

On automorphic cuspidal representations of $U(2,2)$

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Introduction

In this paper, we study the hypercuspidality of automorphic cuspidal representations of $U(2,2)$.

The hypercuspidality in the case of the symplectic group was introduced by I. I. Piatetski-Shapiro [5]. When $G = \mathrm{GSp}_4$, for a given cusp form f on G_A , f is called "hypercuspidal" if the Whittaker function corresponding to f vanishes. Let $L_0^2(G_A)$ be the space of cusp forms on G_A . We denote by $L_{0,1}^2(G_A)$ the orthogonal complement of the space of all hypercuspidal forms in $L_0^2(G_A)$. Then any irreducible cuspidal representation in $L_{0,1}^2$ has a unique non-trivial Whittaker model. Thus, the multiplicity one theorem holds for $L_{0,1}^2$.

Analogously, we define the hypercuspidality in the case of $U(2,2)$ by vanishing of some Whittaker functions occurring in the Fourier expansion of a cusp form. More precisely, for a cusp form f on $U(2,2)$, we consider a Fourier expansion of f with respect to the center of the maximal unipotent subgroup of the Borel subgroup. Then we obtain two Whittaker functions W_f and V_f occurring in the Fourier expansion, where W_f is an ordinary Whittaker function and V_f is defined in §1. We note that in the case of Sp_4 ,

the function V_f did not appear in the similar Fourier expansion of a cusp form f . In terms of these functions, we say f is "U-cuspidal" (resp. "N-cuspidal") if W_f (resp. V_f) vanishes. Moreover, if both function W_f and function V_f vanish, f is called "hypercuspidal".

Next, using the dual reductive pair, we investigate cuspidal representations obtained from the Weil-lifting of those of $U(1,1)$ or $U(2,1)$. Roughly speaking, we have the following:

- (1) Cuspidal representations obtained from the Weil-lifting of those of $U(1,1)$ are U-cuspidal.
- (2) Let τ be a cuspidal representation of $U(2,1)$. Let $\theta(\tau, \psi)$ be a cuspidal representation obtained from the Weil-lifting of τ . Then,
 - (a) if τ is non-hypercuspidal in a sense of [1], then $\theta(\tau, \psi)$ is N-cuspidal, and
 - (b) if τ is hypercuspidal in a sense of [1], then $\theta(\tau, \psi)$ is hypercuspidal.

The details of proof will be given in my Master thesis at Tôhoku University.

Notation

Let F be a global field whose characteristic is different from 2 and let $\mathbb{A}_F$ be the adèle ring of F . Let E be a quadratic extension of F , and denote its Galois involution by $x \mapsto \bar{x}$. We fix once and for all an element i in E such that $\bar{i} = -i$ and a non trivial character ψ of $\mathbb{A}_F/F$.

1. Fourier expansions and the hypercuspidality

In this section, we give a definition of the hypercuspidality for cusp forms on $U(2,2)$.

Let V be a 4-dimensional vector space over E with basis $\{e_1, e_2, e_3, e_4\}$, and $(\ , \)_V$ the skew-hermitian form on V which is represented by the matrix $\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ with respect to $\{e_1, e_2, e_3, e_4\}$.
Let

$$G_F = \left\{ g \in GL_4(E) \mid g \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} {}^t \bar{g} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}$$

and

$$H_F = \left\{ h \in GL_2(E) \mid h \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t \bar{h} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Let B_F be the Borel subgroup of G_F such that its maximal torus is

$$T_F = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & \bar{a}^{-1} & \\ & & & \bar{b}^{-1} \end{pmatrix} \mid a, b \text{ in } E^* \right\},$$

and its unipotent radical is

$$U_F = \left\{ \begin{pmatrix} 1 & a & x - \bar{a}b & b \\ 0 & 1 & \bar{b} - \bar{a}y & y \\ & & 1 & 0 \\ 0 & & -\bar{a} & 1 \end{pmatrix} \mid a, b \text{ in } E, x, y \text{ in } F \right\}.$$

