

Morita equivalences between blocks of finite group algebras

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1. Introduction and notation

In representation theory of finite groups, particularly, in modular representation theory, studying structure of p -blocks (block algebras) of finite groups G , where p is a prime number, is one of the most important and interesting things.

Notation 1.1. Throughout this note we use the following notation and terminology. We denote by G always a finite group, and let p be a prime. Then, a triple $(\mathcal{K}, \mathcal{O}, k)$ is so-called a p -modular system, which is big enough for all finitely many finite groups which we are looking at, including G . Namely, $\mathcal{O}$ is a complete discrete valuation ring, $\mathcal{K}$ is the quotient field of $\mathcal{O}$, $\mathcal{K}$ and $\mathcal{O}$ have characteristic zero, and k is the residue field $\mathcal{O}/\text{rad}(\mathcal{O})$ of $\mathcal{O}$ such that k has characteristic p . We mean by "big enough" above that $\mathcal{K}$ and k are both splitting fields for the finite groups mentioned above. Let A be a block of $\mathcal{O}G$ (and sometimes of kG) with a defect group P . We denote by $\text{mod-}kG$ and by $\text{mod-}A$ the categories of finitely generated right kG - and A -modules, respectively. We write $B_0(kG)$ for the principal block algebra of kG . For the notation and terminology we shall not explain precisely, see the books of [2] and [3].

Setup 1.2. Throughout this note all the time except in Theorem 2.1 our situation is the following: Namely, G and H are finite groups which have the same Sylow p -subgroup P , and hence $P \subseteq G \cap H$. Assume that $\tilde{G}$ is a normal subgroup of G and $\tilde{H}$ is a normal subgroup of H such that $\tilde{G}$ and $\tilde{H}$ have the same Sylow p -subgroup $\tilde{P}$, and hence $\tilde{P} \subseteq \tilde{G} \cap \tilde{H}$, and moreover that $G/\tilde{G} \cong H/\tilde{H}$.

Remark 1.3. If the factor group $G/\tilde{G}$ is p' -groups, then we know essentially by the famous result due to H.Maschke (1898) that the ring extension $k\tilde{G} \subseteq kG$ is a so-called *separable* extension. Then, roughly speaking, $\text{mod-}kG$ and $\text{mod-}k\tilde{G}$ are in some sense *similar* (of course, even the numbers of simples in the two module categories are different, though). Therefore, much more interesting situation should be the case where $|G/\tilde{G}|$ is divisible by p . Then, here comes our situation.

Our situation 1.4. We still keep the setup 1.2. In addition we assume that the factor groups $G/\tilde{G} \cong H/\tilde{H}$ are p -groups. Surely, the factor groups are isomorphic to $P/\tilde{P}$, too. Then, we naturally come to the following questions.

Questions 1.5. Our main concern in this note is the following:

- (i) If there is a *nice* equivalence between $\text{mod-}k\tilde{G}$ and $\text{mod-}k\tilde{H}$, can we lift it to a *nice* equivalence between $\text{mod-}kG$ and $\text{mod-}kH$?
- (ii) If there is a *nice* equivalence between $\text{mod-}kG$ and $\text{mod-}kH$, can we descend it to a *nice* equivalence between $\text{mod-}k\tilde{G}$ and $\text{mod-}k\tilde{H}$?

2. Results

In this short section we shall list two results which come up from Question 1.5.

Theorem 2.1. *Assume 1.4, however, note that we do not assume that P and $\tilde{P}$ are Sylow p -subgroups of G and $\tilde{G}$, respectively. Namely, P is just a p -subgroup of G and also of H , and $\tilde{P}$ is just a p -subgroup of $\tilde{G}$ and also of $\tilde{H}$. We assume then that P is a defect group of A and B , and $\tilde{P}$ is a defect group of $\tilde{A}$ and $\tilde{B}$. Moreover, we suppose*

