

On the b-Function of Nonisolated Hypersurface Singularities

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Let f be a germ of holomorphic function of n variables, and $b_f(s)$ the b-function (i.e. Bernstein polynomial) of f . It is the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$(0.1) \quad b(s)f^s = Pf^{s+1} \quad \text{in } \mathcal{O}_X[f^{-1}][s]f^s$$

for $P \in \mathcal{D}_X[s]$, where $\mathcal{D}_X$ denotes the germs of holomorphic differential operators on $X := (\mathbb{C}^n, 0)$, and $\mathcal{D}_X[s] = \mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[s]$. Substituting $s = -1$, we can check easily that $b_f(s)$ is divisible by $s + 1$. Let $\tilde{b}_f(s) = b_f(s)/(s+1)$, R_f the roots of $\tilde{b}_f(-s)$, $\alpha_f = \min R_f$, and $m_\alpha(f)$ the multiplicity of a root α of $\tilde{b}_f(-s)$. By Kashiwara [7], we have

(0.2) **Theorem.** $\alpha_f > 0$, and $R_f \subset \mathbb{Q}$.

Assume f has isolated singularity and $n > 1$. Let $H_f'' = \Omega_X^n / df \wedge d\Omega_X^{n-2}$, following Brieskorn [2]. Then H_f'' is a free $\mathbb{C}\{\{t\}\}$ -module of rank μ (the Milnor number of f), and has a regular singular meromorphic connection. Let $\tilde{H}_f'' = \sum_{i \geq 0} (t\partial_t)^i H_f'' \subset H_f''[t^{-1}]$ (the saturation of H_f''). By Malgrange [13], we have

(0.3) **Theorem.** $\tilde{b}_f(s)$ is the minimal polynomial of the action of $-\partial_t$ on $\tilde{H}_f'' / t\tilde{H}_f''$.

Combined with a result of Varchenko [29] (and [26]), this implies (see also [17]):

(0.4) **Theorem.** $R_f \subset [\alpha_f, n - \alpha_f]$.

(0.5) **Theorem.** $m_\alpha(f) \leq n - \alpha_f - \alpha + 1$ ($\leq n - 2\alpha_f + 1$).

In the isolated singularity case, we proved also (see [16]):

(0.6) **Proposition.** $Y = f^{-1}(0)$ has rational singularity if and only if $\alpha_f > 1$.

Using the theory of mixed Hodge Modules [18] [19] [20], we extend these to the nonisolated singularity case (see [23] [24]), i.e.

(0.7) **Theorem.** (0.4–6) are valid also in the nonisolated singularity case, where we assume Y reduced in (0.6).

Note that (0.5) is an improvement of $m_\alpha(f) \leq n - \delta_{\alpha,1}$ (where $\delta_{\alpha,1}$ is Kronecker's delta) which is shown in [14] as a corollary of the relation with Deligne's vanishing cycle sheaf

$\varphi_f \mathbb{C}_X$ [3]. See also [8]. This relation implies for example that $\exp(2\pi i \alpha)$ for $\alpha \in R_f$ are the eigenvalues of the monodromy on $\varphi_f \mathbb{C}_X$. But $\varphi_f \mathbb{C}_X$ cannot be replaced with the reduced cohomology of a Milnor fiber at the origin as in the isolated singularity case, because we have to take the Milnor fibration at several points of $\text{Sing } f^{-1}(0)$ even when we consider the b-function of f at the origin.

For the proof of the generalization of (0.4-5), we introduce the notion of microlocal b-function (1.1), and show an assertion (1.2) which may be viewed as a generalization of (0.3). Using this, we can prove the Thom-Sebastiani type theorem for b-function in some case (2.8). In the nondegenerate Newton boundary case [12], we get an estimate of α_f by the Newton polyhedron (2.7). Note that the b-function is also related with the spectrum [27] of f , and with a result of Deligne-Dimca [5]. See [23].

§1. Microlocal b-Function

(1.1) Let $\delta(t-f)$ denote the delta function on $X' := X \times (\mathbb{C}, 0)$ with support $\{f=t\}$, where t is the coordinate of $\mathbb{C}$. Then, setting $s = -\partial_t$, f^s and $\delta(t-f)$ satisfy the same relation (see for example [13]). So f^s in (0.1) can be replaced by $\delta(t-f)$, and f^{s+1} by $t\delta(t-f)$. We define the *microlocal b-function* $\tilde{b}_f(s)$ by the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$(1.1.1) \quad b(s)\delta(t-f) = P\partial_t^{-1}\delta(t-f) \quad \text{in } \mathcal{O}_X[\partial_t, \partial_t^{-1}]\delta(t-f)$$

for $P \in \mathcal{D}_X[\partial_t^{-1}, s]$. Here we can also allow for P a microdifferential operator [7] [9] [10] [25] satisfying a condition on the degree of t and ∂_t (see [24, (1.4)]).