Let P_F be the parabolic subgroup stabilizing the isotropic line Ee

Then P_F is the product $L_F N_F$ of the Levi subgroup

$$L_F = \left\{ \begin{pmatrix} a' & & & \\ & a & & \\ & & \bar{a}', -1 & b \\ c & & & d \end{pmatrix} \mid a' \text{ in } E^*, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } H_F \right\},$$

and the unipotent radical

$$N_F = \left\{ \begin{pmatrix} 1 & a & x-\bar{a}b & b \\ 0 & 1 & \bar{b} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\bar{a} & 1 & 1 \end{pmatrix} \mid a, b \text{ in } E, x \text{ in } F \right\}.$$

Let Z_F be the center of U_F :

$$Z_F = \left\{ \begin{pmatrix} I_2 & x & 0 \\ 0 & 0 & 0 \\ 0 & I_2 & 1 \end{pmatrix} \mid x \text{ in } F \right\}.$$

For each ξ, ζ in E and t in F , we define characters $\psi_{(\xi, t)}$, $\psi_{(\xi, \zeta)}$ and ψ_t of $U_F \backslash U_A$, $N_F \backslash N_A$ and $Z_F \backslash Z_A$, respectively, by

$$\psi_{(\xi, t)} \left(\begin{pmatrix} 1 & a & x-\bar{a}b & b \\ 0 & 1 & \bar{b}-\bar{a}y & y \\ 0 & 1 & 0 & 0 \\ 0 & -\bar{a} & 1 & 1 \end{pmatrix} \right) = \psi(\text{Tr}_{E/F}(\xi a) + ty),$$

$$\psi_{(\xi, \zeta)} \left(\begin{pmatrix} 1 & a & x-\bar{a}b & b \\ 0 & 1 & \bar{b} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\bar{a} & 1 & 1 \end{pmatrix} \right) = \psi(\text{Tr}_{E/F}(\xi a + \zeta b))$$

and

$$\psi_t \left(\begin{pmatrix} I_2 & x & 0 \\ 0 & 0 & 0 \\ 0 & I_2 & 1 \end{pmatrix} \right) = \psi(tx).$$

Further we put $E^1 = \{ a \in E^* \mid a\bar{a} = 1 \}$ and $A_E^1 = \{ a \in A_E^* \mid a\bar{a} = 1 \}$. Then the center $C(G_A)$ of G_A is isomorphic to A_E^1 . For a character χ of $E^1 \backslash A_E^1$, let $\mathcal{A}_\chi(G_A)$ denote the space consisting of cusp forms on G_A which transform according to χ under $C(G_A)$. For each cusp form f on G_A , we define three Whittaker functions corresponding to f by

$$W_f^{\psi_{(\xi, t)}}(g) = \int_{U_F \backslash U_A} \overline{\psi_{(\xi, t)}(u)} f(ug) du,$$

$$V_f^{\psi(\xi, \zeta)}(g) = \int_{N_F \backslash N_{\mathbb{A}}} \overline{\psi(\xi, \zeta)(n)} f(ng) dn$$

and

$$J_f^{\psi_t}(g) = \int_{Z_F \backslash Z_{\mathbb{A}}} \overline{\psi_t(z)} f(zg) dz.$$

First, for a cusp form f on $G_{\mathbb{A}}$, we consider a Fourier expansion of f along Z . Fix g in $G_{\mathbb{A}}$. As a function on the compact abelian group $Z_F \backslash Z_{\mathbb{A}}$, $f(zg)$ has a Fourier expansion of the form

$$f(g) = \int_{Z_F \backslash Z_{\mathbb{A}}} f(zg) dz + \sum_{t \in F^*} J_f^t(g).$$

Let $[F^*]$ (resp. $[E^*]$) be a complete set of representatives of $N_{E/F}(E^*)$ (resp. E^1) in $F^*/N_{E/F}(E^*)$ (resp. E^*/E^1). Then by the analogy to [4] Lemma 6.2, we obtain the following:

