

# Inversion Formulas arising in Inverse Boundary Value Problems

MASARU Ikehata (池 島 優)

Department of Applied Physics, School of Engineering  
Nagoya University,  
Chikusa-Ku, Nagoya 464-01, Japan

**1. Results.** We formulate two inverse problems, which are analogous to the inverse conductivity problem [10].

Notation.  $\Omega$  is a bounded domain of  $\mathbb{R}^2$  with smooth boundary  $\partial\Omega$ ;  $ds$  is the standard measure on  $\partial\Omega$ ;  $\nu$  is the unite outer normal vector field on  $\partial\Omega$ ;  $X = \{H^{1/2}(\partial\Omega)\}^2$  and  $Y = H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ ;  $B(X, X^*)$  is the Banach space of all bounded linear maps from  $X$  to its dual  $X^*$  and  $B(Y, Y^*)$  that of all bounded linear maps from  $Y$  to its dual  $Y^*$ ;  $\nabla \mathbf{u}$  is the Jacobian matrix of a vector valued function  $\mathbf{u} = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$  on  $\Omega$  and  $Sym \nabla \mathbf{u}$  its symmetric part;  $\nabla^2 w$  is the Hessian matrix of a scalar function  $w$  on  $\Omega$ ;  $\mathbf{a} \otimes \mathbf{b} = (a_i b_j)$  for two vectors  $\mathbf{a} = (a_i)$ ,  $\mathbf{b} = (b_j)$ ;  $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ;  $A_{11} = \mathbf{e}_1 \otimes \mathbf{e}_1$ ;  $A_{12} = \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1$ ;  $A_{22} = \mathbf{e}_2 \otimes \mathbf{e}_2$ ;  $\partial_z = \frac{\partial}{\partial x_2} - z \frac{\partial}{\partial x_1}$  for each  $z \in \mathbb{C}$ .

Let  $\mathbf{C} = (\mathbf{C}_{ijkl}(x))_{i,j,k,l=1,2}$  be a fourth-order tensor field over  $\Omega$  with components  $\mathbf{C}_{ijkl} \in L^\infty(\Omega)$ . We denote by  $\mathbf{C}(x)A$  the  $2 \times 2$ -matrix  $(\sum_{k,l} \mathbf{C}_{ijkl}(x)a_{kl})$  for each  $x \in \Omega$  and  $2 \times 2$ -matrix  $A = (a_{kl})$ . We call  $\mathbf{C}$  an elasticity tensor field if

$$\mathbf{C}_{ijkl} = \mathbf{C}_{klij} = \mathbf{C}_{lkij}$$

hold for each  $i, j, k, l = 1, 2$  and there exists a positive number  $\delta$  such that

$$\mathbf{C}(x)A \cdot A \equiv \sum \mathbf{C}_{ijkl}(x)a_{kl}a_{ij} \geq \delta|A|^2$$

holds for almost all  $x \in \Omega$  and all real symmetric  $2 \times 2$ -matrix  $A = (a_{ij})$ .

For each elasticity tensor field  $\mathbf{C}$  we define  $\mathcal{L}_{\mathbf{C}}$ , which is a second order system of partial differential operators acting  $\{H^1(\Omega)\}^2$ , via

$$\mathcal{L}_{\mathbf{C}} \mathbf{u} = \begin{pmatrix} \sum \frac{\partial}{\partial x_j} \{ \mathbf{C}_{i1kl}(x) \frac{\partial u^k}{\partial x_l} \} \\ \sum \frac{\partial}{\partial x_j} \{ \mathbf{C}_{i2kl}(x) \frac{\partial u^k}{\partial x_l} \} \end{pmatrix}, \mathbf{u} = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \in \{H^1(\Omega)\}^2.$$

The associated Dirichlet-to-Neumann map  $\Pi_{\mathbf{C}} \in B(X, X^*)$  is defined by

$$\Pi_{\mathbf{C}}(\varphi) = \{ \mathbf{C}(x) Sym \nabla \mathbf{u} \} \nu|_{\partial\Omega}, \varphi \in X,$$

where  $\mathbf{u} \in \{H^1(\Omega)\}^2$  is the unique solution to

$$\mathcal{L}_{\mathbf{C}} \mathbf{u} = 0 \quad \text{in } \Omega$$

$$\mathbf{u}|_{\partial\Omega} = \varphi.$$

$\Pi_{\mathbf{C}}(\varphi)ds$  is the force exerted across  $ds$  which deforms  $\Omega$  into  $\Omega + \mathbf{u}$ .

