

The geodesic growth series for pure Artin groups of dihedral type

By

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Abstract

We consider the pure Artin group of dihedral type, which is the kernel of the natural projection from the Artin group of dihedral type $I_2(k)$ to the associated Coxeter group. We present a rational function expression for the geodesic growth series of the pure Artin group of dihedral type with respect to a natural generating set, and we explicitly determine the denominator of this rational function expression. Moreover, we show that the growth rate of the series is a Pisot-Vijayaraghavan number.

§ 1. Introduction

For a finitely generated group G with a given generating set Γ , the corresponding *spherical growth series* is defined as

$$\mathcal{S}_{(G,\Gamma)}(t) := \sum_{n=0}^{\infty} \alpha_n t^n,$$

where α_n for $n \in \mathbf{N} \cup \{0\}$ is the number of elements in G whose lengths with respect to Γ are equal to n . The spherical growth series $\mathcal{S}_{(G,\Gamma)}(t)$ is a commonly employed measure of the rate of growth of G with respect to Γ , and has been explored for a number of

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interesting families of pairs (G, Γ) . In particular, $\mathcal{S}_{(G, \Gamma)}(t)$ is known to be rational (i.e., it can be expressed as the quotient of two polynomials with integer-valued coefficients in the ring of formal power series $\mathbf{Z}[[t]]$) in a number of important cases (see, e.g., [3], [4], [5], [6], [7], [9], [10], [11], [12], [16], [18], [20], [22], [26], [27], [28] and [29]).

In the present paper, we are interested in the following series, defined analogously to $\mathcal{S}_{(G, \Gamma)}(t)$:

$$\mathcal{G}_{(G, \Gamma)}(t) := \sum_{n=0}^{\infty} \tilde{\alpha}_n t^n,$$

where $\tilde{\alpha}_n$ for $n \in \mathbf{N} \cup \{0\}$ is the number of geodesic words with respect to Γ whose lengths are equal to n . Recall that a word in the free monoid Γ^* generated by Γ is *geodesic* if the corresponding path in the Cayley graph of G with respect to Γ is a minimal length edge path joining its endpoints. The series $\mathcal{G}_{(G, \Gamma)}(t)$ is called the *geodesic growth series* for the pair (G, Γ) . The growth rate for $\mathcal{G}_{(G, \Gamma)}(t)$ is defined as

$$\tau_{\mathcal{G}} := \limsup_{n \rightarrow \infty} \sqrt[n]{\tilde{\alpha}_n}.$$

We also call $\tau_{\mathcal{G}}$ the *geodesic growth rate* for the pair (G, Γ) . By the Cauchy-Hadamard theorem, the radius of convergence $R_{\mathcal{G}}$ of the series $\mathcal{G}_{(G, \Gamma)}(t)$ is the reciprocal of $\tau_{\mathcal{G}}$.

Although the geodesic growth series $\mathcal{G}_{(G, \Gamma)}(t)$ is not as well understood as the spherical growth series $\mathcal{S}_{(G, \Gamma)}(t)$, there are nonetheless many known pairs (G, Γ) for which the geodesic growth series is rational. The following are some examples: (1) any word-hyperbolic group with respect to an arbitrary generating set (see [5] and [10]); (2) any geometrically finite hyperbolic group with respect to a particular generating set (see [23]); (3) any irreducible affine Coxeter group with respect to the standard generating set (see [24]); (4) any right-angled Artin group with respect to the standard generating set (see [21] and [1]); (5) any Artin group of dihedral type, $G_{I_2(k)}$, with respect to the standard generating set (the so-called ‘set of Artin generators’) (see [26] and [22]); (6) any Artin group of large type with respect to the standard generating set (see [17]); (7) any Garside group with respect to a particular generating set (the so-called ‘set of Garside generators’) (see [8]). For each of the above examples, it has been shown that the set of all geodesic words of the group G with respect to the generating set Γ is a regular language over Γ , which implies the rationality of the geodesic growth series $\mathcal{G}_{(G, \Gamma)}(t)$. The pair (G, Γ) is said to be *strongly geodesic regular* or to form a *Cannon pair* if the set of all geodesic words of G with respect to Γ forms a regular language over Γ . In general, the regularity of a language consisting of geodesic words depends on the generating set Γ (see the example due to Cannon discussed in §4 of [23]).

