

On a constrained noncooperative n-person game

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§1. Introduction

In this paper, a noncooperative equilibrium point in a constrained n-person game is investigated. we show that there exists an equilibrium point in the n-person game if and only if some set valued mapping has a fixed point. Then, using the proposition which is derived from Ekeland's theorem, we shall discuss the conditions under which there exists a fixed point of the mapping.

§2. Formulation of a noncooperative n-person game

We define a noncooperative n-person game by the following strategic form

$$(N, X, F), \quad (2.1)$$

where

(i) $N = \{ 1, 2, \dots, n \}$ is the set of n players.

(ii) $X = \prod_{i=1}^n X^i \subset U = \prod_{i=1}^n U^i$, for each $i \in N$, X^i is the subset of a complete metric space U^i and is called the strategy set of each player i.

(iii) $F = (f^1, f^2, \dots, f^n) : X \rightarrow R^n$, is a multiloss

operator and, for each $i \in N$, $f^i: X \rightarrow R$, denotes a loss function for player i .

In this paper, denoting by $\hat{i} = N - i$ the coalition adverse to each player i , the multistrategy set, $X = \prod_{i=1}^n X^i$ is split as follows

$$X = X^i \times X^{\hat{i}} \quad \text{and} \quad X^{\hat{i}} = \prod_{j \neq i} X^j.$$

If π^i and $\pi^{\hat{i}}$ denote the projection from X into X^i and $X^{\hat{i}}$, we set $x^i = \pi^i x$ and $x^{\hat{i}} = \pi^{\hat{i}} x$ for a multistrategy $x = (x^i, x^{\hat{i}}) \in X$.

Now, we define, for each $i \in N$,

$$\alpha^i = \inf_{x \in X} f^i(x)$$

and, throughout this paper, we assume that $\alpha^i > -\infty$ for all $i \in N$. In this case, the game is bounded below and $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^n)$ is called *shadow minimum* of the game. Then, we have

$$F(X) \subset \alpha + R_+^n,$$

where

$$F(X) = \{ F(x) \in R^n; \text{ for all } x = (x^1, x^2, \dots, x^n) \in X \}$$

and

$$R_+^n = \{ x = (x^1, x^2, \dots, x^n) \in R^n; x^i \geq 0 \text{ for all } i \in N \}.$$

If $\alpha = F(\bar{x})$ belongs to $F(X)$, the multistrategy $\bar{x} \in X$ attains to the minimum of the loss function f^i for each player i . In this case, $\bar{x}$ is the best solution for each player. But, this situation is seldom the case and we have to investigate other solution concepts. So, we consider especially noncooperative equilibrium point.

Definition 2.1 A multistrategy $x = (x^1, x^2, \dots, x^n)$

$\in X$ is said to be a noncooperative equilibrium point (Nash equilibrium point) if, for all $i \in N$,

$$f^i(x) = \inf_{\substack{y \in X \\ \pi^{\hat{i}} y = x^{\hat{i}}}} f^i(y). \quad (2.2)$$

Such a noncooperative equilibrium point shows that given the complementary coalition's choice $x^{\hat{i}}$, the player i responds by playing a strategy $x^i \in X^i$ which minimizes $f^i(\cdot, x^{\hat{i}})$ on X^i .

Remark 2.1 If $N = \{ 1, 2 \}$ and $f^1(x) + f^2(x) = 0$ for all multistrategies $x \in X = X^1 \times X^2$, an equilibrium point is called saddle point.

We define the correspondences U^i mapping $X^{\hat{i}}$ into X^i which assign to the choice $x^{\hat{i}}$ of strategies of players $j \neq i$ the subset of feasible strategies for the i player:

$$U_i(x^{\hat{i}}) = \{ y^i \in X^i ; (y^i, x^{\hat{i}}) \in X \}. \quad (2.3)$$

So, we can rewrite the definition of a noncooperative equilibrium point in the following way:

$$\begin{aligned} (i) \quad & \bar{x}^i \in U_i(x^{\hat{i}}) \quad \text{for all } i \in N \\ (ii) \quad & f_i(x^i, x^{\hat{i}}) = \min_{y^i \in U_i(x^{\hat{i}})} f_i(y^i, x^{\hat{i}}) \quad \text{for all } i \in N. \end{aligned} \quad (2.4)$$

Under such the situation, we introduce the following notations: for all $i \in N$,

$$S_i(x^{\hat{i}}) = \{ x^i \in U_i(x^{\hat{i}}) ; f_i(x^i, x^{\hat{i}}) = \min_{y^i \in U_i(x^{\hat{i}})} f_i(y^i, x^{\hat{i}}) \}$$

and

$$S(x) = \prod_{i=1}^n S_i(x^{\hat{i}}) : X \rightarrow 2^X, \quad (2.5)$$

where 2^X denotes the set of all subsets in X . Then we can show the relationships between a fixed point of $S : X \rightarrow 2^X$ and a

noncooperative equilibrium point in the game.