On the other hand, for each elasticity tensor field  $\mathbf{M}$  we define  $L_{\mathbf{M}}$ , which is a fourth-order partial differential operator acting on  $H^2(\Omega)$ , via

$$L_{\mathbf{M}}w = \sum \frac{\partial^2}{\partial x_i \partial x_j} \{ \mathbf{M}_{ijkl}(x) \frac{\partial^2 w}{\partial x_k \partial x_l} \}, w \in H^2(\Omega).$$

The associated Dirichlet-to-Neumann map  $\Pi_{\mathbf{M}}^* \in B(Y, Y^*)$  is defined by

$$\Pi_{\mathbf{M}}^*(\varphi) = \left( -\left\{ \frac{\partial}{\partial \tau} \mathbf{M}_{\tau}(w) + \mathbf{Q}(w) \right\} \right. \\ \left. \mathbf{M}_{\nu}(w) \right) |_{\partial\Omega}, \varphi \in Y,$$

where  $w \in H^2(\Omega)$  is the unique solution to

$$L_{\mathbf{M}}w = 0 \quad \text{in } \Omega$$

$$\left( \begin{array}{c} w \\ \frac{\partial w}{\partial \nu} \end{array} \right) |_{\partial\Omega} = \phi,$$

$$\mathbf{M}_{\nu}(w) = \mathbf{M}(x) \nabla^2 w \cdot \nu \otimes \nu, \mathbf{M}_{\tau}(w) = \mathbf{M}(x) \nabla^2 w \cdot \nu \otimes \tau,$$

$$\mathbf{Q}(w) = \sum \frac{\partial}{\partial x_{\beta}} \{ (\mathbf{M}(x) \nabla^2 w)_{\alpha\beta} \} \nu_{\alpha}, \tau = \begin{pmatrix} -\nu_2 \\ \nu_1 \end{pmatrix}.$$

$\Pi_{\mathbf{M}}^*(\phi)$  is the external force applied to  $\partial\Omega$  which deforms  $\Omega$  into the graph of  $w$ ;  $\mathbf{M}_{\nu}(w)$  is the bending moment; the first component of  $\Pi_{\mathbf{M}}^*(\phi)$  is the vertical reaction at  $\partial\Omega$ .

This talk is concerned with the following:

#### Inverse Problems.

- I. Determine  $\mathbf{C}$  from  $\Pi_{\mathbf{C}}$ ;
- II. Determine  $\mathbf{M}$  from  $\Pi_{\mathbf{M}}^*$ .

The elasticity tensor field is said to be isotropic if there exist  $\lambda, \mu \in L^{\infty}(\Omega)$ , which are called the Lamé parameters, such that

$$\mathbf{C}(x)A = \lambda(x)\text{Trace}(A)I_2 + 2\mu(x)A$$

holds for almost all  $x \in \Omega$  and all real symmetric  $2 \times 2$ -matrix  $A$ . Since isotropic  $\mathbf{C}$  uniquely determines its Lamé parameters we write  $\mathbf{C}_{(\lambda, \mu)}$  and  $\Pi_{(\lambda, \mu)}$  instead of  $\mathbf{C}$  and  $\Pi_{\mathbf{C}}$ , respectively.

The first problem for isotropic  $\mathbf{C}$  was taken up by the author [2], Akamatsu-Nakamura-Steinberg[1], Nakamura-Uhlmann [8]. In particular, Nakamura-Uhlmann [8] proved that if  $\lambda$  and  $\mu$  are smooth on  $\bar{\Omega}$  and sufficiently close to constants, then  $\Pi_{(\lambda, \mu)}$  uniquely determines  $(\lambda, \mu)$ . In [9] they treated the problem of determining  $D^{\alpha} \mathbf{C}|_{\partial\Omega}$ ,  $|\alpha| = 0, 1, \dots$  from  $\Pi_{\mathbf{C}}$  modulo smoothing operators on  $\partial\Omega$ , where  $\mathbf{C}$  is not necessary isotropic and restricted to being in a class of anisotropic elasticity tensor fields, respectively.

The second problem for isotropic  $\mathbf{M}$  was taken up by the author [3]. In [3] it is proved that if the Lamé parameters  $\lambda, \mu$  of  $\mathbf{M}$  are smooth and sufficiently close to constants on  $\bar{\Omega}$ , then  $\Pi_{(\lambda, \mu)}^*$  together with  $D^\alpha \lambda|_{\partial\Omega}$ ,  $|\alpha| = 0, 1$  and  $D^\beta \mu|_{\partial\Omega}$ ,  $|\beta| = 0, 1, 2, 3$  uniquely determine  $(\lambda, \mu)$ .

In this talk first we shall point out that I and II are equivalent to each other on the simply connected  $\Omega$ ; second we consider the Fréchet derivative  $d\Pi_{\mathbf{C}}$  and  $d\Pi_{\mathbf{M}}^*$  at anisotropic  $\mathbf{C}$  and  $\mathbf{M}$ , respectively; we shall study a relationship between them and give a characterization of the injectivity of  $d\Pi_{\mathbf{C}}$  by the Stroh eigenvalues of  $\mathbf{C}$ .