We can show (see [24, (1.5)]) :

(1.2) **Proposition.** $b_f(s) = (s+1)\tilde{b}_f(s)$.

(1.3) Let $R_X = \mathcal{D}_X[t, \partial_t]$, $\tilde{R}_X = \mathcal{D}_X[t, \partial_t, \partial_t^{-1}]$, and

$$(1.3.1) \quad \underline{B}_f = \mathcal{O}_X[\partial_t]\delta(t-f), \quad \tilde{\underline{B}}_f = \mathcal{O}_X[\partial_t, \partial_t^{-1}]\delta(t-f),$$

where $\mathcal{O}_X[\partial_t]\delta(t-f)$ is a free module of rank one over $\mathcal{O}_X[\partial_t]$ with a basis $\delta(t-f)$ (similarly for $\tilde{\underline{B}}_f$). Then $\underline{B}_f$, $\tilde{\underline{B}}_f$ have naturally a structure of R_X -module and $\tilde{R}_X$ -module respectively.

Let V be the filtration on R_X , $\tilde{R}_X$ by the differences of the degrees of t and ∂_t i.e.,

$$(1.3.2) \quad V^p R_X = \sum_{i-j \geq p} \mathcal{D}_X t^i \partial_t^j \quad (\text{same for } \tilde{R}_X).$$

We define a decreasing filtration G on $\underline{B}_f$, $\tilde{\underline{B}}_f$ by

$$(1.3.3) \quad G^p \underline{B}_f = V^p R_X \delta(t-f), \quad G^p \tilde{\underline{B}}_f = V^p \tilde{R}_X \delta(t-f),$$

and an increasing filtration F by

$$(1.3.4) \quad F_p \underline{B}_f = \bigoplus_{0 \leq i \leq p} \underline{O}_X \partial_t^i \delta(t-f), \quad F_p \tilde{\underline{B}}_f = \bigoplus_{i \leq p} \underline{O}_X \partial_t^i \delta(t-f).$$

Then we have

$$(1.3.5) \quad \partial_t^i : G^p \underline{B}_f \xrightarrow{\sim} G^{p-i} \tilde{\underline{B}}_f, \quad \partial_t^i : F_p \tilde{\underline{B}}_f \xrightarrow{\sim} F_{p+i} \tilde{\underline{B}}_f$$

$$(1.3.6) \quad \underline{D}_X[s](F_p \tilde{\underline{B}}_f) \subset G^{-p} \tilde{\underline{B}}_f.$$

(1.4) *Remark.* $b_f(s)$ and $\tilde{b}_f(s)$ are the minimal polynomial of the action of $s := -\partial_t$ on $\text{Gr}_G^0 \underline{B}_f$ and $\text{Gr}_G^0 \tilde{\underline{B}}_f$ respectively, because s belongs to the center of $\text{Gr}_G^0 R_X = \text{Gr}_G^0 \tilde{R}_X = \underline{D}_X[s]$.

§2. Filtration V

(2.1) Let V denote the filtration of Kashiwara [8] and Malgrange [14] on $\underline{B}_f$ indexed by $\mathbb{Q}$. Here we index V decreasingly so that the action of $\partial_t - \alpha$ on $\text{Gr}_V^\alpha \underline{B}_f$ is nilpotent, where $\text{Gr}_V^\alpha = V^\alpha / V^{>\alpha}$ with $V^{>\alpha} = \bigcup_{\beta > \alpha} V^\beta$. By [7] (see also (0.2) above), we have

$$(2.2.1) \quad F_0 \underline{B}_f \subset V^{>0} \underline{B}_f.$$

We can show (see [24, (2.2) and (2.4)]) :

(2.2) **Lemma.** We have a decreasing filtration V on $\tilde{\underline{B}}_f$ such that

$$(2.2.1) \quad V^\alpha \tilde{\underline{B}}_f = V^\alpha \underline{B}_f + \underline{O}_X [\partial_t^{-1}] \partial_t^{-1} \delta(t-f) \quad \text{for } \alpha \leq 1,$$

$$(2.2.2) \quad \partial_t^j : V^\alpha \tilde{\underline{B}}_f \xrightarrow{\sim} V^{\alpha-j} \tilde{\underline{B}}_f \quad \text{for any } j, \alpha.$$