Proposition 2.1 A multistrategy $\bar{x} \in X$ is a noncooperative equilibrium point if and only if $\bar{x}$ is a fixed point of S , that is,

$$\bar{x} \in S(\bar{x}).$$

Proof. Using (i) and (ii) in (2.4) and the definition of the set valued mapping S , we can easily prove the proposition.

Thus, it is very important to show that there exists a fixed point of the mapping S in X . In next section, we shall study the existence of a fixed point of the mapping S .

§3. A fixed point of the mapping S from U into 2^U

In this section, we assume that the strategy set of each player i is U^i , that is, $X^i = U^i$ for all $i \in N$. In order to show that there exists a noncooperative equilibrium point in the n -person game (2.1), we shall discuss a fixed point of mapping S . Let $P(U)$ be the family of nonempty subset of the complete metric space (U, d) , $2^U - \{\emptyset\}$. For a point $x \in U$ and a member $A \in P(U)$, we define the distance d from x to A as follows:

$$d(x, A) = \inf_{a \in A} d(x, a).$$

Further, given two members $A, B \in P(U)$, we introduce the following notation:

$$\delta(A, B) = \sup_{a \in A} d(a, B). \quad (3.1)$$

The function δ may be infinite valued on $P(U) \times P(U)$. So, throughout the section, we restrict δ to $P'(U) = \{ A \in P(U); A \text{ is close and bounded in } U \}$ and, then we can prove the following

proposition.

Proposition 3.1 For any A, B, and C in $P'(U)$,

- (1) If $\delta(A, B) = 0$, $A \subset B$
- (2) $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$.

Proof. The proof of (1) is obvious from the definition of δ . In order to prove (2), for any $a \in A$, $b \in B$, and $c \in C$, we have

$$d(a, b) \leq d(a, c) + d(c, b). \quad (3.2)$$

Taking infimum of both sides of (3.2) on B, we obtain

$$d(a, B) \leq d(a, c) + d(c, B),$$

that is, for any $c \in C$,

$$d(a, B) - d(c, B) \leq d(a, c). \quad (3.3)$$

So, from (3.3), for any $a \in A$,

$$d(a, B) \leq d(a, C) + \delta(C, B). \quad (3.4)$$

(3.4) shows that (2) holds. Thus, the proof is completed.

Hence, given two members A, B in $P'(U)$, we define

$$H(A, B) = \max [\delta(A, B), \delta(B, A)].$$

Using Proposition 3.1, we show that the function $H: P'(U) \times P'(U) \rightarrow [0, \infty)$ satisfies all properties of a metric on $P'(U)$, that is, $(P'(U), H)$ is a metric space and H is well known as the Hausdorff metric.

Proposition 3.2 For any $x \in U$ and any B, C in $P'(U)$,

- (1) $d(x, B) - d(x, C) \leq \delta(C, B)$
- (2) $|d(x, B) - d(x, C)| \leq H(B, C)$.

Proof. For any $\varepsilon > 0$, there exists $c \in C$ such that

$$d(x, c) \leq d(x, C) + \varepsilon. \quad (3.5)$$

From (3.5), it follows that, for all $b \in B$,

$$\begin{aligned} d(x,B) - d(x,C) &< d(x,b) - d(x,c) + \varepsilon \\ &\leq d(b,c) + \varepsilon. \end{aligned} \quad (3.6)$$

Taking infimum of left side in (3.6) on B and, then, supremum of left side in (3.6) on C, we obtain

$$d(x,B) - d(x,C) \leq \delta(C,B) + \varepsilon.$$

Since ε is arbitrary, we get

$$d(x,B) - d(x,C) \leq \delta(C,B). \quad (3.7)$$

Further, using the definition of H, it follows from (3.7) that

$$d(x,B) - d(x,C) \leq H(C,B). \quad (3.8)$$

Interchanging B with C in (3.8), we have

$$d(x,C) - d(x,B) \leq H(B,C) (= H(C,B)). \quad (3.9)$$

Thus, (3.8) and (3.9) complete the proof of the proposition.

Then, we give the concepts of H-continuity (H-c.), H-upper semicontinuity (H-u.s.c.), and H-lower semicontinuity (H-l.s.c.) of the set valued mapping $S: U \rightarrow P'(U)$.