For each elasticity tensor field  $\mathbf{C}$  denote by  $[\mathbf{C}]$  the symmetric  $3 \times 3$ -matrix

$$[\mathbf{C}] = \begin{pmatrix} C_{1111} & C_{1112} & C_{1122} \\ C_{1211} & C_{1212} & C_{1222} \\ C_{2211} & C_{2212} & C_{2222} \end{pmatrix}.$$

We can define the transform  $\mathbf{C}^*$  of  $\mathbf{C}$  characterized by

$$[\mathbf{C}]^{-1} = P J [\mathbf{C}^*] J P$$

where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the detail of the properties of this transform we refer the readers to [5] and [6]. It follows from the definition that  $(\mathbf{C}^*)^* = \mathbf{C}$  and  $(\mathbf{C}_{(\lambda, \mu)})^* = \mathbf{C}_{(\lambda^*, \mu^*)}$  with  $\lambda^* = -\frac{\lambda}{4\mu(\lambda+\mu)}$ ,  $\mu^* = \frac{1}{4\mu}$ . We prove in §2

**Theorem 1[6, Theorem A].** Let  $\Omega$  be simply connected. Then

$$\Pi_{\mathbf{C}_1} = \Pi_{\mathbf{C}_2} \iff \Pi_{\mathbf{C}_1}^* = \Pi_{\mathbf{C}_2}^*.$$

As a corollary we have immediately

**Corollary 2.** Let  $\Omega$  be simply connected. Then

$$\Pi_{(\lambda_1, \mu_1)} = \Pi_{(\lambda_2, \mu_2)} \iff \Pi_{(\lambda_1^*, \mu_1^*)}^* = \Pi_{(\lambda_2^*, \mu_2^*)}^*.$$

This connects the work done by Nakamura-Uhlmann [8] to that done by the author [3]. Theorem 1 shows the equivalence of I and II on any simply connected  $\Omega$ .

The following is a linearized version of Theorem 1.

**Theorem 3[6, Theorem C].** Let  $\Omega$  be simply connected and  $\mathbf{M} = \mathbf{C}^*$ . Then  $\ker d\Pi_{\mathbf{C}}$  is topologically linear isomorphic to  $\ker d\Pi_{\mathbf{M}}^*$  under the relative topology from  $L^\infty(\Omega)$ .

In the theorem stated below it is not assumed that  $\Omega$  is simply connected.

**Theorem 4[6, Theorem D].** Let  $\mathbf{C}$  be homogeneous and  $\mathbf{M} = \mathbf{C}^*$ . Then,

$$\ker d\Pi_{\mathbf{C}} = 0 \iff \ker d\Pi_{\mathbf{M}}^* = 0 \iff D(P_{\mathbf{M}}) \neq 0$$

where  $D(P_{\mathbf{M}})$  is the discriminant of the polynomial

$$P_{\mathbf{M}}(\tau) = \mathbf{M} \begin{pmatrix} 1 \\ \tau \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \tau \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \tau \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \tau \end{pmatrix};$$

there is an explicit formula of the left inverse of  $d\Pi_M^*$ .

This is proved under  $C_{1112} = C_{1222} = 0$  in [5] and therein we wrote down explicitly the left inverse of  $d\Pi_C$  for such  $C$  with  $D(P_{C^*}) \neq 0$ ; the author does not have the explicit formula of the left inverse  $d\Pi_C$  for general  $C$  with  $D(P_{C^*}) \neq 0$ ; the roots of the algebraic equation  $P_{C^*}(\tau) = 0$  are called the Stroh eigenvalues of  $C$  (see [5]).

In the next section we will give the proofs of Theorems 1 ~ 4.

**2 Proofs.** Throughout this section  $(\cdot)_{,j}$  stands for partial differentiation with respect to  $x_j$  for each  $j = 1, 2$ .

**Proof of Theorem 1.** We study the relationship between three function spaces

$$\mathcal{P}_C \equiv \{u \in H^1(\Omega, \mathbb{C}^2) | \mathcal{L}_C u = 0 \text{ in } \Omega\},$$

$$\mathcal{S}_{C^{-1}} \equiv \{s \in L^2(\Omega, \text{Sym}(\mathbb{C}^2)) | \sum_{\beta} s_{\alpha\beta,\beta} = 0, 2(C^{-1}s)_{12,12} = (C^{-1}s)_{11,22} + (C^{-1}s)_{22,11} \text{ in } \Omega\}.$$

$$\mathcal{A}_{C^*} \equiv \{w \in H^2(\Omega, \mathbb{C}) | L_{C^*} w = 0 \text{ in } \Omega\}.$$

