

The Toeplitz operators on the Bergman spaces with radial symbol

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Abstract

In this paper, we study the invertible (and Fredholm) Toeplitz operators T_φ on the Bergman spaces with radial symbol.

Key Words and Phrases : Bergman spaces, Toeplitz operator, closed range, invertible operator, Fredholm operator, radial symbol.

§1. Introduction

Let D be the open unit disk in complex plane C . Let $H(D)$ be the space of all analytic functions on D .

The space $L^p(dA(z))$ is defined to be the space of Lebesgue measurable functions f on D such that

$$\| f \|_{L^2(dA(z))} = \left\{ \int_D |f(z)|^2 dA(z) \right\}^{\frac{1}{2}} < +\infty,$$

where $dA(z)$ denote the area measure on D . The Bergman space $L_a^2(dA(z))$ is defined by

$$L_a^2(dA(z)) = H(D) \cap L^2(dA(z)).$$

For $\varphi \in L^2(dA(z))$, the Toeplitz operator T_φ with symbol φ is defined on $L_a^2(dA(z))$ by

$$T_\varphi f = P(\varphi f),$$

where $P(f)(z) = \int_D \frac{f(w)}{(1 - \overline{w}z)^2} dA(w)$.

Let X, Y be Banach spaces and let T be a linear operator from X into Y . Then T is called to be bounded below from X to Y if there exists a positive constant $C > 0$ such that $\|Tf\|_Y \geq C \|f\|_X$ for all $f \in X$, where $\|\cdot\|_X, \|\cdot\|_Y$ be the norm of X, Y , respectively.

Let $C(H)$ be the space of the compact operator on the Hilbert space H . If H is a Hilbert space, then a bounded operator T is a Fredholm operator if and only if there exists a bounded operator B such that $TB - I, BT - I \in C(H)$. And a bounded operator T is a Left (Right) Fredholm operator if and only if there exists a bounded operator B such that $BT - I \in C(H) (TB - I \in C(H))$.

The Berezin transform of the Toeplitz operators T_φ is given by

$$\tilde{\varphi}(z) = \widetilde{T_\varphi}(z) = \langle T_\varphi k_z, k_z \rangle$$

, where $k_z(w) = \frac{1 - |z|^2}{(1 - z\overline{w})^2}$.

In [4], B.Korenblum and K.Zhu proved the following result.

Theorem A. Suppose φ is a bounded and radial, that is $\varphi(re^{i\theta}) = \varphi(r)$. Then the following conditions are equivalent :

- (1) T_φ is compact.
- (2) $\tilde{\varphi}(z) \rightarrow 0$ as $|z| \rightarrow 1^-$.
- (3) $\lim_{x \rightarrow 1^-} \frac{1}{1-x} \int_x^1 \varphi(r) dr = 0$.

In [12], N.Zorboska generalized this theorem.

Theorem B. Let φ be a radial function in $L^1(D)$, and that T_φ be bounded on L_a^2 . If $f(r) - \frac{1}{1-r} \int_r^1 \varphi(s) s ds$ is bounded for $0 \leq r < 1$, then the following conditions are equivalent :

- (1) T_φ is compact.
- (2) $\tilde{\varphi}(z) \rightarrow 0$ as $|z| \rightarrow 1^-$.

In [10], the following theorem is well-known.

Theorem C. *Suppose φ is a bounded and nonnegative function. Then the following conditions are equivalent :*

- (1) T_φ is bounded below.
- (2) There is a constant $C > 0$ such that

$$\int_D |f(z)|^2 \varphi(z) dA(z) \geq C \int_D |f(z)|^2 dA(z),$$

for all $f \in L_a^2(dA(z))$.

The following theorem is well-known(see [10]).

Theorem D. *Suppose that $\varphi \in C(\overline{D})$. Then the following conditions are equivalent :*

- (1) T_φ is Fredholm.
- (2) φ is nonvanishing on the unit circle.

The study of Toeplitz operators on the Bergman spaces and Hardy space have been studied by many authors. In this paper, we study when the Toeplitz operators T_φ on the Bergman spaces with radial symbol is invertible or Fredholm.

§2. Statement of main results.

To prove our main theorem, we need the following.

Proposition 1. *Suppose φ is a bounded and radial function. Then the following are equivalent:*

- (1) T_φ is bounded below on $L_a^2(dA(z))$.
- (2) T_φ is an invertible operator on $L_a^2(dA(z))$.
- (3) There exists a positive constant $K > 0$ such that

$$(n+1) \left| \int_0^1 \varphi(t) t^{2n+1} dt \right| \geq K,$$

for $n = 0, 1, 2, \dots$.

Using the above proposition, we can prove the following result.

Proposition 2. Suppose φ is a bounded and radial function. If there exists a positive constant $C > 0$ such that $\frac{1}{1-x} \int_x^1 \varphi(t)dt \geq C$ ($x \in [0, 1)$) or $\frac{1}{1-x} \int_x^1 \varphi(t)dt \leq -C$ ($x \in [0, 1)$), then T_φ is an invertible operator on $L_a^2(dA(z))$.

Moreover, we can also prove the following result.