In this paper, for each integer $k \geq 3$, we consider the *pure Artin group* $P_{I_2(k)}$, which is the kernel of the projection from the Artin group of dihedral type, $G_{I_2(k)}$, to the associated Coxeter group, $\overline{G}_{I_2(k)}$. The group $P_{I_2(k)}$ is geometrically realized as

the fundamental group of the complement of a torus link in the 3-dimensional sphere, which has a natural generating set A (cf. [25]). In particular, in the case $k = 3$, $P_{I_2(3)}$ is the pure braid group with three strands, and A is the standard generating set (cf. [2]). In [13], for any element g of $P_{I_2(k)}$, a particular geodesic representative of g is determined. Then, through analysis of the regularity of the language consisting of such geodesic representatives, a rational function expression for the spherical growth series $\mathcal{S}_{(P_{I_2(k)}, A)}(t)$ of $P_{I_2(k)}$ is derived with respect to the generating set A . Moreover, in [13], all the geodesic representatives of any element of $P_{I_2(k)}$ are determined. From this, it is seen that all geodesic words of a particular type (Type 3 defined in [13]) form a regular language. In the present paper, by using arguments similar to those given in [13], we derive a rational function expression for the geodesic growth series $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$ of $P_{I_2(k)}$ with respect to the generating set A (see Theorem 3.2). Moreover, by using an algebraic argument, we explicitly determine the denominator of this rational function expression (see Theorem 3.3). From this theorem, we can detect a number-theoretic property concerning the geodesic growth rate $\tau_{\mathcal{G}}(k)$ for the pair $(P_{I_2(k)}, A)$: The geodesic growth rate $\tau_{\mathcal{G}}(k)$ is a Pisot-Vijayaraghavan number, i.e., a real algebraic integer $\tau > 1$ whose algebraic conjugates other than τ itself lie in the unit disk (see Corollary 3.5).

§ 2. Geodesic words of pure Artin groups of dihedral type

In this section, we present definitions and basic facts concerning pure Artin groups of dihedral type (see [13] and [15] for details). We inherit all of the notation used in [13].

Let k be an integer greater than 2, $G_{I_2(k)}$ be the *Artin group of dihedral type $I_2(k)$* and $\overline{G}_{I_2(k)}$ be the *Coxeter group of dihedral type $I_2(k)$* . Then there is a natural surjective homomorphism

$$p : G_{I_2(k)} \rightarrow \overline{G}_{I_2(k)}.$$

We call the kernel of p the *pure Artin group of dihedral type* and write it $P_{I_2(k)}$. The group $P_{I_2(k)}$ has the following presentation:

$$P_{I_2(k)} = \langle a_1, \dots, a_k \mid a_1 \cdots a_k = a_2 \cdots a_k a_1 = a_3 \cdots a_k a_1 a_2 = \cdots = a_k a_1 \cdots a_{k-1} \rangle.$$

In this paper, we consider the generating set

$$A = \{a_1, \dots, a_k, a_1^{-1}, \dots, a_k^{-1}\},$$

as in [13], and investigate the so-called ‘geodesic growth series’ of $P_{I_2(k)}$ with respect to A , whose definition is given in §3.

Let A^* and $\{a_1, \dots, a_k\}^*$ denote the free monoids generated by A and $\{a_1, \dots, a_k\}$, respectively. We refer to the finite set A as an *alphabet*, its elements as *letters*, and the elements of A^* (resp., $\{a_1, \dots, a_k\}^*$) as *words* (resp., *positive words*). Let ε denote the *null* word. A subset L of A^* is called a *language* over A . A language L is *regular* if L is recognized by some deterministic, finite-state automaton over A (see [10] or [19] for the definition of automata). The length of a word w is the number of letters it contains, which is denoted by $|w|$. The length of ε is zero. Since A generates the group $P_{I_2(k)}$, there exists a natural surjective monoid homomorphism $\pi : A^* \rightarrow P_{I_2(k)}$. If u and v are words, then $u = v$ means that $\pi(u) = \pi(v)$ and $u \equiv v$ means that u and v are identical letter by letter. A word $w \in \pi^{-1}(g)$ is called a *representative* of g . The length of a group element g is regarded as the quantity

$$\|g\| = \min\{|w| \mid w \in \pi^{-1}(g)\}.$$

A word $w \in A^*$ for which the relation $|w| = \|\pi(w)\|$ holds is termed *geodesic*. A word $w_1 \cdots w_m \in A^*$ is called a *reduced* word if $w_i \neq w_{i+1}^{-1}$ for all $i \in \{1, \dots, m-1\}$. A geodesic representative is a reduced word. Below, we consider some other alphabet, T , which is a subset of A^* (denoted by $\text{FB}_{\leq P}^+ \cup \text{FB}_{\leq N}^-$), and we investigate the regularity of a language over T .