We can easily check that the map

$$f : \mathcal{P}_C \ni u \longmapsto s = C \text{Sym} \nabla u \in \mathcal{S}_{C^{-1}}$$

is well defined. On the other hand, for the check of the well definedness of the map

$$g : \mathcal{A}_{C^*} \ni w \longmapsto s = -J' \nabla^2 w J' \in \mathcal{S}_{C^{-1}}, J' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we need the following

**Lemma 1[6, Lemma A].** For any function  $w$ , put

$$s = -J' \nabla^2 w J'.$$

Then

$$\sum_{\beta} s_{\alpha\beta,\beta} = 0$$

and

$$(C^{-1}s)_{11,22} + (C^{-1}s)_{22,11} - 2(C^{-1}s)_{12,12} = L_{C^*} w.$$

We claim

**Lemma 2[6, Lemma B].** Let  $\Omega$  be simply connected. Then both  $f$  and  $g$  are surjective.

The proof of this lemma is based on two facts stated below.

Let  $E = (E_{ij}(x))_{i,j=1,2}$  be a second-order symmetric tensor field on  $\Omega$ . Then if

$$2E_{12,12} = E_{11,22} + E_{22,11}$$

holds, there exists a vector valued function  $\mathbf{u}$  such that

$$\mathbf{E} = \text{Sym} \nabla \mathbf{u},$$

and vice versa; the equation

$$\sum_{\beta} s_{\alpha\beta,\beta} = 0$$

is equivalent to

$$d(s_{11}dx_2 - s_{12}dx_1) = 0, d(s_{21}dx_2 - s_{22}dx_1) = 0.$$

Now we can give the proof of Theorem 1. Applying Green's theorem to  $\Pi_{\mathbf{C}_1} = \Pi_{\mathbf{C}_2}$  and using Lemma 2, we obtain that

$$\Pi_{\mathbf{C}_1} = \Pi_{\mathbf{C}_2}$$

$$\Longleftrightarrow$$

$$\forall \mathbf{u}_j \in \mathcal{P}_{\mathbf{C}_j} \quad \int_{\Omega} (\mathbf{C}_1 - \mathbf{C}_2) \text{Sym} \nabla \mathbf{u}_1 \cdot \text{Sym} \nabla \mathbf{u}_2 dx = 0$$

$$\Longleftrightarrow$$

$$\forall \mathbf{s}_j \in \mathcal{S}_{\mathbf{C}_j^{-1}} \quad \int_{\Omega} (\mathbf{C}_2^{-1} - \mathbf{C}_1^{-1}) \mathbf{s}_1 \cdot \mathbf{s}_2 dx = 0.$$

Here we note that for any  $\mathbf{H} = (\mathbf{H}_{ijkl}(x))$  satisfying  $\mathbf{H}_{ijkl} = \mathbf{H}_{klij} = \mathbf{H}_{klji}$  there exists a unique  $\mathbf{H}^\dagger$  such that

$$[\mathbf{H}^\dagger] = J[H]J.$$

Then we have

$$\mathbf{H} \mathbf{s}_1 \cdot \mathbf{s}_2 = \mathbf{H}^\dagger \nabla^2 w_1 \cdot \nabla^2 w_2$$

for  $\mathbf{s} = -J' \nabla^2 w_j J'$ . Furthermore, we see that

$$(\mathbf{C}^{-1})^\dagger = \mathbf{C}^*.$$

Therefore  $\Pi_{\mathbf{C}_1} = \Pi_{\mathbf{C}_2}$  is equivalent to

$$\forall w_j \in \mathcal{A}_{\mathbf{C}_j^*} \quad \int_{\Omega} \{(\mathbf{C}_2^{-1})^\dagger - (\mathbf{C}_1^{-1})^\dagger\} \nabla^2 w_1 \cdot \nabla^2 w_2 dx = 0$$

$$\Longleftrightarrow$$

$$\forall w_j \in \mathcal{A}_{\mathbf{C}_j^*} \quad \int_{\Omega} (\mathbf{C}_2^* - \mathbf{C}_1^*) \nabla^2 w_1 \cdot \nabla^2 w_2 dx = 0$$

$$\Longleftrightarrow$$

$$\Pi_{\mathbf{C}_1^*}^* = \Pi_{\mathbf{C}_2^*}^*. \quad \text{Q.E.D.}$$

Since the proof of Theorem 3 can be done in the same way we omit the proof.

**Proof of Theorem 4.** At first we prove

**Proposition 1** [4, Theorem A.] Let  $\mathbf{M}$  be homogeneous. Then

$$\ker d\Pi_{\mathbf{M}}^* = \mathbf{0} \iff D(P_{\mathbf{M}}) \neq 0;$$

there is an explicit formula of the left inverse of  $d\Pi_{\mathbf{M}}^*$  for such  $\mathbf{M}$ .

By this proposition we see that the set of all homogeneous elasticity tensor fields is divided into two groups. This classification just coincides with that done by Lekhnitskii[7].

