

Some remarks on ordered $\ast$ -groupoids

島根大学総合理工学部 今岡 輝男 (Teruo Imaoka)
磯辺 剛司 (Tsuyoshi Isobe)

Department of Mathematics, Shimane University
Matsue, Shimane 690-8504, Japan

A non-empty set G with a partial product, a unary operation $\ast$ and a partial order $\leq$ is called an *ordered $\ast$ -groupoid* if it satisfies the following axioms:

- (A1) $a(bc)$ exists if and only if $(ab)c$ exists, in which case they are equal.
- (A2) $a(bc)$ exists if and only if ab and bc exist.
- (A3) $(a^\ast)^\ast = a$.
- (A4) If ab exists, then $b^\ast a^\ast$ exists and $(ab)^\ast = b^\ast a^\ast$.
- (A5) For any $a \in G$, $a^\ast a$ exists and $a^\ast a$ is the unique projection of G such that there exists $a(a^\ast a)$ and $a(a^\ast a) = a$. We denote $a^\ast a$ by $d(a)$.
- (A6) $a \leq b$ implies $a^\ast \leq b^\ast$.
- (A7) For $a, b, c, d \in G$, if $a \leq b$, $c \leq d$ and there exist ac and bd , then $ac \leq bd$.
- (A8) Let $a \in G$ and $e (= e^2 = e^\ast)$ a projection such that $e \leq d(a)$. Then there exists a unique element $(a|e)$, say, such that $(a|e) \leq a$ and $d(a|e) = e$.
- (A9) $E(G)$ is an order ideal.

Lemma 1. [3] *Let G be an ordered $\ast$ -groupoid.*

- (1) *For any $a \in G$, $aa^\ast$ exists and $aa^\ast$ is the unique element of $P(G)$, the set of all projections of G , such that there exists $(aa^\ast)a$ and $(aa^\ast)a = a$. We denote $aa^\ast$ by $r(a)$.*
- (2) *Let $a \in G$ and $e \in P(G)$ such that $e \leq r(a)$. Then there exists a unique element $(e|a)$, say, such that $(e|a) \leq a$ and $r(e|a) = e$.*

An ordered $\ast$ -groupoid G is called a *locally inductive $\ast$ -groupoid* if it satisfies

- (LG) For any $e, f \in P(G)$, the set of projections of G , there exists the maximum element in $\langle e, f \rangle = \{(g, h) \in P(G) \times P(G) : g \leq e, h \leq f \text{ and } \exists gh\}$.

A regular $*$ -semigroup S is called a locally inverse $*$ -semigroup if eSe is an inverse subsemigroup of S for any projection e in S . Let S be a locally inverse $*$ -semigroup. The representation in [4] raise us a new partial product $\cdot$ on S , which is called a *restricted product*, as follows:

$$a \cdot b = \begin{cases} ab & ab \in R_a \cap L_b \\ \text{undefined} & \text{otherwise} \end{cases}$$

where R_a and L_a denote the $\mathcal{R}$ -class and the $\mathcal{L}$ -class containing a , respectively.

Lemma 2. [3] *Let S be a locally inverse $*$ -semigroup with the natural order $\leq$. Then $S(\cdot, *, \leq)$ is a locally inductive $*$ -groupoid, which is denoted by $\mathbf{G}(S)$.*

Conversely, let $G(\circ, *, \leq)$ be a locally inductive $*$ -groupoid. For any $a, b \in G$, there exists the maximum element (e, f) in $\langle d(a), r(b) \rangle = \{(g, h) \in P(S) \times P(S) : g \leq d(a), h \leq r(b), \exists g \circ h\}$. We define a new product $\otimes$ on G as follows:

$$a \otimes b = (a|e) \circ (f|b),$$

and we call it a *pseudoproduct* of a and b .

Lemma 3. [3] *Let $G(\circ, *, \leq)$ be a locally inductive $*$ -groupoid. The $G(\otimes, *)$ is a locally inverse $*$ -semigroup, which is denoted by $\mathbf{S}(G)$.*

Lemma 4. [3] (1) *For a locally inverse $*$ -semigroup S , we have $\mathbf{S}(\mathbf{G}(S)) = S$.*

(2) *For a locally inductive $*$ -groupoid $G(\circ, *, \leq)$, we have $\mathbf{G}(\mathbf{S}(G(\circ, *, \leq))) = G(\circ, *, \leq)$.*

Let S and T be regular $*$ -semigroups. A mapping $\phi : S \rightarrow T$ is called a *prehomomorphism* if it satisfies that $(ab)\phi \leq (a\phi)(b\phi)$ and $(a\phi)^* = a^*\phi$ for all $a, b \in S$.

Lemma 5. [2] *Let S and T be locally inverse $*$ -semigroups and $\phi : S \rightarrow T$ a mapping.*

- (1) *ϕ is a prehomomorphism if and only if it preserves the restricted product and the natural order.*
- (2) *ϕ is a homomorphism if and only if it is a prehomomorphism which satisfies $(ef)\phi = (e\phi)(f\phi)$ for all $e, f \in E(S)$.*
- (3) *The product of prehomomorphisms between locally inverse $*$ -semigroups is also a prehomomorphism.*

A functor between two ordered $*$ -groupoids is said to be *ordered* if it is order-preserving. An ordered functor between two locally inductive $*$ -groupoids is said to be *inductive* if it preserves the pseudoproduct.

Now, we have the main result.

Theorem 6. (Compare with Theorem 4.1.8 [5]) *The category of locally inverse $*$ -semigroups and prehomomorphisms is isomorphic to the category of locally inductive $*$ -groupoids and ordered functors. Moreover, the category of locally inverse $*$ -semigroups and homomorphisms is isomorphic to the category of locally inductive $*$ -groupoids and inductive functors.*

Proof. Let $\mathbf{G}$ be a function of the category of locally inverse $*$ -semigroups and prehomomorphisms to the category of locally inductive $*$ -groupoids and ordered functors as follows: for any locally inverse $*$ -semigroups S, T and any prehomomorphism $\theta : S \rightarrow T$,

- (1) $\mathbf{G}(S) = S(\cdot, *, \leq)$,
- (2) $\mathbf{G}(\theta) : \mathbf{G}(S) \rightarrow \mathbf{G}(T) (s \mapsto \theta(s))$.

Then it follows from Lemma 2 and Lemma 5 (1) that $\mathbf{G}$ is a functor.

Conversely, define a function $\mathbf{S}$ from the category of locally inductive $*$ -groupoids and ordered functors to the category of locally inverse $*$ -semigroups and prehomomorphisms as follows: for any locally inductive $*$ -groupoids G, H and any ordered functor $\theta : G \rightarrow H$,

- (1) $\mathbf{S}(G) = G(\otimes, *)$,
- (2) $\mathbf{S}(\theta) : \mathbf{S}(G) \rightarrow \mathbf{S}(H) (g \mapsto \theta(g))$.

By Lemma 3 and Lemma 5 (1), $\mathbf{S}$ is a functor. Moreover, it follows from Lemma 4 that $\mathbf{G}(\mathbf{S}(G)) = G$ and $\mathbf{S}(\mathbf{G}(S)) = S$. Thus we have that the category of locally inverse $*$ -semigroups and prehomomorphisms is isomorphic to the category of locally inductive $*$ -groupoids and ordered functors.

By Lemma 5 (2), we can easily obtain the second statement. $\square$

References

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