(2.3) **Proposition.** We have

$$(2.3.1) \quad \text{Gr}_V^\alpha \tilde{\underline{B}}_f = \underline{D}_X(F_p \text{Gr}_V^\alpha \tilde{\underline{B}}_f) \quad \text{if } F_{-p-1} \text{Gr}_V^{n-\alpha} \tilde{\underline{B}}_f = 0.$$

(2.4) *Proof of (0.4) in the general case.* We have $G^1 \text{Gr}_V^\alpha \tilde{\underline{B}}_f \supset \underline{D}_X(F_{-1} \text{Gr}_V^\alpha \tilde{\underline{B}}_f)$ by (1.3.6). So it is enough to show $\text{Gr}_V^\alpha \tilde{\underline{B}}_f = \underline{D}_X(F_{-1} \text{Gr}_V^\alpha \tilde{\underline{B}}_f)$ for $\alpha > n - \alpha_f$ by (1.4), because it implies $\text{Gr}_G^0 \text{Gr}_V^\alpha \tilde{\underline{B}}_f = \text{Gr}_V^\alpha \text{Gr}_G^0 \tilde{\underline{B}}_f = 0$. By definition of α_f , we have

$$(2.4.1) \quad F_0 \text{Gr}_V^\alpha \tilde{\underline{B}}_f = G^0 \text{Gr}_V^\alpha \tilde{\underline{B}}_f = 0 \quad \text{for } \alpha < \alpha_f$$

using (1.3.6). So the assertion follows from (2.3) applied to $p = -1$.

By a similar argument, we prove (0.5) using also the monodromy filtration W . Here W is uniquely characterized by the properties (see [4]) :

$$(2.4.1) \quad N W_i \subset W_{i-2}, \quad N^j : \text{Gr}_f^W \xrightarrow{\sim} \text{Gr}_{-j}^W \quad (j > 0),$$

where $N = s + \alpha$ on $\text{Gr}_V^\alpha \tilde{\underline{B}}_f$. See [24, (2.8)] for the details. We can show also the following:

(2.5) *Remark.* Let $\varphi_f \mathbb{C}_X$ be Deligne's vanishing cycle sheaf [3], and T_u, T_s denote respectively the unipotent and semisimple part of the monodromy T on $\varphi_f \mathbb{C}_X$. Let $\varphi_f^\alpha \mathbb{C}_X = \text{Ker}(T_s - \exp(-2\pi i \alpha))$ (as a shifted perverse sheaf [1]), and $N = \log T_u / 2\pi i$. Then we have $N^{r+1} = 0$ on $\varphi_f^\alpha \mathbb{C}_X$ for $\alpha \in [\alpha_f, \alpha_f + 1)$ and $r = [n - \alpha_f - \alpha]$. In particular, $N^{r+1} = 0$ on $\varphi_f \mathbb{C}_X$ for $r = [n - 2\alpha_f]$. See [24, (0.6)].

(2.6) *Remark.* If $\text{Sing } f^{-1}(0)$ is isolated and f is a quasi-homogeneous polynomial of weight $(w_1, \dots, w_n)$ (i.e. f is a linear combination of monomials $x_1^{m_1} \dots x_n^{m_n}$ such that $m_1 w_1 + \dots + m_n w_n = 1$), then it is well-known that $m_\alpha(f) = 1$ for $\alpha \in R_f$, and α belongs to R_f if and only if the coefficient of t^α in

$$(2.6.1) \quad \prod_i (t^{w_i} - t) / (1 - t^{w_i})$$

is nonzero. This follows for example from Steenbrink [28] (using [13] [29]) and also from Brieskorn or Kashiwara (unpublished). In particular, we have $\max R_f = n - \alpha_f$ in this case.

(2.7) *Remark.* If f has nondegenerate Newton boundary, we can show $\alpha_f \geq 1/t$ for $(t, \dots, t) \in \partial \Gamma_+(f)$ (see [24, (3.3)]), where $\Gamma_+(f)$ is the Newton polygon of f . In the isolated singularity case, it is known that the equality holds. (See also [22].)

(2.8) *Remark.* Let g be a holomorphic function on a germ of complex manifold Y . Let $Z = X \times Y$, and $h = f + g \in \underline{O}_Z$. We define R_g, R_h as in the introduction. Then $R_f R_g \subset R_h + \mathbb{Z}_{\geq 0}$, $R_h \subset R_f R_g + \mathbb{Z}_{\leq 0}$. Furthermore, if there is a holomorphic vector field ξ such that $\xi g = g$, then $R_f R_g = R_h$, and $m_Y(h) = \max_{\alpha+\beta=\gamma} \{m_\alpha(f) + m_\beta(g) - 1\}$. See [24, (4.3-4)]. The last assertion is proved in [30] if f and g have isolated singularities.