Let us first recall the notation and definitions concerning ‘fundamental blocks’ given in [13]. We begin by introducing the word

$$\nabla \equiv a_1 \cdots a_k.$$

It satisfies the following relation:

$$(2.1) \quad \nabla \equiv a_1 \cdots a_k = a_2 \cdots a_k a_1 = a_3 \cdots a_k a_1 a_2 = \cdots = a_k a_1 \cdots a_{k-1}.$$

A *fundamental block* is a word with length smaller than k that appears as a subword in the terms of (2.1). There are $k(k-1)$ fundamental blocks. All of them are listed in [13]. Let $\text{FB}^+(\subset A^*)$ (resp., $\text{FB}^-(\subset A^*)$) denote the set consisting of all the fundamental blocks (resp., all the inverses of fundamental blocks); for $I \in \{0, \dots, k-1\}$, $\text{FB}_I^\pm$ (resp., $\text{FB}_{\leq I}^\pm$) denotes the set consisting of all the elements of $\text{FB}^\pm$ with length equal to I (resp., smaller than or equal to I); for $\mu \equiv a_i \cdots a_k a_1 \cdots a_j \in \text{FB}^+$ (resp., $\mu^{-1} \equiv a_j^{-1} \cdots a_1^{-1} a_k^{-1} \cdots a_i^{-1} \in \text{FB}^-$), we define $\mathcal{L}(\mu) := a_i$ and $\mathcal{R}(\mu) := a_j$ (resp., $\mathcal{L}(\mu^{-1}) := a_j$ and $\mathcal{R}(\mu^{-1}) := a_i$). For $\mu \equiv a_i \cdots a_k a_1 \cdots a_j \in \text{FB}^+$, we call a_{j+1} the *letter subsequent to μ* . When $\mu \equiv a_i \cdots a_k$, we call a_1 the letter subsequent to μ . The letter subsequent to μ is denoted by $\mathcal{N}(\mu)$. For $\mu^{-1} \equiv a_j^{-1} \cdots a_1^{-1} a_k^{-1} \cdots a_i^{-1} \in \text{FB}^-$, we call a_{i-1} the letter subsequent to μ^{-1} , which is denoted by $\mathcal{N}(\mu^{-1})$. When $a_i^{-1} \equiv a_1^{-1}$, we call a_k the letter subsequent to μ^{-1} .

In [13], we introduced the set

$$\tilde{\Gamma} := \{\xi \in A^* \mid |\xi| = \|\pi(\xi)\|\},$$

i.e., the set consisting of all of the geodesic words, and for each (P, N) satisfying $0 \leq P \leq k$ and $0 \leq N \leq k$, the sets

$$\begin{aligned} \tilde{\Gamma}_{P,N} &:= \{\xi \in \text{WT} \mid |\xi| = \|\pi(\xi)\|, (\text{Pos}(\xi), \text{Neg}(\xi)) = (P, N)\}, \\ G_{P,N} &:= \{g \in P_{I_2(k)} \mid (\text{Pos}(g), \text{Neg}(g)) = (P, N)\}, \end{aligned}$$

where $\text{WT}_i = \{w \in A^* \mid w \text{ is a word of Type } i\}$ for each $i \in \{1, 2, 3\}$ (see §3 in [13] for the definition of Type i), $\text{WT} = \text{WT}_1 \cup \text{WT}_2 \cup \text{WT}_3$, and $\text{Pos}(\xi), \text{Neg}(\xi), \text{Pos}(g)$ and $\text{Neg}(g)$ are specific integers between 0 and k (defined in §§2 and 3 of [13]).

By Propositions 3.3, 3.4 and 3.7 and Corollary 3.8 of [13], we have

$$(2.2) \quad \begin{cases} P + N \geq k + 1 \implies \tilde{\Gamma}_{P,N} = \emptyset, & G_{P,N} = \emptyset, \\ P + N \leq k \implies \pi^{-1}(G_{P,N}) \cap \tilde{\Gamma} = \tilde{\Gamma}_{P,N}, \end{cases}$$

$$(2.3) \quad \begin{cases} \tilde{\Gamma} = \bigcup_{P+N \leq k} \tilde{\Gamma}_{P,N} & (\text{disjoint union}), \\ P_{I_2(k)} = \bigcup_{P+N \leq k} G_{P,N} & (\text{disjoint union}), \end{cases}$$

and

$$\bigcup_{\substack{P+N \leq k \\ (P,N) \neq (k,0), (0,k)}} \tilde{\Gamma}_{P,N} = \{\xi \in \text{WT}_3 \mid \text{Pos}(\xi) + \text{Neg}(\xi) \leq k\}.$$

We remark that an element $g \in P_{I_2(k)}$ has more than one geodesic representative if and only if $\text{Pos}(g) + \text{Neg}(g) = k$.

