Lemma 2.1. Let (X, d) be a metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely more generalized hybrid mapping from X into itself. Suppose that there exists $\lambda \in [0, 1]$ such that

$$(1) \quad \alpha + (1-\lambda)\varepsilon + \lambda\zeta + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\} > 0.$$

Then

$$d(T^2x, Tx) \leq \sqrt{A}d(Tx, x)$$

holds for any $x \in X$, where

$$A = \max \left\{ -\frac{\delta + \lambda\varepsilon + (1-\lambda)\zeta + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\}}{\alpha + (1-\lambda)\varepsilon + \lambda\zeta + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\}}, 0 \right\}.$$

Lemma 2.2. Let (X, d) be a metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely more generalized hybrid mapping from X into itself. Suppose that there exists $\lambda \in [0, 1]$ such that

$$(1) \quad \alpha + (1-\lambda)\varepsilon + \lambda\zeta + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\} > 0;$$

$$(2) \quad \alpha + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\} > 0.$$

Then

$$d(T^3x, Tx) \leq \sqrt{B}d(Tx, x)$$

holds for any $x \in X$, where

$$\begin{aligned} B = & \max \left\{ \max \left\{ -\frac{(1-\lambda)\varepsilon + \lambda\zeta}{\alpha + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\}}, 0 \right\} A^2 \right. \\ & \left. + \max \left\{ -\frac{(1-\lambda)\beta + \lambda\gamma + 2\min\{\delta, 0\}}{\alpha + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\}}, 0 \right\} A \right\} \end{aligned}$$

$$-\frac{\lambda\varepsilon + (1-\lambda)\zeta + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\} + 2\min\{\delta, 0\}}{\alpha + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\}}, 0\}.$$

Lemma 2.3. *Let (X, d) be a metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely more generalized hybrid mapping from X into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

- (1) $\alpha + (1-\lambda)\varepsilon + \lambda\zeta + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\} > 0;$
- (2) $\alpha + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\} > 0.$

Then

$$d(T^3x, T^2x) \leq \sqrt{C}d(Tx, x)$$

holds for any $x \in C$, where

$$C = \max\left\{-\frac{\lambda\beta + (1-\lambda)\gamma}{\alpha + (1-\lambda)\varepsilon + \lambda\zeta}, 0\right\}B + \max\left\{-\frac{\delta + \lambda\varepsilon + (1-\lambda)\zeta}{\alpha + (1-\lambda)\varepsilon + \lambda\zeta}, 0\right\}A.$$

By Lemma 2.3, we obtain the following theorem.

Theorem 2.1. *Let (X, d) be a complete metric space and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -widely more generalized hybrid mapping from X into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

- (1) $\alpha + (1-\lambda)\varepsilon + \lambda\zeta + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\} > 0;$
- (2) $\alpha + 2\min\{\lambda\beta + (1-\lambda)\gamma, 0\} > 0;$
- (3) $C < 1.$

Then T has a fixed point u , where $u = \lim_{n \rightarrow \infty} T^n x$ for any $x \in X$. Additionally, if $\alpha + \beta + \gamma + \delta > 0$, then T has a unique fixed point.

Proof. By Lemma 2.3, we obtain

$$d(T^{n+1}x, T^n x) \leq \sqrt{C}d(T^{n-1}x, T^{n-2}x).$$

Since $0 \leq C < 1$, we obtain

$$\begin{aligned} d(T^m x, T^n x) &\leq \sum_{k=n}^{m-1} d(T^{k+1}x, T^k x) \\ &\leq \sum_{k=n, k:\text{even}}^{m-1} C^{\frac{k}{4}} d(Tx, x) + \sum_{k=n, k:\text{odd}}^{m-1} C^{\frac{k-1}{4}} d(T^2x, Tx) \\ &\leq \sum_{k=n, k:\text{even}}^{\infty} C^{\frac{k}{4}} d(Tx, x) + \sum_{k=n, k:\text{odd}}^{\infty} C^{\frac{k-1}{4}} d(T^2x, Tx) \\ &= \sum_{k=\lceil \frac{n+1}{2} \rceil}^{\infty} C^{\frac{k}{2}} d(Tx, x) + \sum_{k=\lceil \frac{n}{2} \rceil}^{\infty} C^{\frac{k}{2}} d(T^2x, Tx) \\ &= \frac{C^{\frac{1}{2}\lceil \frac{n+1}{2} \rceil} d(Tx, x) + C^{\frac{1}{2}\lceil \frac{n}{2} \rceil} d(T^2x, Tx)}{1 - \sqrt{C}} \end{aligned}$$

for any $m, n \in \mathbb{N}$ with $m \geq n$. Therefore $\{T^n x \mid n \in \mathbb{N} \cup \{0\}\}$ is Cauchy. Since X is complete, there exists $u = \lim_{n \rightarrow \infty} T^n x$. Replacing x and y by u and $T^n x$, respectively, we obtain

$$\alpha d(Tu, T^{n+1}x)^2 + ((1-\lambda)\beta + \lambda\gamma)d(u, T^{n+1}x)^2$$

$$\begin{aligned}
& +(\lambda\beta + (1-\lambda)\gamma)d(Tu, T^n x)^2 + \delta d(u, T^n x)^2 \\
& +((1-\lambda)\varepsilon + \lambda\zeta)d(u, Tu)^2 + (\lambda\varepsilon + (1-\lambda)\zeta)d(T^n x, T^{n+1}x)^2 \\
& \leq 0.
\end{aligned}$$

Putting $n \rightarrow \infty$, we obtain

$$(\alpha + \lambda(\beta + \zeta) + (1-\lambda)(\gamma + \varepsilon))d(Tu, u)^2 \leq 0.$$

Since

$$\begin{aligned}
& \alpha + \lambda(\beta + \zeta) + (1-\lambda)(\gamma + \varepsilon) \\
& \geq \alpha + (1-\lambda)\varepsilon + \lambda\zeta + \min\{\lambda\beta + (1-\lambda)\gamma, 0\} \\
& > -\min\{\lambda\beta + (1-\lambda)\gamma, 0\} \\
& \geq 0,
\end{aligned}$$

u is a fixed point.

