

Generic quotient varieties

AKIHIKO GYOJA

Introduction. The purpose of this note is to review [R1, Theorem 2].

Theorem 0.1. *Let k be an algebraically closed field, X an irreducible algebraic variety over k , and G an algebraic group acting on X . Then there exists an open dense $X_0 \subset X$, a variety W_0 , and a morphism $\phi : X_0 \rightarrow W_0$ such that*

- (1) $GX_0 = X_0$,
- (2) every fibre of ϕ is precisely a single G -orbit,
- (3) $\phi : X_0 \rightarrow W_0$ is smooth,
- (4) X_0 and W_0 are non-singular,
- (5) $\phi^* : k(W_0) \xrightarrow{\cong} k(X_0)^G (= k(X)^G)$, and
- (6) $\phi^* : k[W_0] \xrightarrow{\cong} k[X_0]^G$.

(Cf. (0.4) for notation.) In particular, $\phi : X_0 \rightarrow W_0$ is a geometric quotient in the sense of D. Mumford [Mu, p.4].

This theorem can be used in the theory of prehomogeneous vector spaces as follows. Put

$$m := \max\{\dim Gr \mid x \in X\}.$$

Then this maximum is attained by x 's belonging to a dense subset of X . From the above theorem, we get the following results concerning m .

Corollary 0.2. *Let notation be as in (0.1). Then $\dim X - m = \dim W_0 = \text{tr. deg}_k k(X)^G$.*

Corollary 0.3. *Let notation be as in (0.1). The following conditions are equivalent.*

- (1) X has an open dense G -orbit.
- (2) $\text{tr. deg}_k k(X)^G = 0$.
- (3) $k(X)^G = k$.

0.4. Convention and Notation. We fix an algebraically closed field k , and we always assume that an algebraic variety is defined over k unless otherwise stated. We identify a (k) -variety, say X , with the set of its rational points $X(k)$. We denote by $k[X]$ (resp. $k(X)$) the ring of regular functions (resp. the field of rational functions if X is irreducible) on X . For a group Γ acting on a set A , $A^\Gamma := \{a \in A \mid \gamma a = a \text{ for all } \gamma \in \Gamma\}$.

§1.

1.1. Flatness. The concept of ‘flatness’ plays an important role in the algebraic geometry [EGA]. A concise account can be found in [Mi, Chapter 1]. We recall two lemmas from [EGA].

Lemma 1.2. [EGA, (IV, 6.9.1)]. *Let Y be a locally noetherian (I, 2.7.1), integral scheme (I, 2.1.8), and $u : X \rightarrow Y$ a morphism of finite type (I, 6.3.2). Then there exists an open dense $U \subset Y$ such that $u : u^{-1}(U) \rightarrow U$ is flat.*

Lemma 1.3. [EGA, (IV, 2.4.6)]. *Let $f : X \rightarrow Y$ be a flat morphism of locally of finite presentation (I, 6.2.1). Then f is universally open (IV, 2.4.2).*

1.4. Hilbert scheme. [Mu, pp.21–22]. It is known that there exist

- (1) a locally noetherian $\mathbb{Z}$ -scheme $\text{Hilb}_{\mathbb{P}_{\mathbb{Z}}^n}$ whose connected components are projective over $\text{Spec}(\mathbb{Z})$, and
- (2) a closed $\mathbb{Z}$ -subscheme $W_{\mathbb{Z}} \subset \mathbb{P}_{\mathbb{Z}}^n \times \text{Hilb}_{\mathbb{P}_{\mathbb{Z}}^n}$ flat over $\text{Hilb}_{\mathbb{P}_{\mathbb{Z}}^n}$.

such that for

(3) any locally noetherian $\mathbb{Z}$ -scheme $S_{\mathbb{Z}}$, and

(4) any closed $\mathbb{Z}$ -scheme $Z_{\mathbb{Z}} \subset \mathbb{P}_{\mathbb{Z}}^n \times S_{\mathbb{Z}}$, flat over $S_{\mathbb{Z}}$,

there is a unique morphism $f_{\mathbb{Z}} : S_{\mathbb{Z}} \rightarrow \text{Hilb}_{\mathbb{P}_{\mathbb{Z}}^n}$ such that $Z_{\mathbb{Z}} = (1_{\mathbb{P}_{\mathbb{Z}}^n} \times f_{\mathbb{Z}})^*(W_{\mathbb{Z}})$.

