

Infinite-dimensions and rings of continuous functions

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This is a joint study with Takashi Kimura.

We assume that all spaces are normal and all rings are commutative unless otherwise stated. We refer the readers to [3] for dimension theory.

Let aR denote the principal right ideal generated in a ring R by an element a .

Canfoll defined the dimension of a ring as follows.

Definition 1 ([2]). A set of principal ideals a_iR , $i = 1, \dots, n$, is *uniquely generated* if whenever $a_iR = b_iR$, $i = 1, \dots, n$, there exist elements u_i of R such that $a_i = b_iu_i$, $i = 1, \dots, n$, and $\sum_{i=1}^n u_iR = R$. The *dimension* of R — denoted by $\dim R$ — is the least integer n such that every set of $n + 1$ principal ideals is uniquely generated.

Let $C(X)$ be the ring of real-valued continuous functions defined on a space X . For $f \in C(X)$, the zero set $Z(f)$ of f , is defined by $Z(f) = \{x \in X : f(x) = 0\}$. Clearly $\sum_{i=1}^n f_iC(X) = C(X)$ if and only if $\bigcap_{i=1}^n Z(f_i) = \emptyset$.

Canfell [2] proved the following theorem.

Theorem 1 (Canfell [2]). *For every space X we have $\dim X = \dim C(X)$.*

We consider transfinite extensions of the dimension of a ring R .

Borst defined the transfinite dimension of a space X as follows.

Definition 2(cf. [3]). For every set L we denote by $\text{Fin}L$ the collection of all non-empty finite subsets of L and for every $\sigma \in \text{Fin}L$ and $M \subset \text{Fin}L$ let

$$M^\sigma = \{\tau \in \text{Fin}L : \sigma \cup \tau \in M \text{ and } \sigma \cap \tau = \emptyset\}.$$

If $\sigma = \{a\}$, we write M^a instead of $M^{\{a\}}$.

For every subcollection M of $\text{Fin}L$ the *large order* $\text{Ord}M$ of M , which is an ordinal number or the “infinite number” ∞ , is defined by the following conditions:

- (O1) $\text{Ord}M = 0$ if and only if $M = \emptyset$;
- (O2) $\text{Ord}M \leq \alpha$, where α is an ordinal number > 0 , if $\text{Ord}M^a < \alpha$ for every $a \in L$;
- (O3) $\text{Ord}M = \alpha$ if $\text{Ord}M \leq \alpha$ and the inequality $\text{Ord}M \leq \beta$ holds for no $\beta < \alpha$;
- (O4) $\text{Ord}M = \infty$ if $\text{Ord}M \leq \alpha$ holds for no ordinal number α .

Let Γ be an index set. A collection $\tau = \{(A_i, B_i) : i \in \Gamma\}$ of pairs of disjoint closed subsets of X is called *essential* if for every $\{L_i : i \in \Gamma\}$, where L_i is a partition in X between A_i and B_i for every $i \in \Gamma$, we have $\bigcap_{i \in \Gamma} L_i \neq \emptyset$; if τ is not essential then it is called *inessential*.

Definition 3(cf. [3]). For a space X we denote by L the set of all pairs (A, B) of disjoint closed subsets of X . Let us set $M = \{\sigma \in \text{Fin}L : \sigma \text{ is essential}\}$.

The number $\text{Ord}M$ is denoted by $\text{trdim}X$ and called the *transfinite covering dimension* of a space X .

Arenas defined transfinite extensions of the dimension of a ring R .

Definition 4([1]). For a ring R we denote by L the set of principal ideal aR . Let us set $M = \{\sigma \in \text{Fin}L : \sigma \text{ is not uniquely generated}\}$.

The number $\text{Ord}M$ is denoted by $\text{trdim}R$ and called the *transfinite dimension* of a ring R .

Arenas [1] proved the following theorem, which is a partial transfinite generalization of Canfell's theorem.

Theorem 2 (Arenas [1]). *For every space X we have $\text{trdim}X \leq \text{trdim}C(X)$.*

Arenas asked whether the equality in theorem 2 is true.

We gave a negative answer.

Theorem 3. *For every metric space X with $\text{trdim}X \geq \omega$ we have $\text{trdim}C(X) = \infty$.*

We redefine Arenas' definition as follows and prove the transfinite dimensions of the space X to be equal to the transfinite dimensions of the ring $C(X)$.

Definition 5. Let Γ be an index set. A collection $\sigma = \{(a_i, b_i) : i \in \Gamma\}$ of pairs of principal ideals of R is called *essential* if for every $\{u_i : i \in \Gamma\}$, where $u_i \in R$ and $a_i = b_i u_i$ for every $i \in \Gamma$, we have $\sum_{i \in \Gamma} u_i R \neq R$; if σ is not essential then it is called *inessential*.

For a ring R we denote by L the set of all pairs (a, b) of principal ideals of R with $aR = bR$. Let us set $M = \{\sigma \in \text{Fin}L : \sigma \text{ is essential}\}$.

The number $\text{Ord}M$ is denoted by $\text{trdim}R$ and called the *transfinite dimension* of a ring R .

Theorem 4 . *For every space X we have $\text{trdim}X = \text{trdim}C(X)$.*

References

- [1] F. G. Arenas, *Transfinite dimension and rings of continuous functions*, Questions and Answers in General Topology 13(1995), 157-167.
- [2] M. J. Canfell, *Uniqueness of generators of principal ideals in rings of continuous functions*, Proc. Amer. Math. Soc. 26(1970), 571-573.

- [3] R. Engelking, Theory of Dimensions, Finite and Infinite, Heldermann Verlag, 1995.

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