Furthermore, suppose that $\alpha + \beta + \gamma + \delta > 0$ holds. Let u and v be fixed points of T . Then we obtain

$$\begin{aligned}
& \alpha d(Tu, Tv)^2 + \beta d(u, Tv)^2 + \gamma d(Tu, v)^2 + \delta d(u, v)^2 \\
& + \varepsilon d(u, Tu)^2 + \zeta d(v, Tv)^2 \\
& = (\alpha + \beta + \gamma + \delta)d(u, v)^2 \leq 0.
\end{aligned}$$

Since $\alpha + \beta + \gamma + \delta > 0$, we obtain $d(u, v)^2 \leq 0$ and hence $u = v$. $\square$

3. REGARDING PARAMETER CONDITIONS

In this section, we consider families of mappings determined by parameter conditions in the case of metric spaces.

Let

$$\begin{aligned}
& \mathcal{T}(\alpha, \beta, \gamma, \delta, \eta, \zeta) \\
& \stackrel{\text{def}}{=} \left\{ T : X \longrightarrow X \left| \begin{array}{l} \forall x, y \in X, \\ \alpha d(Tx, Ty)^2 + \beta d(x, Ty)^2 + \gamma d(Tx, y)^2 \\ + \delta d(x, y)^2 + \varepsilon d(x, Tx)^2 + \zeta d(y, Ty)^2 \\ \leq 0 \end{array} \right. \right\}.
\end{aligned}$$

If $(\alpha, \beta, \gamma, \delta, \eta, \zeta) \in [0, \infty)^6 \setminus \{(0, 0, 0, 0, 0, 0)\}$, then $\mathcal{T}(\alpha, \beta, \gamma, \delta, \eta, \zeta) = \emptyset$ clearly.

If $(\alpha, \beta, \gamma, \delta, \eta, \zeta) \in (-\infty, 0]^6$, then $\mathcal{T}(\alpha, \beta, \gamma, \delta, \eta, \zeta)$ contains all mappings clearly.

$\alpha + \beta + \gamma + \delta \leq 0 \iff \mathcal{T}(\alpha, \beta, \gamma, \delta, \eta, \zeta)$ contains the identity mapping.

If $\beta + \varepsilon \leq 0$, $\gamma + \zeta \leq 0$, and $\delta \leq 0$, then: let $Tx = c$ (where c is a constant).

Since

$$\begin{aligned}
& \alpha d(Tx, Ty)^2 + \beta d(x, Ty)^2 + \gamma d(Tx, y)^2 + \delta d(x, y)^2 \\
& + \varepsilon d(x, Tx)^2 + \zeta d(y, Ty)^2 \\
& = (\beta + \varepsilon)d(x, c)^2 + (\gamma + \zeta)d(y, c)^2 + \delta d(x, y)^2 \\
& \leq 0,
\end{aligned}$$

Therefore, T is included in $\mathcal{T}(\alpha, \beta, \gamma, \delta, \eta, \zeta)$.

Other than the above, we would like to find conditions under which $\mathcal{T}(\alpha, \beta, \gamma, \delta, \eta, \zeta)$ will not be empty.

Program Results

$(\alpha, \beta, \gamma, \delta, \eta, \zeta) \in [-5, 5]^6$ (step size: 1)

$$(\alpha, \beta, \gamma, \delta, \eta, \zeta) \notin (-\infty, 0]^6 \cup [0, \infty)^6$$

$$\alpha + \beta + \gamma + \delta > 0$$

$$\beta + \varepsilon > 0, \gamma + \zeta > 0, \text{ or } \delta > 0$$

Total number of parameter combinations: 750,987.

Search only for mappings like the following:

$$X = [0, 1] \text{ (step size: 0.1)}$$

$Tx = a + bx$: if the value exceeds the range, it folds up or down to fit within the range. ($a \in [0, 1]$, $b \in [-5, 5]$, step size: 0.1).

Let

$$\mathcal{T}_1(\alpha, \beta, \gamma, \delta, \eta, \zeta) \stackrel{\text{def}}{=} \left\{ \begin{array}{l} T : x \mapsto a + bx \\ \text{folds up or down} \\ \text{if necessary} \end{array} \mid \begin{array}{l} \forall a \in [0, 1], \forall b \in [-5, 5], \forall x, \forall y \in X, \\ \alpha d(Tx, Ty)^2 + \beta d(x, Ty)^2 + \gamma d(Tx, y)^2 \\ + \delta d(x, y)^2 + \varepsilon d(x, Tx)^2 + \zeta d(y, Ty)^2 \\ \leq 0 \end{array} \right\}.$$

Total number of parameter combinations such that $\mathcal{T}_1(\alpha, \beta, \gamma, \delta, \eta, \zeta) \neq \emptyset$ holds: 75,860.

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