

Some geometric constants related with the modulus of convexity of a Banach space

高橋泰嗣

岡山県立大学 名誉教授

Yasuji Takahashi Okayama Prefect. Univ., Professor Emeritus
ym-takahashi@clear.ocn.ne.jp

加藤幹雄

九州工業大学 工学研究院

Mikio Kato

Kyushu Inst. Tech.
katom@mns.kyutech.ac.jp

We shall consider the constant $C_f(X)$ for a Banach space X , where $f(u, v)$ is a real valued continuous function which is non-decreasing in u and v in $[0, 2]$. Some geometric constants of X are unifyingly described by this constant $C_f(X)$ with a suitable f and some previous results are derived.

Let X be a real Banach space with $\dim X \geq 2$. The *modulus of convexity* of X is defined by

$$\delta_X(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in S_X, \|x-y\| = \epsilon \right\} \quad (0 \leq \epsilon \leq 2),$$

where S_X is the unit sphere of X . S_X may be replaced by the unit ball B_X . The function δ_X is continuous on $[0, 2)$, increasing on $[0, 2]$ and strictly increasing on $[\epsilon_0, 2]$, where $\epsilon_0 = \epsilon_0(X) = \sup\{\epsilon \in [0, 2] : \delta_X(\epsilon) = 0\}$ is the *coefficient of convexity* of X . The function $\delta_X(\epsilon)/\epsilon$ is also increasing on $(0, 2]$ (Figiel, 1976).

The *James constant* of X is defined by

$$J(X) = \sup \{ \min(\|x+y\|, \|x-y\|) : x, y \in S_X \}.$$

X is called *uniformly non-square* if $J(X) < 2$. It is well-known that X is uniformly non-square if and only if $\epsilon_0(X) < 2$. If $J(X) < 2$, we have

$$J(X) = 2(1 - \delta_X(J(X)))$$

(Casini [4]).

In this note we shall consider the following constant: Let $f(u, v)$ is a real valued continuous function satisfying $f(u_1, v_1) \leq f(u_2, v_2)$ for all $0 \leq u_1 \leq u_2 \leq 2$ and $0 \leq v_1 \leq v_2 \leq 2$. We define the constant $C_f(X)$ to be

$$C_f(X) = \sup \left\{ f(\|x - y\|, \|x + y\|) : x, y \in S_X \right\}. \quad (1)$$

One should note that

$$\begin{aligned} J(X) &= C_f(X) & \text{if } f(u, v) &= \min(u, v), \\ A_2(X) &= C_f(X) & \text{if } f(u, v) &= (u + v)/2, \\ T(X) &= C_f(X) & \text{if } f(u, v) &= \sqrt{uv}, \\ C'_{NJ}(X) &= C_f(X) & \text{if } f(u, v) &= (u^2 + v^2)/4. \end{aligned}$$

We recall the definitions of these constants. The constant $A_2(X)$ ([3]) is given by

$$A_2(X) := \rho_X(1) + 1,$$

where $\rho_X(\tau)$ is the modulus of smoothness of X ,

$$\rho_X(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S_X \right\} \quad (\tau > 0).$$

The constant $T(X)$ is defined in [1] by

$$T(X) := \sup \{ \sqrt{\|x - y\| \|x + y\|} : x, y \in S_X \}.$$

The *von Neumann-Jordan constant* of X is

$$C_{NJ}(X) := \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \text{ are not both } 0 \right\}, \quad (2)$$

where the supremum can be taken over all $x \in S_X$ and $y \in B_X$. The constant defined by taking supremum over all $x, y \in S_X$ in (2) is denoted by $C'_{NJ}(X)$ ([2]). We have $C'_{NJ}(X) \leq C_{NJ}(X)$ and they do not coincide in general.

It is readily seen that

$$C_f(X) = \sup \left\{ f(\varepsilon, 2(1 - \delta_X(\varepsilon))) : 0 < \varepsilon < 2 \right\}. \quad (3)$$

With regard to a lower bound of $C_f(X)$ we easily have

$$C_f(X) \geq \max \left\{ f(J(X), J(X)), f(\epsilon_0(X), 2) \right\}. \quad (4)$$

In particular we have $C_f(X) = f(2, 2)$ if $J(X) = 2$. It follows from (4) that $T(X) \geq \sqrt{2\epsilon_0(X)}$ ([1]) and $C'_{NJ}(X) \geq 1 + \epsilon_0(X)^2/4$ ([2]), where we have equality in both inequalities if X is not uniformly non-square.