Now, choose any $P, N \in \mathbf{N} \cup \{0\}$ satisfying the conditions $P + N \leq k$ and $(P, N) \notin \{(k, 0), (0, k)\}$, and fix them. Then, any element $w \in \bigcup_{p \leq P, n \leq N} \tilde{\Gamma}_{p,n}$ can be expressed as

$$w \equiv v_1 \cdots v_m \in (\text{FB}_{\leq P}^+ \cup \text{FB}_{\leq N}^-)^*,$$

where $(\text{FB}_{\leq P}^+ \cup \text{FB}_{\leq N}^-)^*$ is the free monoid generated by the finite set $\text{FB}_{\leq P}^+ \cup \text{FB}_{\leq N}^-$, and for each $j \in \{1, \dots, m-1\}$, we have

$$\begin{cases} v_j, v_{j+1} \in \text{FB}^\pm \implies \mathcal{N}(v_j) \neq \mathcal{L}(v_{j+1}), \\ v_j \in \text{FB}^\pm, v_{j+1} \in \text{FB}^\mp \implies \mathcal{R}(v_j) \neq \mathcal{L}(v_{j+1}). \end{cases}$$

Thus, it is seen that the set $\bigcup_{p \leq P, n \leq N} \tilde{\Gamma}_{p,n}$ is a regular language over $\text{FB}_{\leq P}^+ \cup \text{FB}_{\leq N}^-$. In

fact, it is recognized by the deterministic, finite-state automaton $\mathbf{A}_{\leq P, \leq N}$ over $\text{FB}_{\leq P}^+ \cup \text{FB}_{\leq N}^-$ defined as follows:

- (i) **Set of states:** $\{\varepsilon\} \cup \text{FB}_{\leq P}^+ \cup \text{FB}_{\leq N}^- \cup \{\mathbf{fail}\};$
- (ii) **Initial state:** $\{\varepsilon\};$
- (iii) **Set of accept states:** $\{\varepsilon\} \cup \text{FB}_{\leq P}^+ \cup \text{FB}_{\leq N}^-;$
- (iv) **Alphabet:** $\text{FB}_{\leq P}^+ \cup \text{FB}_{\leq N}^-;$
- (v) **Transitions:**
 - (v-1) $\forall v \in \text{FB}_{\leq P}^+ \cup \text{FB}_{\leq N}^-, \varepsilon \xrightarrow{v} v;$
 - (v-2) $\forall u, v \in \text{FB}_{\leq P}^+,$
if $\mathcal{N}(u) \neq \mathcal{L}(v)$, then $u \xrightarrow{v} v$, and if $\mathcal{N}(u) = \mathcal{L}(v)$, then $u \xrightarrow{v} \mathbf{fail};$
 - (v-3) $\forall u, v \in \text{FB}_{\leq N}^-,$
if $\mathcal{N}(u) \neq \mathcal{L}(v)$, then $u \xrightarrow{v} v$, and if $\mathcal{N}(u) = \mathcal{L}(v)$, then $u \xrightarrow{v} \mathbf{fail};$
 - (v-4) $\forall u \in \text{FB}_{\leq P}^+, \forall v \in \text{FB}_{\leq N}^-,$
if $\mathcal{R}(u) \neq \mathcal{L}(v)$, then $u \xrightarrow{v} v$, and if $\mathcal{R}(u) = \mathcal{L}(v)$, then $u \xrightarrow{v} \mathbf{fail};$
 - (v-5) $\forall u \in \text{FB}_{\leq N}^-, \forall v \in \text{FB}_{\leq P}^+,$
if $\mathcal{R}(u) \neq \mathcal{L}(v)$, then $u \xrightarrow{v} v$, and if $\mathcal{R}(u) = \mathcal{L}(v)$, then $u \xrightarrow{v} \mathbf{fail}.$

§ 3. Geodesic growth series for $P_{I_2(k)}$

In this section, by considering the structure of the automaton $\mathbf{A}_{\leq P, \leq N}$ given in §2, we determine a rational function expression for the geodesic growth series of the group $P_{I_2(k)}$ with respect to the generating set A .

The *geodesic growth series* of the group $P_{I_2(k)}$ with respect to the generating set A is defined by the following formal power series:

$$(3.1) \quad \mathcal{G}_{(P_{I_2(k)}, A)}(t) := \sum_{q=0}^{\infty} \tilde{\alpha}_q t^q,$$

where for each $q \in \mathbf{N} \cup \{0\}$, we define

$$(3.2) \quad \tilde{\alpha}_q := \sharp\{\xi \in A^* \mid |\xi| = \|\pi(\xi)\| = q\}.$$

Note that the radius of convergence of the growth series $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$ is greater than or equal to that of the growth series of the free group of rank k , which is equal to $\frac{1}{2k-1}$ (cf. Chapter VI of [9]). Thus, $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$ is a holomorphic function near the origin, $0 \in \mathbf{C}$.