1.5. If we put $\text{Hilb}_{\mathbb{P}_k^n} := \text{Hilb}_{\mathbb{P}_{\mathbb{Z}}^n} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(k)$ and $W_k := W_{\mathbb{Z}} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(k)$, we may replace $\mathbb{Z}$ with k everywhere in (1.3).

§2.

We start this section with the following simple lemma.

Lemma 2.1. *Let $f : X \rightarrow Y$ be a morphism between algebraic varieties, and $Z \subset X$ a constructible subset (i.e., a finite disjoint union of locally closed subsets with respect to the Zariski topology.) We further assume that Z is irreducible and $\overline{f(Z)} = Y$. Then*

$$(1) \quad Y_0 := \{y \in Y \mid \overline{f^{-1}(y) \cap Z} = f^{-1}(y) \cap \overline{Z}\}$$

contains an open dense subset of Y . (Let $Y^{\sharp}$ denote the largest open subset of Y contained in Y_0 .)

Proof. Let Z_0 be the largest subset of Z which is open in $\overline{Z}$. Since Z is constructible, Z_0 is open dense in $\overline{Z}$. Let $\{Z_i\}_{i \in I}$ (resp. $\{Z'_j\}_{j \in J}$) be all the irreducible components of $\overline{Z} \setminus Z_0$ such that $\overline{f(Z_i)} = Y$ (resp. $\overline{f(Z_i)} \subsetneq Y$). Let Y_1 be the totality of $y \in Y$ such that

$$(2) \quad \dim(f^{-1}(y) \cap Z_i) = \dim Z_i - \dim Y \quad \text{for all } i \in I \sqcup \{0\}, \text{ and}$$

$$(3) \quad f^{-1}(y) \cap Z'_j = \emptyset \quad \text{for all } j \in J.$$

Then Y_1 is a constructible dense subset of Y . (The condition (2) gives an open subset, while (3) gives a constructible one in general.) For $y \in Y_1$, we have

$$(4) \quad f^{-1}(y) \cap Z \subset \overline{f^{-1}(y) \cap Z} \subset f^{-1}(y) \cap \overline{Z} = \bigcup_{i \in I \sqcup \{0\}} (f^{-1}(y) \cap Z_i).$$

by (3). Especially,

$$\begin{aligned} \dim(f^{-1}(y) \cap \overline{Z}) &= \max_{i \in I \sqcup \{0\}} \dim(f^{-1}(y) \cap Z_i) \\ &= \max_{i \in I \sqcup \{0\}} (\dim Z_i - \dim Y) \text{ by (2)} \\ &= \dim Z_0 - \dim Y = \dim \overline{Z} - \dim Y. \end{aligned}$$

(Indeed, $\dim Z_i < \dim Z_0$ for all $i \in I$.) In other words, the fibres of $f : \overline{Z} \rightarrow Y$ attain the minimum dimension at $y \in Y_1$. Hence all the irreducible components of $f^{-1}(y) \cap \overline{Z}$ are of the same dimension $\dim \overline{Z} - \dim Y$. (See [EGA, (IV, 13.2)] for the related generality.) Since

$$\dim(f^{-1}(y) \cap Z_i) = \dim Z_i - \dim Y < \dim \overline{Z} - \dim Y$$

for $i \in I$, $f^{-1}(y) \cap Z_i$ ($i \in I$) are nowhere dense in $f^{-1}(y) \cap \overline{Z}$, and consequently (1) yields that $f^{-1}(y) \cap Z_0$ ($\subset f^{-1}(y) \cap Z$) is dense in $f^{-1}(y) \cap \overline{Z}$. Hence $\overline{f^{-1}(y) \cap Z} = f^{-1}(y) \cap \overline{Z}$, i.e., $Y_1 \subset Y_0$. Since Y_1 is constructible and dense in Y , we get the desired result. ■

