

作用素不等式二題

Exponential operator inequalities

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Section 1.

Let X be a unital Banach algebra over $\mathbf{R}$ or $\mathbf{C}$, that is, a complete normed algebra with a unit 1 such that $\|1\| = 1$.

The aim of this note is, roughly speaking, to show that if $f : [0, \infty) \rightarrow X$ satisfies $f(0) = 1$, $f'(0) = a$, then $f(\frac{t}{n})^n$ converges to e^{ta} as $n \rightarrow \infty$, where

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

by definition.

If $X = \mathbf{R}$, this assertion clearly follows from the L'hospital theorem. Since a set of all bounded operators on a Banach space is a unital Banach algebra, for a bounded operator A , e^A is defined as above. In this case for bounded operators A, B the Lie product formula:

$$\exp(A + B) = (n) \lim_{n \rightarrow \infty} \left\{ \exp\left(\frac{A}{n}\right) \exp\left(\frac{B}{n}\right) \right\}^n$$

is well-known, where (n) means that the limit is in the sense of the (operator) norm topology. This implies that the above assertion holds for $f(t) = \exp(tA) \exp(tB)$ as well. The above definition e^x is not useful for unbounded operator. However it is well-known that if A is a generator of (C_0) contractive semigroup, then

$$e^{tA} = (s) \lim_{n \rightarrow \infty} \left(1 - \frac{t}{n} A\right)^{-n} \text{ for } t > 0,$$

where (s) means that the limit is in the sense of strong topology. The Lie product formula was extended to the case of unbounded operators on a Banach space in [2][4].

Chernoff [1] showed a product formula in a more general form as follows :

Let $f(t)$ be a strongly continuous function from $[0, \infty)$ to the linear contractions on a Banach space. Suppose that $f(0) = 1$ and the strong derivative $f'(0)$ has a closure A which is a generator of a (C_0) contractive semigroup. Then $f(t/n)^n$ strongly converges to e^{tA} .

In the proof of this theorem the condition $\|f(t)\| \leq 1$ plays an important role, so it is not easy to relax it. However we encounter many cases where $f(t)$ is not a contraction and the derivative A is bounded : in this case

$$\frac{f(t)}{\|f(t)\|}$$

is a contraction, but may not be differentiable at $t = 0$; so we can not use the Chernoff's theorem. Therefore we need to make a new product formula for bounded operators. See [3] for product formulas.

Theorem 1. *Let X be a unital Banach algebra, and let $f(t)$ be a function from an interval $0 \leq t < \zeta$ to X . If $f(0) = 1$ and $f(t)$ has a norm right derivative a at $t = 0$, then*

$$\|f(\frac{t}{n})^n - \exp(ta)\| \rightarrow 0 \quad (n \rightarrow \infty) \quad \text{for } 0 \leq t < \infty.$$

Proof. For every $t : 0 \leq t < \infty$, $f(\frac{t}{n})$ is defined for sufficiently large n , so we may assume f is defined on $[0, \infty)$. We claim that

there is $r > 0$ such that $\|f(t)\|^{\frac{1}{t}}$ is bounded on $(0, r)$.

To see this we may show that $\frac{1}{t} \log \|f(t)\|$ is bounded above on $0 < t < r$.

Since

$$\left\| \frac{f(t) - 1}{t} - a \right\| \rightarrow 0 \quad (t \rightarrow +0),$$

$\frac{1}{t}(\|f(t)\| - 1)$ is bounded, and $\|f(t)\| \rightarrow 1$ ($t \rightarrow +0$). Thus

$$\frac{\log \|f(t)\|}{t} = \begin{cases} \frac{\log \|f(t)\| - \log 1}{\|f(t)\| - 1} \cdot \frac{\|f(t)\| - 1}{t} & (\|f(t)\| \neq 1) \\ 0 & (\|f(t)\| = 1) \end{cases}$$

is bounded on some interval $(0, r)$.

Now take an arbitrary $t : 0 < t < \infty$, and fix it. By the claim above, we can see that $\{\|f(\frac{t}{n})\|^n\}_n$ is bounded. Thus there is $M > 0$ such that

$$e^{t\|a\|} \leq M, \quad \|f(\frac{t}{n})\|^n \leq M \quad \text{for every } n.$$

From

$$f\left(\frac{t}{n}\right)^n - e^{ta} = \sum_{m=0}^{n-1} f\left(\frac{t}{n}\right)^m \left\{ f\left(\frac{t}{n}\right) - e^{\frac{t}{n}a} \right\} (e^{\frac{t}{n}a})^{n-1-m},$$

it follows that

$$\begin{aligned} \|f\left(\frac{t}{n}\right)^n - e^{ta}\| &\leq \|f\left(\frac{t}{n}\right) - e^{\frac{t}{n}a}\| \sum_{m=0}^{n-1} M^{\frac{m}{n}} (e^{\frac{t}{n}\|a\|})^{n-1-m} \\ &= n \|f\left(\frac{t}{n}\right) - e^{\frac{t}{n}a}\| \cdot \frac{M - e^{t\|a\|}}{n(M^{\frac{1}{n}} - e^{\frac{t}{n}\|a\|})}. \end{aligned}$$