For each pair (P, N) , we define

$$\mathcal{G}_{P,N}(t) := \sum_{q=0}^{\infty} \sharp\{\xi \in \tilde{\Gamma}_{P,N} \mid |\xi| = q\} t^q.$$

Then, from the partition (2.3), we have

$$(3.3) \quad \mathcal{G}_{(P_{I_2(k)}, A)}(t) = \mathcal{G}_{k,0}(t) + \mathcal{G}_{0,k}(t) + \sum_{\substack{P+N \leq k \\ (P,N) \neq (k,0), (0,k)}} \mathcal{G}_{P,N}(t),$$

and from Proposition 3.7 of [13], we obtain

$$(3.4) \quad \mathcal{G}_{P,N}(t) = \mathcal{S}_{P,N}(t), \text{ if } P + N \leq k - 1,$$

where $\mathcal{S}_{P,N}(t)$ is the spherical growth series for the set $\Gamma_{P,N}$ (see §§4 and 5 of [13] for their definitions).

In order to simplify the presentation of the growth series, for each $q \in \mathbf{N} \cup \{0\}$, we introduce the following:

$$\begin{cases} T_q := t + t^2 + \cdots + t^q, & \text{for } q \geq 1, \\ T_0 := 0. \end{cases}$$

First, let us consider the case in which $P + N \leq k$ and $(P, N) \notin \{(k, 0), (0, k)\}$. In this case, we have the following proposition.

Proposition 3.1. *For each P, N satisfying $P + N \leq k$ and $(P, N) \notin \{(k, 0), (0, k)\}$, we have*

$$\sum_{0 \leq p \leq P, 0 \leq n \leq N} \mathcal{G}_{p,n}(t) = \frac{1 + T_P + T_N}{1 - (k-1)(T_P + T_N)}.$$

Proof. From (3.4), if $P + N \leq k - 1$, the assertion is identical to that of Proposition 5.1 in [13]. Hence, we need only consider the case in which $P + N = k$ and $(P, N) \notin \{(k, 0), (0, k)\}$, that is, $(P, N) \in \{(1, k-1), (2, k-2), \dots, (k-1, 1)\}$. For $q \in \mathbf{N} \cup \{0\}$, we define

$$\tilde{B}_q(P; N) := \{\xi \in \bigcup_{0 \leq p \leq P, 0 \leq n \leq N} \tilde{\Gamma}_{p,n} \mid |\xi| = q\}$$

and

$$\tilde{\beta}_q(P; N) := \sharp \tilde{B}_q(P; N).$$

Then, we have

$$\sum_{0 \leq p \leq P, 0 \leq n \leq N} \mathcal{G}_{p,n}(t) = \sum_{q=0}^{\infty} \tilde{\beta}_q(P; N) t^q.$$

Further, note that for $q = 0$, we have

$$\tilde{\beta}_0(P; N) = 1.$$

Then, by considering the structure of the automaton $\mathbf{A}_{\leq P, \leq N}$, we obtain the same recursive formula for $\tilde{\beta}_q(P, N)$ as for $\beta_q(P, N)$ given in Lemma 5.2 of [13]. Moreover, we obtain the same equalities for $\tilde{\beta}_q(P, N)$ as for $\beta_q(P, N)$ appearing in Lemma 5.3 of [13]. Therefore, we obtain the desired result. $\square$

Next, consider the case in which $(P, N) \in \{(k, 0), (0, k)\}$. Because all positive words are geodesic with respect to A (see Lemma 3.1 of [13]), the set of all positive words, i.e., $\{a_1, \dots, a_k\}^*$, is equal to $\bigcup_{0 \leq p \leq k} \tilde{\Gamma}_{p,0}$. Hence, with $\sharp\{a_1, \dots, a_k\} = k$, we obtain

$$(3.5) \quad \sum_{p=0}^k \mathcal{G}_{p,0}(t) = \frac{1}{1 - kt}.$$

Thus, from (2.3), (3.5) and Proposition 3.1, we have

$$(3.6) \quad \begin{aligned} \mathcal{G}_{k,0}(t) &= \sum_{p=0}^k \mathcal{G}_{p,0}(t) - \sum_{p=0}^{k-1} \mathcal{G}_{p,0}(t) \\ &= \frac{1}{1 - kt} - \frac{1 + T_{k-1}}{1 - (k-1)T_{k-1}} \\ &= \frac{kt^k}{(1 - kt)\{1 - (k-1)T_{k-1}\}}. \end{aligned}$$

Then, by considering the inverses of positive words, we obtain

$$(3.7) \quad \mathcal{G}_{0,k}(t) = \mathcal{G}_{k,0}(t).$$

We are now ready to state the first main result of this paper. From (3.3), (3.6), (3.7) and Proposition 3.1, and employing the trick in Lemma 5.3 of [22], we obtain the following:

Theorem 3.2. *The geodesic growth series for the pure Artin group $P_{I_2(k)}$ of dihedral type with respect to the generating set A possesses the rational function expression*

$$(3.8) \quad \begin{aligned} \mathcal{G}_{(P_{I_2(k)}, A)}(t) &= \frac{2kt^k}{(1 - kt)\{1 - (k-1)T_{k-1}\}} \\ &\quad + \sum_{p=1}^{k-1} \frac{1 + T_p + T_{k-p}}{1 - (k-1)(T_p + T_{k-p})} - \sum_{p=1}^{k-2} \frac{1 + T_p + T_{k-1-p}}{1 - (k-1)(T_p + T_{k-1-p})}. \end{aligned}$$

It is easy to verify that for each term on the right-hand side, the numerator and denominator have no common zero.