**Proof of Proposition 1.** We can write  $P_{\mathbf{M}}(\tau)$  in the form

$$M_{2222}(\tau - \alpha)(\tau - \bar{\alpha})(\tau - \beta)(\tau - \bar{\beta})$$

with some  $\alpha, \beta$  satisfying  $\operatorname{Im} \alpha \cdot \operatorname{Im} \beta > 0$ . Hence

$$D(P_{\mathbf{M}}) \neq 0 \iff \alpha \neq \beta$$

and  $L_{\mathbf{M}}$  can be factorized as follows:

$$M_{2222} \partial_{\alpha} \partial_{\bar{\alpha}} \partial_{\beta} \partial_{\bar{\beta}}.$$

Hence if  $P_{\mathbf{M}}(z) = 0$ , the function

$$\exp\{-ic(x_1 + zx_2)\} \quad (c \in \mathbb{C})$$

is a solution of  $L_{\mathbf{M}}w = 0$ .

(i)  $\Leftarrow$

Assume  $\alpha \neq \beta$ . Let  $\xi \in \mathbb{R}^2 \setminus \{0\}$  and

$$\{z_1, z_2\} = \{\alpha, \bar{\alpha}\}, \{\alpha, \bar{\beta}\}, \{\alpha, \beta\}, \{\beta, \bar{\beta}\}, \{\bar{\alpha}, \beta\}, \{\bar{\alpha}, \bar{\beta}\}.$$

Then

$$\mathbf{E}_{\xi}(x; z_1, z_2) := \exp\left\{-i \frac{\xi_2 - z_1 \xi_1}{z_2 - z_1} (x_1 + z_2 x_2)\right\}$$

is a solution of  $L_{\mathbf{M}}w = 0$  and

$$\mathbf{E}_{\xi}(x; z_1, z_2) \mathbf{E}_{\xi}(x; z_2, z_1) = e^{-ix \cdot \xi}$$

holds. Let  $d\Pi_{\mathbf{M}}^*(\mathbf{H}) = 0$ . This is equivalent to

$$\int_{\Omega} \mathbf{H}(x) \nabla^2 u \cdot \nabla^2 v dx = 0$$

for any  $u, v$ ; the solutions of  $L_{\mathbf{M}}w = 0$  in  $\Omega$ . Substitue  $\mathbf{E}_{\xi}(x; z_1, z_2)$  and  $\mathbf{E}_{\xi}(x; z_2, z_1)$  for  $u, v$ . Then we obtain

$$\mathbf{S} \mathbf{A} \tilde{\mathbf{H}}(\xi) = 0$$

where

$$\mathbf{S} = \begin{pmatrix} 1 & \alpha + \bar{\alpha} & \alpha^2 + \bar{\alpha}^2 & \alpha\bar{\alpha} & \alpha\bar{\alpha}^2 + \alpha^2\bar{\alpha} & \alpha^2\bar{\alpha}^2 \\ 1 & \alpha + \bar{\beta} & \alpha^2 + \bar{\beta}^2 & \alpha\bar{\beta} & \alpha\bar{\beta}^2 + \alpha^2\bar{\beta} & \alpha^2\bar{\beta}^2 \\ 1 & \alpha + \beta & \alpha^2 + \beta^2 & \alpha\beta & \alpha\beta^2 + \alpha^2\beta & \alpha^2\beta^2 \\ 1 & \beta + \bar{\beta} & \beta^2 + \bar{\beta}^2 & \beta\bar{\beta} & \beta\bar{\beta}^2 + \beta^2\bar{\beta} & \beta^2\bar{\beta}^2 \\ 1 & \bar{\alpha} + \beta & \bar{\alpha}^2 + \beta^2 & \bar{\alpha}\beta & \bar{\alpha}\beta^2 + \bar{\alpha}^2\beta & \bar{\alpha}^2\beta^2 \\ 1 & \bar{\alpha} + \bar{\beta} & \bar{\alpha}^2 + \bar{\beta}^2 & \bar{\alpha}\bar{\beta} & \bar{\alpha}\bar{\beta}^2 + \bar{\alpha}^2\bar{\beta} & \bar{\alpha}^2\bar{\beta}^2 \end{pmatrix},$$

$$A\tilde{\mathbf{H}}(\xi) = \begin{pmatrix} \tilde{\mathbf{H}}A_{11} \cdot A_{11} \\ \tilde{\mathbf{H}}A_{11} \cdot A_{12} \\ \tilde{\mathbf{H}}A_{11} \cdot A_{22} \\ \tilde{\mathbf{H}}A_{12} \cdot A_{12} \\ \tilde{\mathbf{H}}A_{12} \cdot A_{22} \\ \tilde{\mathbf{H}}A_{22} \cdot A_{22} \end{pmatrix},$$

and

$$\tilde{\mathbf{H}}(\xi) = \int_{\Omega} e^{-ix \cdot \xi} \mathbf{H}(x) dx \quad (\xi \in \mathbb{R}^2).$$