Definition 3.1 The mapping S is called H-c. at $x \in U$ if for any each sequence $\{x^k\}, k=1,2,\dots$, in U converging to x, it follows that $\{H(S(x^k), S(x))\}, k=1,2,\dots$, converges to 0. Further, when S is H-c. at every point in U, S is called H-c. in U.

Definition 3.2 The mapping S is called H-u.s.c. at $x \in U$ if for each sequence $\{x^k\}, k=1,2,\dots$, in U converging to x, it follows that $\{\delta(S(x^k), S(x))\}, k=1,2,\dots$, converges to 0. The mapping S is called H-l.s.c. at $x \in U$ if for each sequence $\{x^k\}, k=1,2,\dots$, in U converging to x, it follows that $\{\delta(S(x), S(x^k))\}, k=1,2,\dots$, converges to 0. The mapping S is called H-u.s.c. (resp. H-l.s.c.) in U if it is H-u.s.c. (resp. H-l.s.c.) at every point in U.

Remark 3.1 If the mapping S is H-c., it is obvious that S is H-u.s.c. and H-l.s.c..

The mapping S is called a contraction one if there exists a real number $r \in [0,1)$ such that for all $x, y \in U$,

$$H(S(x), S(y)) \leq rd(x, y). \quad (3.10)$$

Then, introducing the function $G(x) = d(x, S(x))$: $U \rightarrow \mathbb{R}$, which plays an important role in the section, we can prove H-c. of G .

Proposition 3.3 Suppose S is a contraction mapping. Then, the function G is H-c. in U .

Proof. From the propositions 3.1 and 3.2, it follows that

$$\begin{aligned} |G(x) - G(y)| &\leq |d(x, S(x)) - d(y, S(y))| \\ &\leq |d(x, S(x)) - d(y, S(x))| + |d(y, S(x)) - d(y, S(y))| \\ &\leq d(x, y) + H(S(x), S(y)). \end{aligned} \quad (3.11)$$

Since H is contraction, we obtain

$$|G(x) - G(y)| \leq (1+r)d(x, y),$$

which completes the proof.

Further, in order to show that there exists a fixed point of the mapping S , we shall need the following proposition which is derived from Ekeland's theorem (see [4] and [5]).

Proposition 3.4 Let (U, d) be a complete metric space, and $G: U \rightarrow \mathbb{R} \cup \{+\infty\}$, a l.s.c. function, $\neq +\infty$, bounded from below. For any $\varepsilon > 0$, there is some point $\bar{x} \in U$ with:

$$\begin{aligned} G(\bar{x}) &\leq \inf_{x \in U} G(x) + \varepsilon \\ G(x) &\geq G(\bar{x}) - \varepsilon d(\bar{x}, x). \end{aligned}$$

Theorem 3.1 Suppose that $S: U \rightarrow P'(U)$, is a contraction mapping, that is, S satisfies (3.10). Then, S has a fixed point

$\bar{x} \in U$ satisfying $\bar{x} \in S(\bar{x})$.

Proof. Since $G(x) = d(x, S(x))$ is H-c. in U by Proposition 3.3 and $G(x) \geq 0$, using Proposition 3.4 for any $\varepsilon \in (0, 1-r)$, there exists a point $\bar{x}$ such that for all $x \in U$,

$$G(x) + \varepsilon d(\bar{x}, x) \geq G(\bar{x}). \quad (3.12)$$

From (3.10), it follows that for all $x \in S(\bar{x})$,

$$G(\bar{x}) \leq (r+\varepsilon)d(\bar{x}, x),$$

that is,

$$G(\bar{x}) \leq (r+\varepsilon)G(\bar{x}).$$

Thus, $G(\bar{x}) = 0$ because $r+\varepsilon < 1$. This completes the proof.

Proposition 3.5 If S is H-u.s.c. in U , $G(x) = d(x, S(x))$ is l.s.c..

Proof. For any sequence $\{x^k\}, k=1, 2, \dots$, in U converging to x , we make use of the following notations:

$$F_1(x^k) = G(x^k) - d(x, x^k)$$

$$F_2(x^k) = d(x, S(x^k)) - G(x).$$

Then, we have

$$\begin{aligned} G(x^k) - G(x) &= d(x^k, S(x^k)) - d(x, S(x)) \\ &= G(x^k) - d(x, S(x^k)) + d(x, S(x^k)) - G(x) \\ &= F_1(x^k) + F_2(x^k). \end{aligned} \quad (3.13)$$

Using (3.4) in the proof of Proposition 3.1 for $F_1(x^k)$ and $F_2(x^k)$, we get

$$F_1(x^k) \geq -d(x, x^k), \quad F_2(x^k) \geq -\delta(S(x^k), S(x)). \quad (3.14)$$

From (3.13) and (3.14), it follows that

$$G(x^k) - G(x) \geq -d(x, x^k) - \delta(S(x^k), S(x)). \quad (3.15)$$

Since S is H-u.s.c., (3.15) shows that

$$\liminf_{k \rightarrow \infty} G(x^k) \geq G(x).$$

Whence, the proof is complete.