Next, we rewrite the right-hand side of (3.8) using the common denominator

$$G(t) := (1 - kt) \prod_{\substack{a+b=k \\ b \geq a \geq 1}} \{1 - (k-1)(T_a + T_b)\} \prod_{\substack{a+b=k-1 \\ b \geq a \geq 0}} \{1 - (k-1)(T_a + T_b)\},$$

and sum the terms. Then we obtain a single fraction expression for $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$. Let $H(t)$ be its numerator. Then we have $\mathcal{G}_{(P_{I_2(k)}, A)}(t) = \frac{H(t)}{G(t)}$. Now, we state the second main result of this paper:

Theorem 3.3. *The two polynomials $G(t)$ and $H(t)$ do not have a common zero.*

This theorem is proved in the next section.

Example 3.4.

$$\mathcal{G}_{(P_{I_2(3)}, A)}(t) = \frac{(1+2t)(1-9t+28t^2-36t^3+16t^4+12t^5)}{(1-3t)(1-4t)(1-2t-2t^2)(1-4t-2t^2)},$$

$$\mathcal{G}_{(P_{I_2(4)}, A)}(t) = \frac{(1-17t+87t^2-60t^3-432t^4-153t^5+2007t^6+1512t^7-297t^8-1026t^9-702t^{10}-216t^{11})}{(1-4t)(1-6t-3t^2)(1-6t-6t^2)(1-3t-3t^2-3t^3)(1-6t-3t^2-3t^3)}.$$

From Theorem 3.3, the radius of convergence of the series $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$ is realized as the absolute value of a zero of the polynomial $G(t)$. Hence, only from Lemma 3.1(i) and (ii) of [14], we obtain the following:

Corollary 3.5. *The geodesic growth rate $\tau_{\mathcal{G}}(k)$ for the pair $(P_{I_2(k)}, A)$ is a Pisot-Vijayaraghavan number.*

See Theorem 3.2 of [14]. Another demonstration of this corollary derived from Lemma 3.1(i) -(iv) of [14] is given there.

§ 4. Denominator of the geodesic growth series

In this section, we consider the denominators of the terms in the formula for $\mathcal{G}_{(P_{I_2(k)}, A)}(t)$ given in Theorem 3.2, and through this consideration we demonstrate Theorem 3.3.

Let $k \geq 3$ be an integer. Define

$$f_i(t) := 1 - (k-1)(T_{i-1} + T_{k-i}) \quad \text{for } i \in \{1, \dots, k\},$$

and

$$\begin{aligned} g_0(t) &:= 1 - kt, \\ g_i(t) &:= 1 - (k-1)(T_i + T_{k-i}) \quad \text{for } i \in \{1, \dots, k-1\}. \end{aligned}$$

Then, the formula in Theorem 3.2 can be written as

$$(4.1) \quad \mathcal{G}_{(P_{I_2(k)}, A)}(t) = \frac{2kt^k}{g_0(t)f_1(t)} + \sum_{p=1}^{k-1} \frac{1 + T_p + T_{k-p}}{g_p(t)} - \sum_{p=1}^{k-2} \frac{1 + T_p + T_{k-1-p}}{f_{p+1}(t)}.$$

Next, we prove the following lemma, from which Theorem 3.3 follows immediately.

Lemma 4.1.

1. No two mutually different polynomials $f_i(t)$ and $f_j(t)$ have a common zero.
2. No two mutually different polynomials $g_i(t)$ and $g_j(t)$ have a common zero.
3. No two polynomials $f_i(t)$ and $g_j(t)$ have a common zero.

Proof. First note that we have

$$(4.2) \quad \begin{aligned} f_{\frac{k+1}{2}-j}(t) &= f_{\frac{k+1}{2}+j}(t) & \text{for } j \in \{1, \dots, \frac{k-1}{2}\}, & \text{ if } k \text{ is odd,} \\ f_{\frac{k}{2}+1-j}(t) &= f_{\frac{k}{2}+j}(t) & \text{for } j \in \{1, \dots, \frac{k}{2}\}, & \text{ if } k \text{ is even,} \end{aligned}$$

and

$$(4.3) \quad \begin{aligned} g_{\frac{k+1}{2}-j}(t) &= g_{\frac{k-1}{2}+j}(t) & \text{for } j \in \{1, \dots, \frac{k-1}{2}\}, & \text{ if } k \text{ is odd,} \\ g_{\frac{k}{2}-j}(t) &= g_{\frac{k}{2}+j}(t) & \text{for } j \in \{1, \dots, \frac{k}{2}-1\}, & \text{ if } k \text{ is even.} \end{aligned}$$

Also, we know that

$$(4.4) \quad \begin{aligned} f_i(1) &\neq 0, & f_i(0) &\neq 0, \\ g_i(1) &\neq 0, & g_i(0) &\neq 0, \end{aligned}$$

for all $i \in \{1, \dots, k-1\}$.