Since

$$\det \mathbf{S} = -\{(\alpha - \bar{\alpha})(\alpha - \bar{\beta})(\alpha - \beta)(\bar{\alpha} - \bar{\beta})(\bar{\alpha} - \beta)(\bar{\beta} - \beta)\}^2 \quad ([4, \text{Lemma A)]},$$

we can conclude  $A\tilde{\mathbf{H}}(\xi) = 0$  and it is possible to write down the left inverse of  $d\Pi_{\mathbf{M}}^*(H)$  explicitly. The result is as follows. Put

$$C(\xi; \{z_1, z_2\}) := \left\{ \frac{(z_2 - z_1)^2}{(\xi_2 - z_1\xi_1)(\xi_2 - z_2\xi_1)} \right\}^2,$$

$$\mathbf{D}(\xi; \{z_1, z_2\}) := C(\xi; \{z_1, z_2\}) \int_{\partial\Omega} d\Pi_{\mathbf{M}}^*(\mathbf{H}) \left( \frac{u|_{\partial\Omega}}{\frac{\partial u}{\partial \nu}|_{\partial\Omega}} \right) \cdot \left( \frac{v|_{\partial\Omega}}{\frac{\partial v}{\partial \nu}|_{\partial\Omega}} \right) ds$$

for  $u = \mathbf{E}_{\xi}(x; z_1, z_2)$  and  $v = \mathbf{E}_{\xi}(x; z_2, z_1)$ . Then

$$\mathbf{D}(0; \{z_1, z_2\}) := \lim_{\xi \rightarrow 0} \mathbf{D}(\xi; \{z_1, z_2\})$$

exists and the left inverse of  $d\Pi_{\mathbf{M}}^*$  is given by the following formula:

$$A\tilde{\mathbf{H}}(\xi) = \mathbf{S}^{-1} \mathbf{D}(\xi)$$

where

$$\mathbf{D}(\xi) := \begin{pmatrix} \mathbf{D}(\xi; \{\alpha, \bar{\alpha}\}) \\ \mathbf{D}(\xi; \{\alpha, \bar{\beta}\}) \\ \mathbf{D}(\xi; \{\alpha, \beta\}) \\ \mathbf{D}(\xi; \{\beta, \bar{\beta}\}) \\ \mathbf{D}(\xi; \{\bar{\alpha}, \beta\}) \\ \mathbf{D}(\xi; \{\bar{\alpha}, \bar{\beta}\}) \end{pmatrix}.$$

(ii)  $\Rightarrow$

Let  $Im \ z \neq 0$ . We note that the identity

$$\nabla^2 w = \frac{1}{(\bar{z} - z)^2} \{ \partial_{\bar{z}}^2 w A'_{11} + (-\partial_z \partial_{\bar{z}} w A'_{12} + \partial_z^2 w A'_{22}) \} \quad ([4, 1.13])$$

holds for any scalar function  $w$  where

$$\begin{aligned} A'_{11} &= \begin{pmatrix} 1 \\ z \end{pmatrix} \otimes \begin{pmatrix} 1 \\ z \end{pmatrix}, \\ A'_{12} &= \begin{pmatrix} 1 \\ z \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix} + \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ z \end{pmatrix}, \\ A'_{22} &= \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix}. \end{aligned}$$

This yields

$$\mathbf{H}(x) \nabla^2 u \cdot \nabla^2 v = \frac{1}{(\bar{z} - z)^4} \sigma^t(\mathbf{H}A) \sigma \begin{pmatrix} \partial_{\bar{z}}^2 u \\ -\partial_z \partial_{\bar{z}} u \\ \partial_z^2 u \end{pmatrix} \cdot \begin{pmatrix} \partial_{\bar{z}}^2 v \\ -\partial_z \partial_{\bar{z}} v \\ \partial_z^2 v \end{pmatrix}$$

where

$$\begin{aligned} \sigma &= \begin{pmatrix} 1 & 2 & 1 \\ z & z + \bar{z} & \bar{z} \\ z^2 & 2z\bar{z} & \bar{z}^2 \end{pmatrix}, \\ \mathbf{H}A &= \begin{pmatrix} \mathbf{H}A_{11} \cdot A_{11} & \mathbf{H}A_{11} \cdot A_{12} & \mathbf{H}A_{11} \cdot A_{22} \\ & \mathbf{H}A_{12} \cdot A_{12} & \mathbf{H}A_{12} \cdot A_{22} \\ & & \mathbf{H}A_{22} \cdot A_{22} \end{pmatrix} \in Sym(\mathbb{R}^3). \end{aligned}$$