Since

$$n(M^{\frac{1}{n}} - e^{\frac{t}{n}\|a\|}) \rightarrow \log M - t\|a\|$$

and

$$n \|f\left(\frac{t}{n}\right) - e^{\frac{t}{n}a}\| \leq t \left\| \frac{n}{t} \left\{ f\left(\frac{t}{n}\right) - 1 \right\} - a \right\| + t \left\| \frac{n}{t} (-e^{\frac{t}{n}a} + 1) + a \right\| \rightarrow 0 \quad (n \rightarrow \infty),$$

we get

$$\|f\left(\frac{t}{n}\right)^n - e^{ta}\| \rightarrow 0 \quad (n \rightarrow \infty).$$

This concludes the proof. $\square$

Corollary 1. For $a_i \in X$ ($i = 1, \dots, m$)

$$\| \left\{ \left(1 + \frac{a_1}{n}\right) \cdots \left(1 + \frac{a_m}{n}\right) \right\}^n - \exp(a_1 + \cdots + a_m) \| \rightarrow 0,$$

$$\| (e^{\frac{a_1}{n}} \cdots e^{\frac{a_m}{n}})^n - \exp(a_1 + \cdots + a_m) \| \rightarrow 0.$$

Proof. By setting $f(t) = (1 + ta_1) \cdots (1 + ta_m)$ or $f(t) = e^{ta_1} \cdots e^{ta_m}$, these follows from the theorem. $\square$

Let $\phi(z)$ be a holomorphic function in a neighborhood $|z - 1| < \delta$. Then for $a \in X$: $\|a - 1\| < \delta$, $\phi(a)$ is defined by

$$\phi(a) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(1)}{n!} (a - 1)^n,$$

which converges in the norm. Thus for $f(t)$ with the property set out in the theorem $\phi(f(t))$ is well-defined for sufficiently small t . Since $\phi(f(0)) = \phi(1)$ and the right norm

derivative of $\phi(f(t))$ at $t = 0$ is $\phi'(1)f'(0)$, we have

Corollary 2. *If $\phi(z)$ is a scalar valued holomorphic function in a neighborhood of $z = 1$, with $\phi(1) = 1$, then for $f(t)$ which has the property set out in the theorem,*

$$\|\phi(f(\frac{t}{n}))^n - \exp(t\phi'(1)a)\| \rightarrow 0 \quad (n \rightarrow \infty) \quad \text{for } 0 \leq t < \infty.$$

In particular, we have

Corollary 3.

$$\|\{(1 + \frac{a_1}{n})^{\lambda_1} \cdots (1 + \frac{a_m}{n})^{\lambda_m}\}^n - \exp(\lambda_1 a_1 + \cdots + \lambda_m a_m)\| \rightarrow 0 \quad \text{for } \lambda_i \in \mathbb{R}.$$

In the proof of Theorem 1 that the domain of f is the right half real line is not essential. We can get the same result as above even if the domain of f is a half line with end point 0 in $\mathbb{C}$. More generally we show

Theorem 4. *Let X be a unital Banach algebra, set $D = \{z \in \mathbb{C} : \alpha \leq \arg z \leq \beta, 0 \leq \alpha \leq 2\pi\}$. If a function $f : D \rightarrow X$ satisfies $f(0) = 1$ and $f'(0) = a$, that is,*

$$\|\frac{f(z) - f(0)}{z} - a\| \rightarrow 0 \quad (z \in D, z \rightarrow 0),$$

then for every $z \in D$, $\|f(\frac{z}{n})^n - \exp za\| \rightarrow 0 \quad (n \rightarrow \infty)$.

Proof. In the same way as the proof of Theorem 1 one can easily show that $\|f(z)\|^{\frac{1}{|z|}}$ is bounded on a neighborhood of $0 \in D$, and that, for fixed $z \in D$,

$$\|f(\frac{z}{n})^n - e^{za}\| \leq \|f(\frac{z}{n}) - e^{\frac{z}{n}a}\| \sum_{m=0}^{n-1} M^{\frac{m}{n}} (e^{\frac{|z|}{n}\|a\|})^{n-1-m},$$

from which the theorem follows. □

References

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Section 2.

Let A and B be bounded selfadjoint operators on a Hilbert space. The following celebrated inequality was found by Furuta in [4] and simply proved in [5].

$$A \geq B \geq 0 \text{ implies } A^{(p+r)/q} \geq (A^{r/2} B^p A^{r/2})^{1/q} \quad (1)$$

for $p \geq 0, q \geq 1, r \geq 0$ such that $(1+r)q \geq p+r$.