Theorem 3.2 Suppose that S is H-u.s.c. and there exist two positive real numbers $\alpha_1, \alpha_2, \alpha_1 + \alpha_2 < 1$, such that for all $x, y \in U$,

$$H(S(x), S(y)) \leq \alpha_1 G(x) + \alpha_2 G(y). \quad (3.16)$$

Then, there exists a fixed point $\bar{x} \in U$ of S satisfying

$$\bar{x} \in S(\bar{x}).$$

Proof. Since S is H-u.s.c., $G(x) = d(x, S(x))$ is l.s.c. from Proposition 3.5. Using Proposition 3.4 for $\varepsilon \in (0, 1 - \alpha_1 / (1 - \alpha_2))$, we have a point $\bar{x} \in U$ such that for all $x \in U$,

$$G(x) \geq G(\bar{x}) - \varepsilon d(\bar{x}, x). \quad (3.17)$$

From (3.17), it follows that for all $x \in S(\bar{x})$,

$$\begin{aligned} G(\bar{x}) &\leq G(x) + \varepsilon d(\bar{x}, x) \\ &\leq H(S(\bar{x}), S(x)) + \varepsilon d(\bar{x}, x) \\ &\leq \alpha_1 G(\bar{x}) + \alpha_2 G(x) + \varepsilon d(\bar{x}, x) \\ &\leq \alpha_2 H(S(\bar{x}), S(x)) + \alpha_1 G(\bar{x}) + \varepsilon d(\bar{x}, x) \\ &\leq \alpha_2 [\alpha_1 G(\bar{x}) + \alpha_2 G(x)] + \alpha_1 G(\bar{x}) + \varepsilon d(\bar{x}, x) \\ &\leq \alpha_2^2 G(x) + \alpha_1 (\alpha_2 + 1) G(\bar{x}) + \varepsilon d(\bar{x}, x) \end{aligned}$$

$$\leq \alpha_2^{n+1} G(x) + \alpha_1 (\alpha_2^n + \alpha_2^{n-1} + \dots + \alpha_2 + 1) G(\bar{x}) + \varepsilon d(\bar{x}, x).$$

Since $\alpha_2^{n+1} \rightarrow 0$ as $n \rightarrow \infty$, we arrive at

$$G(\bar{x}) \leq \alpha_1 / (1 - \alpha_2) G(\bar{x}) + \varepsilon d(\bar{x}, x) \quad \text{for all } x \in S(\bar{x}). \quad (3.18)$$

From (3.18), we get

$$G(\bar{x}) \leq \alpha_1 / (1 - \alpha_2) G(\bar{x}) + \varepsilon G(\bar{x}),$$

that is,

$$[1 - \alpha_1/(1-\alpha_2) - \varepsilon]G(\bar{x}) \leq 0.$$

This shows that $G(\bar{x}) = 0$ because $[1 - \alpha_1/(1-\alpha_2) - \varepsilon] > 0$. Thus, the proof is completed.

Theorem 3.3 Suppose that $F: U \rightarrow R$, a l.s.c. function, bounded from below and there exists a positive real number r such that

$$F(x) - F(y) \geq rd(x,y) \quad \text{for all } y \in S(x) \text{ and all } x \in U. \quad (3.19)$$

Then, S has a fixed point $\bar{x} \in U$ satisfying

$$S(\bar{x}) = \{\bar{x}\}.$$

Proof. Using Proposition 3.4 for $\varepsilon \in (0,r)$, there exists some point $\bar{x} \in U$ such that

$$F(x) \geq F(\bar{x}) - \varepsilon d(\bar{x},x) \quad \text{for all } x \in U.$$

Thus, we get

$$F(y) + \varepsilon d(\bar{x},y) \geq F(\bar{x}) \quad \text{for all } y \in S(\bar{x}). \quad (3.20)$$

From (3.19) and (3.20), it follows that

$$(r-\varepsilon)d(\bar{x},y) \leq 0 \quad \text{for all } y \in S(\bar{x}).$$

This shows that $d(\bar{x},y) = 0$ for all $y \in S(\bar{x})$, because $r-\varepsilon > 0$.

Whence the proof is completed.

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