Now assume  $z = \alpha = \beta$ . If we take  $\mathbf{H}$  such that

$$\sigma^t(\mathbf{H}A) \sigma = \begin{pmatrix} \partial_z^3 \varphi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_{\bar{z}}^3 \bar{\varphi} \end{pmatrix} \quad (\iff \mathbf{H}A = 2Re\{ \partial_z^3 \varphi \begin{pmatrix} \bar{z}^2 \\ -2\bar{z} \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \bar{z}^2 \\ -2\bar{z} \\ 1 \end{pmatrix} \})$$

with some  $\varphi$  satisfying  $D^\alpha \varphi = 0$  for  $|\alpha| = 0, 1, 2$  on  $\partial\Omega$ , integration by parts tells us that

$$\begin{aligned} \int_{\Omega} \mathbf{H}(x) \nabla^2 u \cdot \nabla^2 v dx &= \frac{1}{(\bar{z} - z)^4} \int_{\Omega} (\partial_z^3 \varphi \partial_{\bar{z}}^2 u \partial_{\bar{z}}^2 v + \partial_{\bar{z}}^3 \bar{\varphi} \partial_z^2 u \partial_z^2 v) dx \\ &= \frac{2}{(\bar{z} - z)^4} \int_{\Omega} (\partial_z \varphi \partial_z \partial_{\bar{z}}^2 u \partial_z \partial_{\bar{z}}^2 v + \partial_{\bar{z}} \bar{\varphi} \partial_{\bar{z}} \partial_z^2 u \partial_{\bar{z}} \partial_z^2 v) dx = 0, \end{aligned}$$

where  $\partial_z^2 \partial_{\bar{z}}^2 u = \partial_z^2 \partial_{\bar{z}}^2 v = 0$  in  $\Omega$ . Since  $L_{\mathbf{M}} = \mathbf{M}_{2222} \partial_z^2 \partial_{\bar{z}}^2$ , we have  $\mathbf{H} \in \ker d\Pi_{\mathbf{M}}^*$ . Q.E.D.

In [4, Proposition A], we gave how to find such  $\varphi$  appeared above for each  $\mathbf{H} \in \ker d\Pi_{\mathbf{M}}^*$  when  $\Omega$  is simply connected.

**Proposition 2** [5, Theorem B.] Let  $\mathbf{C}$  be homogeneous. Then

$$D(P_{\mathbf{C}^*}) = 0 \implies \ker d\Pi_{\mathbf{C}} \neq 0.$$

**Proof.** By assumption, we can write  $P_{\mathbf{C}^*}(\tau)$  in the form

$$\mathbf{C}_{2222}^*(\tau - z)^2(\tau - \bar{z})^2$$

with  $\operatorname{Im} z \neq 0$ . Then we can show that

$$\mathbf{K}_z := \{\mathbf{H}|\sigma^t(\mathbf{H}\mathbf{A})\sigma = \begin{pmatrix} \partial_z^3 \varphi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_{\bar{z}}^3 \bar{\varphi} \end{pmatrix}, D^\alpha \varphi|_{\partial\Omega} = 0 \text{ for } |\alpha| = 0, 1, 2\} \subset \ker d\Pi_{\mathbf{C}}$$

and

$$\ker d\Pi_{\mathbf{C}} \subset \ker d\Pi_{\mathbf{C}^*}.$$

These are proved as follows.

First we show

$$\mathbf{H} \in \ker d\Pi_{\mathbf{C}}$$

if and only if

$$\int_{\Omega} \mathbf{H}(x) \operatorname{Sym} \nabla \mathcal{F}_{\mathbf{C}} \mathbf{w}_1 \cdot \operatorname{Sym} \nabla \mathcal{F}_{\mathbf{C}} \mathbf{w}_2 dx = 0$$

for any  $\mathbf{w}_j = (w_j^1, w_j^2)^t$  satisfying  $L_{\mathbf{C}} \cdot \mathbf{w}_j = 0$  in  $\Omega$  ( $j = 1, 2$ ), where

$$\mathcal{L}_{\mathbf{C}} \mathcal{F}_{\mathbf{C}} = \mathcal{F}_{\mathbf{C}} \mathcal{L}_{\mathbf{C}} \sim L_{\mathbf{C}} \cdot \mathbf{I}_2.$$

Second we write  $\mathbf{C}$  in terms of  $z$  in the form

$$\mathbf{C} \sim \mathbf{C}_z(\theta) \quad ([5, \text{Proposition 3}]),$$

where

$$A\mathbf{C}_z(\theta) \sim \begin{pmatrix} t^2 \\ -st \\ 2t(\theta - \frac{1}{2}) \\ 4t(1 - \theta + \frac{s^2}{4t}) \\ -s \\ 1 \end{pmatrix},$$

$$t = z \cdot \bar{z}, s = z + \bar{z}, \frac{s^2}{4t} < \theta < 1.$$

Note that we ignored the nonzero constant multiplication factor.