Proposition 1. For each cusp form f on $G_{\mathbb{A}}$, one has

$$\begin{aligned} f(g) = & \sum_{t \in [F^*]} \left\{ \sum_{\gamma \in R_F \backslash L_F} W_f^{\psi(1, t)}(\gamma g) + \sum_{\gamma \in L(1, ti) \backslash L_F} V_f^{\psi(1, ti)}(\gamma g) + \right. \\ & \left. + \sum_{a \in [E^*]} J_f^{\psi_t} \left(\begin{pmatrix} a & & & \\ & 1 & & \\ & & \bar{a}^{-1} & \\ & & & 1 \end{pmatrix} g \right) \right\}, \end{aligned}$$

where

$$R_F = \left\{ \begin{pmatrix} a & & & \\ & a & & \\ & & ab & \\ & & & a \end{pmatrix} \mid a \text{ in } E^1, b \text{ in } F \right\}$$

and

$$L(1, ti) = \left\{ \begin{pmatrix} a' & & & \\ & a & & \\ & & \bar{a}', -1 & \\ & & & (a' - d)t^{-1}i^{-1} \end{pmatrix} \in L_F \right\}.$$

Now let

$$W(\psi) = \{ (W_f^\psi(1, t))_{t \in [F^*]} \mid f \in \mathcal{A}_0(G_{\mathbb{A}})_\chi \}$$

and

$$V(\psi) = \{ (V_f^\psi(1, ti))_{t \in [F^*]} \mid f \in \mathcal{A}_0(G_{\mathbb{A}})_\chi \}.$$

We define a linear map D from $\mathcal{A}_0(G_{\mathbb{A}})_\chi$ to $W(\psi) \oplus V(\psi)$ by

$$D(f) = ((W_f^\psi(1, t))_t, (V_f^\psi(1, ti))_t).$$

In terms of this linear map, we give the following

Definition. Let f be a cusp form on $G_{\mathbb{A}}$. We say f is *N-cuspidal* (resp. *U-cuspidal*) if f is contained in $D^{-1}(W(\psi))$ (resp. $D^{-1}(V(\psi))$). Further we say f is *hypercuspidal* if f is contained in $\text{Ker}(D)$.

We can show that these spaces are invariant by the action of the Hecke algebra of $G_{\mathbb{A}}$ and independent of a choice of a character ψ and a representative set $[F^*]$.

2. Lifting from $U(1,1)$ to $U(2,2)$

In this section, we consider the Weil-lifting $\theta(\tau, \psi)$ of an irreducible automorphic cuspidal representation τ of $H_{\mathbb{A}}$ to $G_{\mathbb{A}}$, and investigate the cuspidality of $\theta(\tau, \psi)$.

Let W be a 2-dimensional vector space over E , $(\ , \)_W$ the skew-hermitian form on W which is represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with respect to a suitable basis. We consider the symplectic space $X_F = (V \otimes W)_F$ obtained by taking the imaginary part of the hermitian form $(\ , \)_W \cdot (\ , \)_V$. Thus X_F is a 16-dimensional space over F , and we have a dual reductive pair $(H, G) \subset \text{Sp}_{16}(F)$.

In the same manner as in [1], §6 and §8, we choose and fix one Weil-representation ω_ψ of $G_{\mathbb{A}} H_{\mathbb{A}}$. Let $X_F = X_1 \oplus X_2$ be a complete paroralization of X_F and $S(X_{1,\mathbb{A}})$ the Schwarz - Bruhat space on $X_{1,\mathbb{A}}$.

Now suppose (τ, V_τ) is an automorphic cuspidal representation of $H_{\mathbb{A}}$ in the space of cusp forms on $H_{\mathbb{A}}$. For each ϕ in V_τ and Φ in $S(X_{1,\mathbb{A}})$, we put

$$\theta_\psi^\Phi(g, h) = \sum_{v \in X_{1,F}} \omega_\psi(gh) \Phi(v) \quad (h \in H_{\mathbb{A}}, g \in G_{\mathbb{A}}),$$

$$f_\phi^\Phi(g) = \int_{H_F \backslash H_{\mathbb{A}}} \theta_\psi^\Phi(g, h) \phi(h) dh.$$

We call the representation of $G_{\mathbb{A}}$ realized on

$$\theta(\tau, \psi) = \{ f_\phi^\Phi \mid \phi \text{ in } V_\tau, \Phi \text{ in } S(X_{1,\mathbb{A}}) \}$$

the "Weil-lifting" of τ .

