

On a decomposition space of a weak self-similar set^{*)}

早稲田大学・物質開発工学科 北田韶彦 (Akihiko Kitada)

Department of Materials Science and Engineering,

Waseda University

1. Introduction

We call the function f_j defined on a metric space (X, d) weak contraction provided that there exists a function $\alpha_j(l)$, $l > 0$ such that $0 < \alpha_j(l) < 1$, $\inf_{l>0} \alpha_j(l) > 0$ and $d(f_j(x), f_j(y)) \leq \alpha_j(l)d(x, y)$ for $d(x, y) < l$, $j = 1, \dots, m$ ($2 \leq m < \infty$). In a complete metric space, there exists a unique nonempty compact weak self-similar set S generated by the system $\{f_j; j = 1, \dots, m\}$ of weak contractions, namely, there exists a set S satisfying the relation $\bigcup_{j=1}^m f_j(S) = S$ [1,2,3].

In the present article, we will search the conditions for the weak contractions f_j , $j = 1, \dots, m$ defined on a compact metric space X to generate a weak self-similar set S which has the property that every nonempty compact metric space is a continuous image of it, and then, we will construct a decomposition space of S , which is homeomorphic to the compact space X .

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2. Construction of a decomposition space of the set S

Every nonempty, perfect, zero-dimensional, compact metric space is known [4,5] to have the above mentioned property that any nonempty compact metric space is a continuous image of it. If there exists a nonempty, perfect, zero-dimensional, compact weak self-similar set S in a compact metric space X , then, X must be a continuous image of S .

To attain our main results, we prepare the following lemmata.

Lemma 1. Let X be a nonempty compact metric space with a metric d . Assume that the following conditions hold for the weak contraction f_j , $j = 1, \dots, m$.

- i) Each f_j is one to one.
- ii) The set $\{x \in X; f_j(x) = x, j = 1, \dots, m\}^{1)}$ is not degenerate. That is, there exist at least two points x_0 and x'_0 such that $f_{j_0}(x_0) = x_0$ and $f_{j'_0}(x'_0) = x'_0$.
- iii) There exists a point $l_0 > 0$ such that $\sum_{j=1}^m \alpha_j(l_0) < 1$.

Then, there exists in X a perfect, zero-dimensional compact set S such that $\bigcup_{j=1}^m f_j(S) = S$.

Proof. Let us define a set S by $\bigcap_n X^n = S$. Here, $X^n = \bigcup_{j_1 \dots j_n \in W_n} X_{j_1 \dots j_n}$ where $X_{j_1 \dots j_n} = f_{j_1} \circ \dots \circ f_{j_n}(X)$ and W_n denotes the set of all words $j_1 \dots j_n$ with length n on symbols $\{1, \dots, m\}$. Since $x_0 = f_{j_0} \circ \dots \circ f_{j_0}(x_0) \in X_{j_0 \dots j_0} \subset X^n$ for any n , the compact set S is nonempty. We note that for any n and for any $j_1 \dots j_n \in W_n$, $f_{j_1} \circ \dots \circ f_{j_n}(x_0) = f_{j_1} \circ \dots \circ f_{j_n} \circ f_{j_0} \circ \dots \circ f_{j_0}(x_0) \in X_{j_1 \dots j_n j_0 \dots j_0} \subset X^k$ for any $k \geq n + 1$. Since the relation $X^{n+1} \subset \dots \subset X^1$ is obvious, $f_{j_1} \circ \dots \circ f_{j_n}(x_0) \in \bigcap_k X^k = S$. In the same way, concerning another fixed point x'_0 , the relation $f_{j_1} \circ \dots \circ f_{j_n}(x'_0) \in S$ must hold for any $j_1 \dots j_n \in W_n$.

¹⁾ It is known [1] that any weak contraction f defined on a complete metric space X has a fixed point in X .

Next we note that the diameter of $X_{j_1 \dots j_n} \rightarrow 0$ ($n \rightarrow \infty$), that is, for any $\varepsilon > 0$, there exists N such that for any $n \geq N$ and for any $j_1 \dots j_n \in W_n$, the diameter $X_{j_1 \dots j_n} < \varepsilon$. In fact, since the relation $d(f_j(x), f_j(y)) \leq \tilde{\alpha}_j(d(x, y))d(x, y)$ holds for $\tilde{\alpha}_j(l) = \inf_{p>l} \alpha_j(p)$, each diameter of $X_{j_1 \dots j_n}$ is dominated from above as follows.

$$\text{diameter of } X_{j_1 \dots j_n} \leq \left(\max_j \{ \tilde{\alpha}_j(\text{diameter of } X) \} \right)^n \text{diameter of } X.$$