2.2. Let k, G, X be as in the introduction, and $\overline{X}$ an irreducible projective variety containing X as an open dense subset. (Such $\overline{X}$ exists [N], but possibly the G -action on X can not be extended to $\overline{X}$. See [S] for equivariant completions.) Let Z be the Zariski closure of $\{(gx, x) \mid x \in X, g \in G\}$ in $\overline{X} \times X$, and $\pi : Z \rightarrow X$ the second projection. Intuitively, $\pi : \pi^{-1}(X^\sharp) \rightarrow X^\sharp$ is the family of orbit closures $\overline{Gx}$ in $\overline{X}$

parametrized by $x \in X^\sharp$, where $X^\sharp$ is defined as in (2.1) using Z and $\pi : \overline{X} \times X \rightarrow X$ in place of Z and $f : X \rightarrow Y$. Let X_0 be the largest open (dense) subset of $X^\sharp$ such that $\pi : \pi^{-1}(X_0) \rightarrow X_0$ is flat (cf. (1.2)). Then applying (1.5) to $S = X_0$ and $Z = \pi^{-1}(X_0) (\subset \overline{X} \times X_0 \subset \mathbb{P}_k^n \times X_0$ for some n), we get a morphism $f : X_0 \rightarrow \text{Hilb}_{\mathbb{P}_k^n}$ which makes the following diagram cartesian.

$$\begin{array}{ccc} \pi^{-1}(X_0) & \longrightarrow & W \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow{f} & \text{Hilb}_{\mathbb{P}_k^n} \end{array}$$

Let Y_1 be the largest open subset of $\overline{f(X_0)}$ such that $f : f^{-1}(Y_1) \rightarrow Y_1$ is flat (cf. (1.2)) and surjective. Put $X_1 = f^{-1}(Y_1)$.

Lemma 2.3. (1) The open dense subset $X_1 (\subset X)$ is preserved by G .

(2) The fibres of $f : X_1 \rightarrow Y_1$ are precisely the G -orbit in X_1 .

(3) $f : X_1 \rightarrow Y_1$ is universally open.

Proof. (1) is obvious. (3) follows from (1.3). For $x \in X_1$, let $i_x : \text{Spec}(k) \rightarrow X_1$ be the corresponding geometric point. Then we get cartesian squares

$$\begin{array}{ccccc} \overline{Gx} & \longrightarrow & \pi^{-1}(X_1) & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(k) & \xrightarrow{i_x} & X_1 & \xrightarrow{f} & \text{Hilb}_{\mathbb{P}_k^n} \end{array}$$

Hence for $x, x' \in X_1$,

$$Gx = Gx' \Leftrightarrow \overline{Gx} = \overline{Gx'} \Leftrightarrow f \circ i_x = f \circ i_{x'} \Leftrightarrow f(x) = f(x').$$

(To see the first equivalence, note that Gx is the unique G -orbit which is open in $\overline{Gx} \cap X$. The second equivalence follows from the uniqueness part in (1.4).) ■

§3.

We need some preliminary from the field theory.

Lemma 3.1. *If L/K is a finitely generated field extension, and M a field such that $K \subset M \subset L$, then M/K is also finitely generated.*

Proof. Let K' be a purely transcendental extension of K , contained in M , and such that the transcendental degree $\text{tr. deg}_K K' (< \text{tr. deg}_K L < +\infty)$ is maximal among such extensions. Replacing K' with K , we may assume that M/K is an algebraic extension.