We define an embedding $H_{\mathbb{A}} \hookrightarrow \text{Sp}_8(\mathbb{A}_F)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} \alpha(a) & 0 & 0 & \beta(b) \\ 0 & \alpha(a) & -\beta(b) & 0 \\ 0 & \gamma(c) & \delta(d) & 0 \\ -\gamma(c) & 0 & 0 & \delta(d) \end{pmatrix},$$

where for any x in E

$$\alpha(x) = \begin{pmatrix} \text{Re}(x) & \text{Im}(x) \\ -N_{E/F}(i)\text{Im}(x) & \text{Re}(x) \end{pmatrix}, \quad \beta(x) = \begin{pmatrix} \text{Im}(x) & -\text{Re}(x) \\ \text{Re}(x) & N_{E/F}(i)\text{Im}(x) \end{pmatrix},$$

$$\gamma(x) = \begin{pmatrix} N_{E/F}(i)\text{Im}(x) & -\text{Re}(x) \\ \text{Re}(x) & \text{Im}(x) \end{pmatrix}, \quad \delta(x) = \begin{pmatrix} \text{Re}(x) & N_{E/F}(i)\text{Im}(x) \\ -\text{Im}(x) & \text{Re}(x) \end{pmatrix}.$$

According to this embedding, the Weil-representation ω_ψ° of $\text{Sp}_8(\mathbb{A}_F)$ can be restricted to $\text{SU}(1,1)$. Furthermore, in the same manner as in [1], it can be extended to an ordinary representation ω_ψ° of $H_{\mathbb{A}}$. This extension is determined only up to twisting by a character of $\mathbb{A}_E^1$ composed with the determinant. Therefore we choose

one such extension ω_ψ° in accordance with the choice of the ordinary representation ω_ψ of $H_A G_A$. Then the Weil-representation ω_ψ° can be realized on the Schwarz - Bruhat space $S(W_A)$ of W_A . Hence, for each $\phi \in S(W_A)$, we put

$$\theta_\phi(h) = \sum_{w \in W_F} \omega_\psi^\circ(h) \phi(w)$$

and denote by $\Theta(\psi, \chi^{-1})$ the space consisting of theta-series θ_ϕ which transform according to χ^{-1} under the center of H_A .

Theorem 2. Let (τ, V_τ) be an irreducible cuspidal representation of H_A in $\mathcal{A}_\bullet(H_A)_\chi$.

- (1) If τ is non-trivial, then $\Theta(\tau, \psi)$ is also non-trivial.
- (2) $\Theta(\tau, \psi)$ is cuspidal if and only if τ is orthogonal to $\Theta(\psi, \chi^{-1})$.
- (3) If $\Theta(\tau, \psi)$ is cuspidal and non-trivial, then it is U-cuspidal, but not hypercuspidal.

3. Lifting from $U(2,1)$ to $U(2,2)$

We use the similar argument as in §2.

Let W be a 3-dimensional vector space over E with a basis $\{w_{-1}, w_0, w_1\}$ and $(\ , \)_W$ the hermitian form which is represented by the matrix

$$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

with respect to $\{w_{-1}, w_0, w_1\}$. Let G° be the corresponding unitary group, and N° the maximal unipotent subgroup of G° :

$$N_F^\circ = \left\{ \begin{pmatrix} 1 & a & z \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{pmatrix} \mid a, z \text{ in } E, z + \bar{z} = -a\bar{a} \right\}.$$

In the same manner as in §2, we have a dual reductive pair $(G, G^\circ) \subset \mathrm{Sp}_{24}(F)$. Further, for an irreducible cuspidal representation (τ, V_τ) of $G_{\mathbb{A}}^\circ$, we denote by $\Theta(\tau, \psi)$ the Weil-lifting of it.

For the general theory of cusp forms on $G_{\mathbb{A}}^\circ$, we refer to [1]. We define a character $\psi_\circ$ of $N_F^\circ \backslash N_{\mathbb{A}}^\circ$ by

$$\psi_\circ \left(\begin{pmatrix} 1 & a & z \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{pmatrix} \right) = \psi(\mathrm{Tr}_{E/F}(a)).$$

For $\phi \in L_{0,0}^2(G_F^\circ \backslash G_{\mathbb{A}}^\circ)$, we put

$$W_\phi^\psi(g) = \int_{N_F^\circ \backslash N_{\mathbb{A}}^\circ} \overline{\psi_\circ(n)} \phi(ng) dn.$$