§3. Rational Singularity

(3.1) Let Y be a reduced complex analytic space. We say that Y has *rational singularity*, if the natural morphism

$$(3.1.1) \quad \underline{O}_Y \rightarrow R\pi_* \underline{O}_{Y'}$$

is an isomorphism for a resolution of singularity $\pi : Y' \rightarrow Y$. If Y is Cohen-Macaulay and pure dimensional, it is equivalent to the bijectivity of the trace morphism

$$(3.1.2) \quad \pi_* \omega_{Y'} \rightarrow \omega_Y$$

by duality [15], because $R^i \pi_* \omega_{Y'} = 0$ for $i > 0$ by [6] (this follows also from [11] [21]) where π is assumed projective. Here ω_Y denotes the dualizing sheaf (i.e., the dualizing complex [15] shifted by the dimension to the right). The trace morphism (3.1.2) is injective, and its image is independent of the choice of resolution, because (3.1.2) is an isomorphism if

Y is smooth. We will denote by $\tilde{\omega}_Y$ the image of (3.1.2).

(3.2) Assume Y is a reduced divisor D on the germ of complex manifold X in the introduction. Let f be a reduced defining equation of D .

Using the coordinate system $(x_1, \dots, x_n)$ of X , we have the involution of $\underline{D}_X$ such that $(PQ)^* = Q^*P^*$, $(x_i)^* = x_i$, $(\partial/\partial x_i)^* = -\partial/\partial x_i$. So the right $\underline{D}$ -module ω_X is identified with the left $\underline{D}$ -module $\underline{Q}_X$ using the basis $dx = dx_1 \wedge \dots \wedge dx_n$ of ω_X , and we get isomorphisms

$$(3.2.2) \quad \underline{B}_f = \omega_X[\partial_t]\delta(t-f), \quad \tilde{\underline{B}}_f = \omega_X[\partial_t, \partial_t^{-1}]\delta(t-f).$$

We can show (see [23]) :

(3.3) Theorem. We have a commutative diagram

$$(3.3.1) \quad \begin{array}{ccccccc} 0 & \rightarrow & \tilde{\omega}_D & \rightarrow & \omega_D & \rightarrow & \omega_D/\tilde{\omega}_D \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & F_0 W_0 \text{Gr}_V^1 \underline{B}_f & \rightarrow & F_0(\underline{B}_f/V^{>1} \underline{B}_f) & \rightarrow & F_0(\tilde{\underline{B}}_f/V^{>1} \tilde{\underline{B}}_f) \rightarrow 0, \end{array}$$

such that the vertical morphisms are isomorphisms.

(3.4) Remark. The horizontal short exact sequences correspond to the short exact sequence of mixed Hodge modules [19] :

$$(3.4.1) \quad 0 \rightarrow \mathbb{Q}_D^H[n-1] \rightarrow \psi_f \mathbb{Q}_X^H[n] \rightarrow \varphi_f \mathbb{Q}_X^H[n] \rightarrow 0.$$

In fact, taking Gr_V of (3.3.1), we get F_{1-n} of the underlying filtered $\underline{D}$ -module of (3.4.1) (using (2.2.1)), because the underlying filtered $\underline{D}$ -modules $\psi_f \omega_X, \varphi_f \omega_X$ of $\psi_f \mathbb{Q}_X^H[n], \varphi_f \mathbb{Q}_X^H[n]$ are defined by

$$(3.4.2) \quad \psi_f \omega_X = \bigoplus_{0 < \alpha \leq 1} \text{Gr}_V^\alpha \underline{B}_f, \quad \varphi_f \omega_X = \bigoplus_{0 < \alpha \leq 1} \text{Gr}_V^\alpha \tilde{\underline{B}}_f.$$

Here we have a shift of the filtration F coming from the transformation of left and right filtered $\underline{D}$ -modules (see [23]). Furthermore, $\tilde{\omega}_D$ is F_{1-n} of the underlying filtered $\underline{D}$ -module of the intersection complex $\text{IC}_D^{\mathbb{Q}^H}$ which is a quotient of $\mathbb{Q}_D^H[n-1]$.

As a corollary of (3.3), we get (0.6) and the following

(3.5) Corollary. We have a canonical isomorphism

$$(3.5.1) \quad F_{1-n}(\varphi_f \omega_X) = \bigoplus_{0 < \alpha \leq 1} \text{Gr}_V^\alpha(\omega_D/\tilde{\omega}_D),$$

such that $\text{Gr}_V^\alpha(\omega_D/\tilde{\omega}_D)$ corresponds to the $\exp(-2\pi i \alpha)$ -eigenspace of $\varphi_f \omega_X$ with respect to the action of monodromy.

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