Let N be a purely transcendental extension of K , contained in L , and such that $\text{tr. deg}_K N$ is maximal among such extensions. Then L/N is an algebraic, finitely generated extension, i.e., $[L : N] < +\infty$. On the other hand, M/K is algebraic, N/K is purely transcendental, and hence they are linearly disjoint. Therefore $[M : K] = [MN : N] \leq [L : N] < +\infty$. ■

3.2. Separably generated extension. ([W, p.14]) A finitely generated extension is called a *separably generated extension* if it is a separably algebraic extension of a purely transcendental extension.

Concerning this concept, we need the following easier half of [W, Chap.1, Prop.19].

Lemma 3.3. *Let L/K be a finitely generated field extension contained in a fixed algebraically closed field. If $K^{p^{-1}}$ and L are linearly disjoint over K , then L/K is a separably generated extension.*

Proof. (An extract from [W].) Let $L = K(a_1, \dots, a_n)$, and let us prove the lemma by induction on n . Let $I := \{f \in K[x_1, \dots, x_n] \mid f(a_1, \dots, a_n) = 0\}$, where $x_1, \dots, x_n$ are indeterminates. If $I = 0$, then our conclusion holds. If not, let

$P(x_1, \dots, x_n) \in I \setminus \{0\}$ be a polynomial of minimal degree. Put $P_i := \partial P / \partial x_i$. If $P_1 = \dots = P_n = 0$, then $P = Q^p$ with some $Q \in K^{p^{-1}}[x_1, \dots, x_n]$. Then $Q \in \{g \in K^{p^{-1}}[x_1, \dots, x_n] \mid g(a_1, \dots, a_n) = 0\}$, and, by the sublemma below (with $K' = K^{p^{-1}}$), we can see that the right hand side is $\{\sum_i \lambda_i g_i \mid \lambda_i \in K^{p^{-1}}, g_i \in I\}$. This is impossible since $\deg Q < \deg P$. Therefore, we may assume that $P_n \neq 0$. Since $\deg P_n < \deg P$, $P_n(a_1, \dots, a_n) \neq 0$. This means that a_n is separable over $L' := K(a_1, \dots, a_{n-1})$. Since $K^{p^{-1}}$ and L' are linearly disjoint, L'/K is separably generated extension by the induction hypothesis. Hence $L = L'(a_n)$ is separably generated over K .

Sublemma. *Let L/K and K'/K be field extensions in a fixed algebraically closed field, and assume that L and K' are linearly disjoint over K . Let $a_1, \dots, a_n \in L$ and $g \in K'[x_1, \dots, x_n]$, and assume $g(a_1, \dots, a_n) = 0$. Then there exists $\kappa_i \in K'$ and $g_i \in K[x_1, \dots, x_n]$ ($1 \leq i \leq n$) such that $g_i(a_1, \dots, a_n) = 0$ and $g(x_1, \dots, x_n) = \sum_i \kappa_i g_i(x_1, \dots, x_n)$.*

Proof. Let $\{\kappa_i\}_i$ be a K -linear basis of K' . Then g can be uniquely expressed as $g = \sum_i \kappa_i g_i$ (finite sum) with $g_i \in K[x_1, \dots, x_n]$. Since

$$(1) \ 0 = g(a_1, \dots, a_n) = \sum_i \kappa_i g_i(a_1, \dots, a_n),$$

$$(2) \ \kappa_i \in K' \text{ are linearly independent over } K,$$

$$(3) \ g_i(a_1, \dots, a_n) \in L, \text{ and}$$

$$(4) \ L \text{ and } K' \text{ are linearly disjoint over } K,$$

it follows that $g_i(a_1, \dots, a_n) = 0$. ■

Lemma 3.4. *Let $f : X \rightarrow Y$ be a dominant morphism between irreducible varieties. Then there exists an open dense $U \subset X$ such that $f|_U$ is étale (resp. smooth) [EGA, (IV, §17)] if and only if $k(X)/k(Y)$ is a separably algebraic extension (resp. a separably generated extension).*

What is necessary for our present purpose is the 'if part' whose proof is an easy exercise. For the 'only if part', see [Mi, Chap.1, §3] and [SGA, exposé II]. ■

We also need the following lemma of M. Rosenlicht [R2, p.4, ↑ §.8, p.5, ↓ §.9].