that the factor groups $Q := G/\tilde{G} \cong H/\tilde{H} \cong P/\tilde{P}$ are just cyclic group C_p of order p , and that $A, \tilde{A}, B, \tilde{B}$ respectively are block algebras of $kG, k\tilde{G}, kH, k\tilde{H}$, such that A covers $\tilde{A}$ and B covers $\tilde{B}$. Set $\Delta Q := \{(u, u) \in Q \times Q | u \in Q\}$. We assume, in addition, that $\tilde{A}$ and $\tilde{B}$ are both ΔQ -invariant, that is, they are stable under conjugation action by all elements in Q . Set furthermore that $\Delta := (\tilde{G} \times \tilde{H})\Delta Q = (\tilde{G} \times \tilde{H})\Delta P = (\tilde{G} \times \tilde{H})\Delta G = (\tilde{G} \times \tilde{H})\Delta H$. Then, we get the following: Suppose that there is a bounded complex $\tilde{M}^\bullet \in C^b(\mathcal{O}\tilde{A}\text{-mod-}\mathcal{O}\tilde{B})$ of finitely generated $(\mathcal{O}\tilde{A}, \mathcal{O}\tilde{B})$ -bimodules such that

- (1) $\tilde{M}^\bullet \otimes_{\mathcal{O}} \mathcal{K}$ induces an isometry $\tilde{I}$ from $\mathbb{Z}\text{Irr}(\tilde{A})$ to $\mathbb{Z}\text{Irr}(\tilde{B})$,
- (2) $\tilde{M}^\bullet$ is perfect (exact), that is, all terms in the complex $\tilde{M}^\bullet$ are projective as left $\mathcal{O}\tilde{G}$ -modules and also as right $\mathcal{O}\tilde{H}$ -modules (and hence the isometry $\tilde{I}$ above is perfect),
- (3) the complex $\tilde{M}^\bullet$ extends from $\tilde{G} \times \tilde{H}$ to Δ .

Then, we can define a bounded complex $M^\bullet := \tilde{M}^\bullet_{\tilde{G} \times \tilde{H} \rightarrow \Delta} \uparrow^{G \times H} \in C^b(\mathcal{O}A\text{-mod-}\mathcal{O}B)$, and the new complex $M^\bullet$ induces a perfect isometry from $\mathbb{Z}\text{Irr}(A)$ to $\mathbb{Z}\text{Irr}(B)$. where $M^\bullet := \tilde{M}^\bullet_{\tilde{G} \times \tilde{H} \rightarrow \Delta} \uparrow^{G \times H}$ is an induced complex by applying the functor $- \otimes_{\mathcal{O}\Delta} \mathcal{O}[G \times H]$ to the bounded complex $\tilde{M}^\bullet$.

Corollary 2.2. *We easily get [1, Example 4.3] in our previous paper by making use of Theorem 2.1.*

Theorem 2.2. *Assume 1.4. Here we assume that P is a Sylow p -subgroup of G and H , and also $\tilde{P}$ is a Sylow p -subgroup of $\tilde{G}$ and $\tilde{H}$. Moreover, we suppose that the factor groups $Q := G/\tilde{G} \cong H/\tilde{H} \cong P/\tilde{P}$ are isomorphic finite p -groups, and that $A, \tilde{A}, B, \tilde{B}$ respectively are principal block algebras of $kG, k\tilde{G}, kH, k\tilde{H}$. Set $\Delta P := \{(u, u) \in P \times P | u \in P\}$. Moreover, we denote by $\text{Scott}(G \times H, \Delta P)$ the (Alperin-)Scott module in $G \times H$ with respect to a subgroup ΔP of $G \times H$, see [2, Chap.4 Theorem 8.4, Corollary 8.5]. Then, we get the following: If $A M_B := \text{Scott}(G \times H, \Delta P)$ induces a Morita equivalence (and hence it is a Puig equivalence) between A and B , then ${}_{\tilde{A}}\tilde{M}_{\tilde{B}} := \text{Scott}(\tilde{G} \times$*

$\tilde{H}, \Delta\tilde{P})$ induces a Morita equivalence (and hence it is a Puig equivalence) between $\tilde{A}$ and $\tilde{B}$. (Recall that $A := B_0(kG) = \text{Scott}(G \times G, \Delta\tilde{P})$ and $B := B_0(kH) = \text{Scott}(H \times H, \Delta\tilde{P})$).

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