Also we put

$$L_{0,0}^2(G_F^\circ \backslash G_{\mathbb{A}}^\circ) = \{ \phi \in L_{0,0}^2(G_F^\circ \backslash G_{\mathbb{A}}^\circ) \mid W_\phi^\psi \equiv 0 \},$$

$$L_{0,1}^2(G_F^\circ \backslash G_{\mathbb{A}}^\circ) = \text{the orthocomplement of } L_{0,0}^2 \text{ in } L_0^2.$$

These spaces are invariant under $G_{\mathbb{A}}^\circ$ and independent of ψ . Clearly, we have an orthogonal decomposition

$$L_0^2(G_F^\circ \backslash G_{\mathbb{A}}^\circ) = L_{0,0}^2(G_F^\circ \backslash G_{\mathbb{A}}^\circ) \oplus L_{0,1}^2(G_F^\circ \backslash G_{\mathbb{A}}^\circ).$$

We know from [1] that the multiplicity one theorem holds for $L_{0,1}^2$.

Now for each x in F^* , we take a vector w_x in W such that $(w_x, w_x)_W = x$, and let $G_{x,F}^\circ$ be the stabilizer of w_x in G_F° . Then we obtain a following

Proposition 3. $\Theta(\tau, \psi)$ is cuspidal if and only if

$$\int_{G_{x,F}^\circ \backslash G_{x,\mathbb{A}}^\circ} \phi(gh) dg = 0$$

for any x in F^* , ϕ in V_τ and h in $G_{\mathbb{A}}^\circ$.

In particular, if we take $w_x = \frac{1}{2}w_{-1} + xw_1$, then for any x in F^*

$$G_{x,F}^\circ \supset \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a \text{ in } E^1 \right\}.$$

Thus if V_τ satisfies the condition

$$(\#) \quad \int_{E^1 \backslash \mathbb{A}_E^1} \varphi \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} g \right) da = 0 \quad \text{for any } \varphi \text{ in } V_\tau \text{ and } g \text{ in } G_{\mathbb{A}}^\circ,$$

then $\theta(\tau, \psi)$ is cuspidal.

Theorem 4. (1) Suppose $(\tau, V_\tau) \in L_{0,1}^2(G_F^\circ \backslash G_{\mathbb{A}}^\circ)$. If τ is non-trivial, then $\theta(\tau, \psi)$ is also non-trivial. Moreover, if V_τ satisfies the condition (#), then $\theta(\tau, \psi)$ is N-cuspidal, but not hypercuspidal.
 (2) Suppose $(\tau, V_\tau) \in L_{0,0}^2(G_F^\circ \backslash G_{\mathbb{A}}^\circ)$. If V_τ satisfies the condition (#), then $\theta(\tau, \psi)$ is hypercuspidal.

In the proof, we take a complete polarization of $X_F = (V \otimes W)_F$ by $X_F = X_1 \oplus X_2$, where $X_1 = e_1 \otimes W + e_2 \otimes W$ and $X_2 = e_3 \otimes W + e_4 \otimes W$. Under this decomposition of X_F , we can give explicitly the action of the Weil-representation ω_ψ of $G_{\mathbb{A}} G_{\mathbb{A}}^\circ$ to Schwarz-Bruhat space $S(X_1, \mathbb{A}) \simeq S(W_{\mathbb{A}} \oplus W_{\mathbb{A}})$. In the case (1), we put $f = f_\varphi^\Phi \in \theta(\tau, \psi)$, where $\Phi \in S(W_{\mathbb{A}} \oplus W_{\mathbb{A}})$ and $\varphi \in V_\tau$. Then by computing W_f^ψ directly, we have

$$W_f^\psi(1, \frac{1}{2}t) \equiv 0 \quad \text{for } 1 \neq t \in [F^*]$$

$$W_f^\psi(1, \frac{1}{2})(g) = \int_{Z_{\mathbb{A}}^\circ \backslash G_{\mathbb{A}}^\circ} \omega_\psi(gh) \Phi(w_1, w_0) W_\varphi^\psi(h) dh,$$

where Z° is the center of N° . In particular, the latter formula defines the "local Weil-lifting" of a non-degenerate admissible representation of $G_{F_V}^\circ$ to G_{F_V} .

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