1. From (4.2), it is sufficient to consider the polynomials $f_1(t), \dots, f_{\frac{k+1}{2}}(t)$ (resp., $f_1(t), \dots, f_{\frac{k}{2}}(t)$) if k is odd (resp., k is even).

Suppose that $f_i(t)$ and $f_j(t)$ ($1 \leq i < j$) have a common zero ρ . Then we have

$$f_i(\rho) = f_j(\rho).$$

From this equality, we obtain

$$(4.5) \quad (\rho^i - \rho^{k-j+1})(\rho^{j-i-1} + \rho^{j-i-2} + \cdots + \rho + 1) = 0.$$

Also, from (4.4), we have $\rho \neq 0$. Hence, (4.5) implies that ρ is an algebraic integer over $\mathbf{Q}$. Next, note that from $f_i(\rho) = 0$, we also have

$$(4.6) \quad \rho + \cdots + \rho^{i-1} + \rho + \cdots + \rho^{k-i} = \frac{1}{k-1}.$$

The left-hand side of (4.6) is an algebraic integer over $\mathbf{Q}$. However, $\frac{1}{k-1}$ is not an algebraic integer for $k \geq 3$. Thus, we obtain a contradiction. Hence, $f_i(t)$ and $f_j(t)$ do not have a common zero.

2. From (4.3), it is sufficient to consider the polynomials $g_0(t), g_1(t), \dots, g_{\frac{k-1}{2}}(t)$ (resp., $g_0(t), g_1(t), \dots, g_{\frac{k}{2}}(t)$) if k is odd (resp., k is even).

Let $i \in \{1, \dots, k-1\}$. Then, $g_i(t)$ and $g_0(t)$ do not have a common zero, because $g_i(\frac{1}{k}) \neq 0$. The result for $g_i(t)$ and $g_j(t)$ ($1 \leq i < j$) is obtained by an argument similar to that given in Part 1.

3. From (4.2) and (4.3), we can assume that $i \leq \frac{k+1}{2}$ and $j \leq \frac{k-1}{2}$ (resp., $i \leq \frac{k}{2}$ and $j \leq \frac{k}{2}$) if k is odd (resp., k is even).

The fact that $f_i(\frac{1}{k}) \neq 0$ implies that no $f_i(t)$ has a common zero with $g_0(t)$. Next, suppose that $f_i(t)$ and $g_j(t)$ ($j \geq 1$) have a common zero ρ . Then we have

$$(4.7) \quad f_i(\rho) = g_j(\rho).$$

Also, from (4.4), we know that $\rho \neq 0$. Thus, from (4.7), we obtain

$$(4.8) \quad \rho^{i-1} + \rho^{k-i} = \rho^j + \rho^{k-j}.$$

We now show that ρ is an algebraic integer over $\mathbf{Q}$. This is done by considering the following four cases.

Case 1: $i = j$. Here, from (4.8), we have

$$\rho^i - \rho^{i-1} = 0.$$

Case 2: $i < j$. Here, from (4.8), we have

$$\rho^{k-i} - \rho^{k-j} - \rho^j + \rho^{i-1} = 0.$$

If k is odd, then $j \leq \frac{k-1}{2}$. Hence, we have

$$k-i > k-j > j > i-1.$$

If k is even, then $j \leq \frac{k}{2}$. Hence, we have

$$k - i > k - j \geq j > i - 1.$$

Case 3: $i > j$ and $j < \frac{k}{2}$. Here, from (4.8), we have

$$\rho^{k-j} + \rho^j - \rho^{k-i} - \rho^{i-1} = 0.$$

From $j < \frac{k}{2}$, we have

$$k - j > j.$$

Because $i \leq \frac{k+1}{2}$ (resp., $i \leq \frac{k}{2}$) if k is odd (resp., even), we have

$$k - j > k - i \geq i - 1.$$

Case 4: $i > j$ and $j = \frac{k}{2}$. Here, from (4.8), we have

$$2\rho^j - \rho^{i-1} - \rho^{2j-i} = 0.$$

If $j = i - 1$, we have

$$\rho^{i-1} - \rho^{i-2} = 0.$$

If $j < i - 1$, we have

$$\rho^{i-1} - 2\rho^j + \rho^{2j-i} = 0$$

and

$$i - 1 > j, \quad i - 1 > 2j - i.$$

Therefore, because ρ is not equal to zero, in each case, ρ is an algebraic integer over $\mathbf{Q}$. This implies a contradiction for the same reason as in Part 1. Hence, $f_i(t)$ and $g_j(t)$ do not have a common zero. $\square$

References

- [1] Y. Antolin and L. Ciobanu, *Geodesic growth in right-angled and even Coxeter groups*, European J. Combin. **34** (2013), 859–874.