Theorem 1. *Let $J(X) < 2$ and assume that $f(u, v) = f(v, u)$ for all $u, v \in [0, 2]$. Then*

$$C_f(X) = \sup \left\{ f(\epsilon, 2(1 - \delta_X(\epsilon))) : J(X) \leq \epsilon < 2 \right\}. \quad (5)$$

We shall present some applications of (5): Let $J(X) < 2$. Then

$$\rho_X(1) = \sup \left\{ \frac{\epsilon}{2} - \delta_X(\epsilon) : J(X) \leq \epsilon < 2 \right\} \leq 2 \left(1 - \frac{1}{J(X)} \right) \quad (6)$$

and

$$C'_{NJ}(X) = \sup \left\{ \frac{\epsilon^2}{4} + (1 - \delta_X(\epsilon))^2 : J(X) \leq \epsilon < 2 \right\} \leq 1 + 4 \left(1 - \frac{1}{J(X)} \right)^2. \quad (7)$$

We shall give simple proofs of (6) and (7). We write J and $\delta(\epsilon)$ for $J(X)$ and $\delta_X(\epsilon)$ respectively. Since $\delta(\epsilon)/\epsilon$ is increasing, $\delta(\epsilon) \geq \delta(J)\epsilon/J$ for all $J \leq \epsilon < 2$. Noting $2\delta(J) = 2 - J$ we have

$$\frac{\epsilon}{2} - \delta(\epsilon) \leq \frac{\epsilon}{2} - \delta(J)\epsilon/J \leq 1 - 2\delta(J)/J = 1 - (2 - J)/J = 2(1 - 1/J),$$

which proves (6). Similarly we have

$$\frac{\epsilon^2}{4} + (1 - \delta_X(\epsilon))^2 \leq \frac{\epsilon^2}{4} + (1 - \delta(J)\epsilon/J)^2 \leq 1 + (1 - 2\delta(J)/J)^2 = 1 + 4(1 - 1/J)^2,$$

which proves (7).

In 2008 Alonso et al. [2] showed that

$$C'_{NJ}(X) \leq J(X),$$

which is useful to estimate the von Neumann-Jordan constant $C_{NJ}(X)$ by $J(X)$.

It was shown in [2] that

$$C_{NJ}(X) \leq 1 + (\sqrt{2C'_{NJ}(X)} - 1)^2 \leq 1 + (\sqrt{2J(X)} - 1)^2,$$

while by using (7) we easily have

$$C'_{NJ}(X) \leq 1 + 4(1 - 1/J(X))^2 \leq (1 + \sqrt{J(X) - 1})^2/2,$$

which yields that

$$C_{NJ}(X) \leq 1 + (\sqrt{2C'_{NJ}(X)} - 1)^2 \leq J(X)$$

(Kato-Takahashi [6]; see also [8], [9]). The simple inequality

$$C_{NJ}(X) \leq J(X) \tag{8}$$

concerning the von Neumann-Jordan and James constants was first proved by Takahashi and Kato [7] in 2009, which answered affirmatively a question posed in Alonso et al. [2]. In [7] they proved (8) as

$$C_{NJ}(X) \leq \frac{2}{2 - \rho_X(1)} \leq J(X),$$

where the second inequality is equivalent to (6).

References

- [1] J. Alonso and E. Llorens-Fuster, Geometric mean and triangles inscribed in a semicircle in Banach spaces, *J. Math. Anal. Appl.* **340** (2008), 1271-1283.
- [2] J. Alonso, P. Martín and P. L. Papini, Wheeling around von Neumann-Jordan constant in Banach spaces, *Studia Math.* **188** (2008), 135-150.
- [3] M. Baronti, E. Casini and P. L. Papini, Triangles inscribed in a semicircle, in Minkowski planes, *J. Math. Anal. Appl.* **252** (2000), 124-146.
- [4] E. Casini, About some parameters of normed linear spaces, *Atti. Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **80** (1986), 11-15.
- [5] M. Kato, L. Maligranda and Y. Takahashi, On James, Jordan-von Neumann constants and the normal structure coefficients of Banach spaces, *Studia Math.* **144** (2001), 275-295.

- [6] M. Kato and Y. Takahashi, On sharp estimates concerning von Neumann-Jordan and James constants for a Banach space, *Rend. Circ. Mat. Palermo Serie II, Suppl.* **82** (2010), 75-91.
- [7] Y. Takahashi and M. Kato, A simple inequality for the von Neumann-Jordan and James constants of a Banach space, *J. Math. Anal. Appl.*, **359** (2009), 602-609.
- [8] F. Wang, On the James and von Neumann-Jordan constants in Banach spaces, *Proc. Amer. Math. Soc.* **138** (2010), 695-701.
- [9] C. Yang and H. Li, An inequality between Jordan-von Neumann constant and James constant, *Appl. Math. Letters* **23** (2010), 277-281.