Lemma 3.5. *Let L be a field, G a group of field automorphisms of L , and $K = L^G$ the subfield of L consisting of all elements of L left fixed by each automorphism of G . Then L/K is separably generated.*

Proof. (An extract from [R2].) By (3.3), it suffices to show that $K^{p^{-1}}$ and L are linearly disjoint over K , i.e., that if we have a relation $\sum_{i=1}^n \kappa_i \lambda_i^p = 0$, where $\kappa_i \in K$, $\lambda_i \in L$ and where not all κ_i 's are 0, then $\lambda_1, \dots, \lambda_n$ are linearly dependent over K . Clearly we may take $n > 1$. If $\sigma_1, \dots, \sigma_n \in G$, we have $\sum_{i=1}^n \kappa_i \sigma_j(\lambda_i^p) = 0$ ($j = 1, \dots, n$), so $\det(\sigma_j(\lambda_i^p))_{1 \leq i, j \leq n} = 0$ and hence $\det(\sigma_j(\lambda_i))_{1 \leq i, j \leq n} = 0$. Let r be the maximal rank that $(\sigma_j(\lambda_i))_{1 \leq i, j \leq n}$ can assume for $\sigma_1, \dots, \sigma_n \in G$; then $1 \leq r < n$. Reorder λ_i 's and choose $\sigma_1, \dots, \sigma_r \in G$ so that $\det(\sigma_j(\lambda_i))_{1 \leq i, j \leq r} \neq 0$. Hold $\sigma_1, \dots, \sigma_r$ fixed, and let $\sigma_{r+1} \in G$ be arbitrary. Then $\det(\sigma_j(\lambda_i))_{1 \leq i, j \leq r+1} = 0$, so there exist $\mu_1, \dots, \mu_r \in L$ such that

$$(1)_j \quad \sigma_j(\lambda_{r+1}) = \sum_{i=1}^r \mu_i \sigma_j(\lambda_i)$$

for all $j = 1, \dots, r+1$. Here $(1)_{r+1}$ is redundant, and $\mu_1, \dots, \mu_r$ are uniquely determined only by $(1)_j$, $1 \leq j \leq r$. Therefore these μ_i 's are independent of the choice of σ_{r+1} , and hence we have $\sigma(\lambda_{r+1}) = \sum_{i=1}^r \mu_i \sigma(\lambda_i)$ for any $\sigma \in G$. If $\tau \in G$, we have

$$\sigma(\lambda_{r+1}) = \tau(\tau^{-1}\sigma(\lambda_{r+1})) = \tau\left(\sum_{i=1}^r \mu_i \cdot \tau^{-1}\sigma(\lambda_i)\right) = \sum_{i=1}^r \tau(\mu_i) \cdot \sigma(\lambda_i).$$

By the uniqueness of $\mu_1, \dots, \mu_r$, we have $\tau(\mu_i) = \mu_i$, so each $\mu_i \in K = L^G$. Hence any one of $(1)_j$ yields $\lambda_{r+1} = \sum_{i=1}^r \mu_i \lambda_i$ with $\mu_1, \dots, \mu_r \in K$. Hence $\lambda_1, \dots, \lambda_n$ are linearly dependent over K . ■

3.6. Proof of Theorem 0.1. Now let us return to (2.3). Put $L = k(X_1)$ and $K = L^G$. By (3.1), K/k is finitely generated. Hence there is an irreducible k -variety W_1 such that $K = k(W_1)$, and we get a G -equivariant dominant rational morphism $\phi : X_1 \rightarrow W_1$. (The G -action on W_1 is trivial.) Let X_2 be the locus where ϕ is defined and smooth. Put $\phi(X_2) =: W_2$. Then $W_2 \subset W_1$ is open dense (cf. (3.1) and (3.5)), $\phi : X_2 \rightarrow W_2$ is an open mapping (cf. (1.3)), and $GX_2 = X_2$.