- [2] J. S. Birman, *Braids, Links, and Mapping Class Groups*, Ann. Math. Studies **82**, Princeton Univ. Press, Princeton, 1975.
- [3] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Math. **231**, Springer, New York, 2005.
- [4] N. Bourbaki, *Groupes et Algèbres de Lie*, Chapitres IV, V et VI, *Eléments de Mathématique XXXIV*, Hermann, Paris, 1968.
- [5] J. W. Cannon, *The combinatorial structure of cocompact discrete hyperbolic groups*, Geom. Dedicata **16** (1984), 123–148.
- [6] J. W. Cannon and Ph. Wagreich, *Growth functions of surface groups*, Math. Ann. **293** (1992), 239–257.
- [7] R. Charney, *Geodesic automation and growth functions for Artin groups of finite type*, Math. Ann. **301** (1995), 307–324.
- [8] R. Charney and J. Meier, *The language of geodesics for Garside groups*, Math. Z. **248** (2004), 495–509.
- [9] P. de la Harpe, *Topics in Geometric Group Theory*, Chicago Lectures in Math., The Univ. Chicago Press, 2000.
- [10] D. B. A. Epstein with J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson and W. P. Thurston, *Word Processing in Groups*, Jones and Bartlett Publishers, Boston, 1992.
- [11] D. B. A. Epstein, A. R. Iano-Fletcher and U. Zwick, *Growth functions and automatic groups*, Experiment. Math. **5** (1996), 297–315.
- [12] W. J. Floyd and S. P. Plotnick, *Growth functions on Fuchsian groups and the Euler characteristic*, Invent. Math. **88** (1987), 1–29.
- [13] M. Fujii, *The spherical growth series for pure Artin groups of dihedral type*, Publ. Res. Inst. Math. Sci. **49** (2013), 497–530.
- [14] M. Fujii, *The growth rates for pure Artin groups of dihedral type*, European J. Combin. **40** (2014), 108–115.
- [15] M. Fujii and T. Satoh, *The growth series for pure Artin monoids of dihedral type*, RIMS Kôkyûroku Bessatsu **B48** (2014), 111–130.
- [16] M. Fujii, *Computation of the spherical growth series of finitely generated groups and monoids by using automata*, in *Handbook of Group Actions*, Vol. 2, Adv. Lect. Math. **32** (2015), 479–521, edited by L. Ji, A. Papadopoulos and S.-T. Yau, Higher Educational and International Press, Beijing and Boston.
- [17] D. F. Holt and S. Rees, *Artin groups of large type are shortlex automatic with regular geodesics*, Proc. Lond. Math. Soc. **104** (2012), 486–512.
- [18] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Univ. Press, Cambridge, 1990.
- [19] D. C. Kozen, *Automata and Computability*, Undergraduate Texts in Computer Science, Springer, 1997.
- [20] J. Lewin, *The growth function of a graph group*, Comm. Algebra **17** (1989), 1187–1191.
- [21] J. Loeffler, J. Meier and J. Worthington, *Graph products and Cannon pairs*, Internat. J. Algebra Comput. **12(7)** (2002), 747–754.
- [22] J. Mairesse and F. Mathéus, *Growth series for Artin groups of dihedral type*, Internat. J. Algebra Comput. **16** (2006), 1087–1107.
- [23] W. D. Neumann and M. Shapiro, *Automatic structures, rational growth, and geometrically finite hyperbolic groups*, Invent. Math. **120** (1995), 259–287.
- [24] G. A. Noskov, *Growth of certain non-positively curved cube groups*, European J. Combin. **21** (2000), 659–666.

- [25] R. Randell, *The fundamental group of the complement of a union of complex hyperplanes*, Invent. Math. **69** (1982), 103–108.
- [26] L. Sabalka, *Geodesics in the braid group on three strands*, in Group Theory, Statistics and Cryptography, Contemp. Math. **360** (2004), 133–150.
- [27] L. Solomon, *The orders of the finite Chevalley groups*, J. Algebra **3** (1966), 376–393.
- [28] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc. **80**, 1968.
- [29] M. Stoll, *Rational and transcendental growth series for the higher Heisenberg groups*, Invent. Math. **126** (1996), 85–109.