Now, let x be a point of $S = \bigcap_n X^n$. For any $\varepsilon > 0$, there exists a number N such that for any $j_1 \dots j_N \in W_N$, the diameter of $X_{j_1 \dots j_N} < \varepsilon$. Then, there exists $j_1 \dots j_N \in W_N$ such that $x \in X_{j_1 \dots j_N} \subset \text{open sphere } S(x, \varepsilon)$. It is evident that both $f_{j_1} \circ \dots \circ f_{j_N}(x_0)$ and $f_{j_1} \circ \dots \circ f_{j_N}(x'_0)$ are contained in $S(x, \varepsilon)$. But the condition (i) guarantees that the point $f_{j_1} \circ \dots \circ f_{j_N}(x_0) \in S$ is different from the point $f_{j_1} \circ \dots \circ f_{j_N}(x'_0) \in S$. Therefore, at least one of the two points $f_{j_1} \circ \dots \circ f_{j_N}(x_0)$ and $f_{j_1} \circ \dots \circ f_{j_N}(x'_0)$ is different from the point x . This means that the point x is an accumulation point of S , and then, it is convinced that the nonempty compact set S must be perfect. Taking the condition (i) into account, we can easily verify the relation $\bigcup_{j=1}^m f_j(S) = S$. As this relation implies that $\bigcup_{j_1 \dots j_n \in W_n} f_{j_1} \circ \dots \circ f_{j_n}(S) = S$ for any $n \geq 1$, the estimate [2] concerning the Hausdorff dimension of S , $\dim_H S \leq \inf_{l>0} x(l)$ where $\sum_{j=1}^m \tilde{\alpha}_j(l)^{x(l)} = 1$, holds. Then, since the condition (iii) implies $\dim_H S < 1$, the dimension of $S = 0$, immediately follows from the fact [6] that the dimension²⁾ of a set does not exceed its Hausdorff dimension. ■

Next, let us recall the definition of the decomposition space [8] and that of the quotient map [9].

Let $\mathbf{D}$ be a partition of a topological space (X, τ) , *i.e.* $\mathbf{D}$ is a collection of nonempty, mutually disjoint subsets of X such that $\bigcup \mathbf{D} = X$. The topological space $(\mathbf{D}, \tau(\mathbf{D}))$ defined by the decomposition topology $\tau(\mathbf{D}) = \{U \subset \mathbf{D}; \bigcup U \in$

²⁾ Since S is a separable metric space, $\text{ind} S = \text{Ind} S = \dim S$ [7].

$\tau\}$, is called a decomposition space of (X, τ) . An onto map $f : (X, \tau) \rightarrow (Y, \tau')$ is said to be a quotient map provided that $\tau' = \tau_f$. Here, $\tau_f = \{u' \subset Y; f^{-1}(u') \in \tau\}$. Concerning the relationships between the decomposition space and the quotient map, we have the following general lemma.

Lemma 2. Let f be a quotient map from a space (X, τ) onto a space (Y, τ') . Then, (Y, τ') is homeomorphic to the decomposition space $(\mathbf{D}_f, \tau(\mathbf{D}_f))$ of (X, τ) . Here, $\mathbf{D}_f = \{f^{-1}(y); y \in Y\}$.

Proof. Let us show that $h : (Y, \tau') \rightarrow (\mathbf{D}_f, \tau(\mathbf{D}_f)), y \mapsto f^{-1}(y)$ is an homeomorphism. First, it is evident that the map h is one to one and onto. Then, it is sufficient to show that the map is continuous and open. Let U be a nonempty open set of $\mathbf{D}_f$. There exists a nonempty subset B of Y such that $U = \{f^{-1}(y); y \in B\}$. $\cup U = \cup \{f^{-1}(y); y \in B\} = f^{-1}(B)$. Since the subset U is $\tau(\mathbf{D}_f)$ -open, $f^{-1}(B) \in \tau$. The relation $B = h^{-1} \circ h(B) = h^{-1}(U)$ follows immediately from $h(B) = \{f^{-1}(y); y \in B\} = U$. This means the continuity of the map h . Next, let B be a nonempty τ' -open, that is, $f^{-1}(B) \in \tau$. Since $\cup h(B) = \cup \{f^{-1}(y); y \in B\} = f^{-1}(B)$, $h(B) \in \tau(\mathbf{D}_f)$. Hence, the map h is an open map. ■

It is an elemental fact that the existence of a continuous map f from a compact space (X, τ) onto a T_2 space (Y, τ') implies that $\tau' = \tau_f$. Therefore, the above lemma can be rewritten as follows.

Lemma 3. If there exists a continuous map f from a compact space (X, τ) onto a T_2 space (Y, τ') , then, (Y, τ') is homeomorphic to the decomposition space $(\mathbf{D}_f, \tau(\mathbf{D}_f))$, $\mathbf{D}_f = \{f^{-1}(y); y \in Y\}$.

Under the Lemma 3, we can state our results as the following proposition.

Proposition 1. Let X be a nonempty compact metric space, and let S be a weak self-similar set generated from f_j , $j = 1, \dots, m$ satisfying the three conditions in Lemma 1. Then, there exists a continuous function f from S onto X , and the decomposition space $(\mathbf{D}_f, \tau(\mathbf{D}_f))$, $\mathbf{D}_f = \{f^{-1}(x) \subset S; x \in X\}$, of S is homeomorphic to X .

3. Metrizability of the decomposition space

It is known [10,11] that i) If a T_2 space Y is an image of a continuous function f defined on a compact metric space X , then, the decomposition space $(\mathbf{D}_f, \tau(\mathbf{D}_f))$, $\mathbf{D}_f = \{f^{-1}(y) \subset X; y \in Y\}$, of X is upper semi continuous [12], and ii) Any upper semicontinuous decomposition of a compact metric space is metrizable.

Therefore, from the above discussions, our decomposition space of S is easily verified to be metrizable.

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