Third we show that for any  $\mathbf{w} = (w^1, w^2)$ , the factorization

$$\operatorname{Sym} \nabla \mathcal{F}_{\mathbf{C}} \mathbf{w} \sim \partial_{\bar{z}}^2 u A'_{11} - \frac{1}{2} \partial_z \partial_{\bar{z}} (u + v) A'_{12} + \partial_z^2 v A'_{22} \quad ([5, \text{Proposition 5}])$$

holds where

$$\begin{pmatrix} u \\ v \end{pmatrix} \sim \begin{pmatrix} \theta - \frac{s^2}{4t} & 2 - \theta - \frac{s^2}{4t} \\ -(2 - \theta - \frac{s^2}{4t}) & -(\theta - \frac{s^2}{4t}) \end{pmatrix} \begin{pmatrix} \partial_{\bar{z}} w_z \\ \partial_z w_{\bar{z}} \end{pmatrix},$$

$$\begin{pmatrix} w_z \\ w_{\bar{z}} \end{pmatrix} = \begin{pmatrix} 1 & z \\ 1 & \bar{z} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Fourth we show that  $H \in \ker d\Pi_C$  is equivalent to

$$\int_{\Omega} \sigma^t(HA)\sigma \begin{pmatrix} \partial_{\bar{z}}^2 u \\ -\frac{1}{2}\partial_z \partial_{\bar{z}}(u+v) \\ \partial_z^2 v \end{pmatrix} \cdot \begin{pmatrix} \partial_{\bar{z}}^2 u' \\ -\frac{1}{2}\partial_z \partial_{\bar{z}}(u'+v') \\ \partial_z^2 v' \end{pmatrix} dx = 0 \quad ([5, \text{Proposition 6}])$$

for any  $u, u', v, v'$ ; the solutions of  $L_C \cdot w = 0$  in  $\Omega$ . From this we obtain immediately  $K_z \subset \ker d\Pi_C$ . Finally put  $u = v$  and  $u' = v'$ . Then we obtain  $\ker d\Pi_C \subset \ker d\Pi_C^*$ . Q.E.D.

**Proposition 3 [6]** Let  $C$  be homogeneous. Then

$$D(P_C \cdot) \neq 0 \implies \ker d\Pi_C = \mathbf{O}.$$

**Proof.** Take a open ball  $B$  such that  $\Omega \subset B$ . Then

$$\ker d\Pi_C(\text{on } \Omega) \subset \ker d\Pi_C(\text{on } B)$$

by zero extension of  $H \in \ker d\Pi_C(\text{on } \Omega)$  outside  $\Omega$ . Since  $B$  is simply connected, Theorem 2 and Proposition 1 imply

$$\ker d\Pi_C(\text{on } B) \simeq \ker d\Pi_C^*(\text{on } B) = \mathbf{O}$$

and hence  $\ker d\Pi_C = \mathbf{O}$  on  $\Omega$ . Q.E.D.

This completes the proof of Theorem 4.

#### Acknowledgement

The author wishes to express his thanks to Prof. T. Suzuki for his sincere encouragement and to Prof. S. Saitoh for his careful reading of the original manuscript and suggesting some improvements.

#### REFERENCES

1. Akamatsu M., Nakamura G. and Steinberg S., *Identification of Lamé coefficients from boundary observations*, Inverse Problems **7** (1991), 335–354.
2. Ikehata M., *Inversion formulas for the linearized problem for an inverse boundary value problem in elastic prospection*, SIAM J. Appl. Math. **50** (1990), 1635–1644.
3. Ikehata M., *An inverse problem for the plate in the Love-Kirchhoff theory*, SIAM J. Appl. Math. **53** (1993), 942–970.
4. Ikehata M., *The linearization of the Dirichlet to Neumann map in anisotropic plate theory*, Inverse Problems (to appear).

5. Ikehata M., *The linearization of the Dirichlet to Neumann map in the anisotropic Kirchhoff-Love plate theory*, submitted.
6. Ikehata M., *A relationship between two Dirichlet to Neumann maps in anisotropic elastic plate theory*, submitted.
7. Lekhnitskii S. G., "Anisotropic Plates," Gordon and Breach, New York, 1968.
8. Nakamura G. and Uhlmann G, *Identification of Lamé parameters by boundary measurements*, Amer.J.Math. 115, 1161–1187.
9. Nakamura G. and Uhlmann G, *Inverse problems at the boundary for an elastic medium*, SIAM J.Math.Anal. (to appear).
10. Sylvester J. and Uhlmann G, *The Dirichlet to Neumann map and applications*, in "Inverse Problems in P.D.E.," SIAM, Philadelphia, 1990, pp. 101–139.