By (2.3, (2)), $k(Y_1) \subset k(X_1)^G = k(W_1) = k(W_2)$. Hence we get a dominant rational morphism $\psi : W_2 \rightarrow Y_1$. Take open dense $W_3 \subset W_2$ and $Y_3 \subset Y_1$ so that $\psi : W_3 \rightarrow Y_3$ is surjective regular morphism. Put $X_3 := \phi^{-1}(W_3)$.

Since $\phi : X_3 \rightarrow W_3$ is G -equivariant, each fibre of ϕ is a union of G -orbits. But each fibre of $f = \psi \circ \phi$ is precisely a G -orbit. Hence each fibre of ϕ is also precisely a G -orbit.

Hence all the assertions of (2.3) remain valid when $f : X_1 \rightarrow Y_1$ is replaced with $\phi : X_3 \rightarrow W_3$. Moreover $\phi : X_3 \rightarrow W_3$ is smooth, and $k(W_3) = k(X_3)^G$.

Let W_4 be the non-singular locus of W_3 , and put $X_4 := \phi^{-1}(W_4)$. Then all the conditions (0.1, (1)–(5)) are satisfied, and

$$\phi^*k[W_4] \subset k[X_4]^G \subset k(X_4)^G = \phi^*k(W_4).$$

In order to prove (0.1, (6)), let us assume the contrary, i.e., that there exists $\alpha \in k[X_4]^G \setminus \phi^*k[W_4]$. Then $\alpha = \phi^*\beta$ with some $\beta \in k(W_4) \setminus k[W_4]$. Let W'_4 be the locus where $1/\beta$ is regular. Since W_4 is a normal variety, $Z := \{w \in W'_4 \mid \beta(w)^{-1} = 0\}$ is a non-empty subvariety of a pure codimension one. Take $x_0 \in X_4$ so that $\phi(x_0) \in Z$. Then both $\phi^*\beta$ and $1/\phi^*\beta$ are regular on $\phi^{-1}(W_4)$, and

$$1 = (\phi^*\beta)(x_0) \cdot (\phi^*\beta)(x_0)^{-1} = (\phi^*\beta)(x_0) \times 0.$$

Thus we get a contradiction, and get (0.1, (6)). ■

3.7. Remark. In order to simplify the exposition, we assumed in (0.1) that k is algebraically closed and that X is irreducible, but these assumptions are not essential, and indeed are not assumed in [R1].

References

- [EGA] A.Grothendieck, *Eléments de géométrie algébrique, Chapter I*, (1970), Springer; *Chapters 2-4*, Publ. IHES (1961-1967).
- [SGA] A.Grothendieck et al., *Séminaire de géométrie algébrique, SGA I*, Lecture Notes in Math. **224** (1971), Springer.
- [Mi] J.S.Milne, *Etale cohomology*, (1980), Princeton University Press.
- [Mu] D.Mumford et al., *Geometric invariant theory, 3rd ed.*, (1994), Springer.
- [N] M.Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **2-1** (1961), 1-10.
- [R1] M.Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401-443.
- [R2] M.Rosenlicht, *Automorphisms of function fields*, Trans AMS **79** (1955), 1-11.
- [S] H.Sumihiko, *Equivariant completion, I*, J.Math. Kyoto Univ. **14** (1974), 1-28; *II*, ibid. **15** (1975), 573-605.
- [W] A.Weil, *Foundations of algebraic geometry, 2nd ed.*, (1962), Amer. Math. Soc. Colloq. Publ.