Ando [1] showed the following theorem in the case of $s = p = r$ with a splended idea. Then Fujii, Furuta, Kamai [2], by making use of Ando's result, proved that $A \geq B$ implies (2).

Theorem A. $A \geq B$ implies that for $p \geq 0, r \geq s \geq 0$

$$e^{sA} \geq (e^{\frac{r}{2}A} e^{pB} e^{\frac{r}{2}A})^{\frac{s}{r+p}}. \quad (2)$$

In [1] Ando also showed the converse :

Theorem B. *If*

$$e^{tA} \geq (e^{\frac{t}{2}A} e^{tpB} e^{\frac{t}{2}A})^{\frac{t}{r+p}} \quad \text{for every } t > 0,$$

then $A \geq B$.

The aim of this note is to give a new way to get exponential inequalities from operator inequalities like (1), and to extend Theorems A, B.

We start with a quite simple proof of Theorem A. This technique seems to be very effective

to study operator inequality.

Another proof of Theorem A. For sufficiently large n we have $1 + \frac{A}{n} \geq 1 + \frac{B}{n} \geq 0$. By substituting np and nr to p and r of (1), respectively, we get,

$$\left(1 + \frac{A}{n}\right)^{\frac{n(p+r)}{q}} \geq \left\{\left(1 + \frac{A}{n}\right)^{\frac{nr}{2}} \left(1 + \frac{B}{n}\right)^{np} \left(1 + \frac{A}{n}\right)^{\frac{nr}{2}}\right\}^{1/q}, \text{ for } rq \geq p+r.$$

Since for selfadjoint operator X , $(1 + \frac{X}{n})^n$ converges to e^X in the operator norm as $n \rightarrow \infty$, we gain (2) by setting $s = \frac{p+r}{q}$. $\square$

We slightly extend Theorem A by using itself.

Proposition 1. $A \geq B$ implies

$$e^{sA} \geq \left\{e^{\frac{r}{2}A} e^{(qA+pB)} e^{\frac{r}{2}A}\right\}^{\frac{s}{(p+q+r)}} \quad (3)$$

for p, q, r, s with $0 \leq s \leq r$, $0 \leq p, p+q$, and $0 < p+q+r$.

Proof. If $p+q=0$, then $e^{(qA+pB)}$ is contractive, so that the above inequality follows. Therefore we assume that $p+q > 0$. Since

$$\frac{qA+pB}{q+p} \leq A,$$

by using (2), we gain (3). $\square$

Now we extend Theorem B:

Theorem 2. If there are p, q, r, s with $p > 0, p+q \geq 0, r \geq s > 0$ such that

$$e^{stA} \geq \left\{e^{\frac{rt}{2}A} e^{t(qA+pB)} e^{\frac{rt}{2}A}\right\}^{\frac{s}{(p+q+r)}}$$

for every $t > 0$, then $A \geq B$.

Proof. If $p+q+r=s$, then the above inequality implies that $e^{t(qA+pB)}$ is contractive because of $p+q=0$. Hence $A \geq B$. Suppose $p+q+r > s$. Set

$$f(t) = e^{\frac{-rt}{2}A} e^{-t(qA+pB)} e^{\frac{-rt}{2}A}, \quad g(t) = e^{-stA}.$$

Then we get

$$(f(t)^{\frac{s}{(p+q+r)}} x, x) \geq (g(t)x, x) \quad (\|x\| = 1, \quad t > 0),$$

from which it follows that

$$(f(t)x, x)^{\frac{s}{(p+q+r)}} \geq (g(t)x, x) \quad (t > 0)$$

because of Jensen's inequality. Since the values of both sides of the inequality above at $t = 0$ are 1, the right derivative of the left hand side at $t = 0$ is greater than or equal to the one of the right hand side. Since the norm derivative of e^{tT} at $t = 0$ is generally T , we have

$$\frac{s}{(p+q+r)} \left(\left(-\frac{r}{2}A - (qA + pB) - \frac{r}{2}A \right) x, x \right) \geq (-sAx, x).$$

Hence we gain $A \geq B$. □

We end this note with referring to an exponential inequality which appeared in [3]:

If $A - B \geq \delta > 0$, then $e^{tA} - e^{tB} \geq \delta/2 > 0$ for some $t > 0$.

This seems to be useful, so that we give a more generalized result, which we can see by a simple calculation.

Let $f(t), g(t)$ be selfadjoint operator valued functions defined in a neighborhood of $t = 0$. If $f(0) = g(0)$ and $f'(0) - g'(0) \geq \delta > 0$, where the derivative is in the sense of norm, then $f(t) - g(t) \geq \delta/2$ for t in a neighborhood of 0.

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