Proposition 3. Suppose $\varphi \in C([0, 1))$ is a bounded and radial real-valued function. If there exists a positive constant $C > 0$ such that $\inf_{x \in [0, 1)} \frac{1}{1-x} \left| \int_x^1 \varphi(t)dt \right| \geq C$, then T_φ is an invertible operator on $L_a^2(dA(z))$.

The following is our main result.

Theorem 4. Suppose $\varphi \in C(\overline{D})$ is a bounded and radial, and $\varphi \geq 0$ (or $\varphi \leq 0$). Then the following are equivalent:

- (1) T_φ is bounded below on $L_a^2(dA(z))$
- (2) T_φ is an invertible operator on $L_a^2(dA(z))$
- (3) There exists a positive constant $C > 0$ such that

$$\inf_{z \in D} |\tilde{\varphi}(z)| \geq C.$$

- (4) There exists a positive constant $C > 0$ such that

$$\inf_{x \in [0, 1)} \left| \frac{1}{1-x} \int_x^1 \varphi(t)dt \right| \geq C.$$

- (5) There exists a positive constant $K > 0$ such that

$$(n+1) \left| \int_0^1 \varphi(t)t^{2n+1}dt \right| \geq K,$$

for $n = 0, 1, 2, \dots$.

Remark 5. There exists an example that T_φ is invertible on $L_a^2(dA(z))$ and that (4) of the above theorem does not hold. For example, let $\varphi(t) = t - \frac{7}{10}$. Since there exists a positive constant $K > 0$ such that

$$(n+1) \left| \int_0^1 \varphi(t)t^{2n+1}dt \right| \geq K,$$

for $n = 0, 1, 2, \dots$, T_φ is invertible on $L_a^2(dA(z))$. But for $x = \frac{2}{5}$, an elementary calculation implies that $\frac{1}{1-x} \int_x^1 \varphi(t)dt = 0$. $\square$

Using Theorem D and Theorem 4, we can prove the following.

Theorem 6. *Suppose $\varphi \in \mathcal{C}(\overline{D})$ is a bounded and radial, and $\varphi \geq 0$ (or $\varphi \leq 0$). Then the following are equivalent:*

- (1) T_φ is an invertible operator on $L_a^2(dA(z))$
- (2) T_φ is a Fredholm operator on $L_a^2(dA(z))$.
- (3) φ is nonvanishing on the unit circle.

The following is well-known(see [2]).

Proposition E. *Suppose φ is a bounded function. Then the following are equivalent:*

- (1) T_φ is a Left Fredholm operator on $L_a^2(dA(z))$
- (2) $\liminf_{n \rightarrow \infty} \|T_\varphi e_n\|_{L_a^2} > 0$, where e_n be an orthonormal basis of L_a^2 .

When φ is a bounded and radial function, T_φ is a normal operator. So we see the following.

Proposition F. *Suppose φ is a bounded and radial function. Then the following are equivalent:*

- (1) T_φ is a Fredholm operator on $L_a^2(dA(z))$
- (2) $\liminf_{n \rightarrow \infty} (n+1) \left| \int_0^1 \varphi(t) t^{2n+1} dt \right| > 0$.

The problem which we must consider next is following.

Problem 7. *Suppose φ is a bounded and radial function. Then the following are equivalent:*

- (1) T_φ is a Fredholm operator on $L_a^2(dA(z))$
- (2) $\liminf_{n \rightarrow \infty} \|T_\varphi e_n\|_{L_a^2} > 0$, where e_n be an orthonormal basis of L_a^2 .
- (3) $\liminf_{n \rightarrow \infty} (n+1) \left| \int_0^1 \varphi(t) t^{2n+1} dt \right| > 0$.
- (4) $\liminf_{x \rightarrow 1^-} \frac{1}{1-x} \left| \int_x^1 \varphi(t) dt \right| > 0$.
- (5) $\liminf_{|z| \rightarrow 1^-} |\tilde{\varphi}(z)| > 0$.

The following results were obtained.

Theorem 8. *Suppose φ is a bounded and positive radial function. If $\liminf_{|z| \rightarrow 1^-} \varphi(|z|) > 0$, then T_φ is a Fredholm operator on $L_a^2(dA(z))$.*

Theorem 9. *Suppose φ is a bounded and radial function and that $\lim_{x \rightarrow 1^-} \frac{1}{1-x} \int_x^1 \varphi(t) dt = A$.*

Then T_φ is a Fredholm operator on $L_a^2(dA(z))$ if and only if

$$\liminf_{x \rightarrow 1^-} \frac{1}{1-x} \left| \int_x^1 \varphi(t) dt \right| = \lim_{x \rightarrow 1^-} \frac{1}{1-x} \left| \int_x^1 \varphi(t) dt \right| > 0 .$$

Theorem 10. *Suppose φ is a bounded radial function and that $\lim_{x \rightarrow 1^-} \frac{1}{1-x} \int_x^1 \varphi(t) dt = A$. Then T_φ is a Fredholm operator on $L_a^2(dA(z))$ if and only if $\liminf_{|z| \rightarrow 1^-} |\tilde{\varphi}(z)| > 0$.*

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