

Deformations of hyperbolic structures on the figure eight knot complement and an elliptic curve associated to them

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1. Introduction.

In this paper, we consider a hyperbolic structure on a manifold to be a (not necessarily complete) Riemannian metric of constant sectional curvature -1 . A hyperbolic 3-manifold is an orientable 3-manifold with a hyperbolic structure. It is well known that the complement of the figure eight knot K in the 3-dimensional sphere S^3 admits a complete, volume finite hyperbolic structure σ_∞ . By Mostow-Prasad rigidity, such a hyperbolic structure on $S^3 - K$ is unique. On the other hand, Thurston [11] analyzed flexibility of the hyperbolic structure σ_∞ by allowing incomplete hyperbolic structures on $S^3 - K$. In fact, he showed that deformations of σ_∞ on $S^3 - K$ (no longer complete) can be holomorphically parametrized by points in an open subset U of a complex affine plane curve C . Its is given in (1) below.

When the parameters of deformations get closer to the boundary ∂U or get into $C - U$, degenerations of the hyperbolic structures occur. There are finitely many degenerations which induce closed 3-manifolds with other geometric structures by Dehn filling along K . In fact, there are twenty such degenerations and such a closed 3-manifold is one of the following types : a Sol-manifold, a $\widetilde{\mathrm{PSL}}_2(\mathbf{R})$ -manifold, a Haken manifold which is decomposed into a $\widetilde{\mathrm{PSL}}_2(\mathbf{R})$ -manifold and a Euclidean manifold along an embedded torus, or a Euclidean orbifold.

We give explicitly a birational map from C to a non-singular plane cubic curve in Weierstrass form E which is given in (3) below. The curve E is what is called an elliptic curve defined over $\mathbf{Q}$. It is well known that any elliptic curve is an abelian group under an addition law. In this paper, we see that a concrete correspondence holds between the closed 3-manifolds mentioned above (except the $\widetilde{\mathrm{PSL}}_2\mathbf{R}$ -manifolds) and some points of finite order

on the elliptic curve E . Especially, the Haken manifolds above exactly correspond to the rational points of E which form the cyclic group of order four.

2. Deformations of hyperbolic structures and hyperbolic Dehn filling.

In this section, we briefly describe a concrete treatment of deformations of hyperbolic structures on 3-manifolds which is given by deforming ideal tetrahedra and also well-known results on the hyperbolic Dehn filling for the figure eight knot complement. (See Thurston [11] [12] [13], Cooper-Hodgson-Kerckhoff [1] and Neumann-Zagier [9].)

Each ideal tetrahedron S in the 3-dimensional hyperbolic space $\mathbf{H}^3$ is determined up to isometry by a complex number as follows (Fig. 1, see [13]). Take a 3-dimensional simplex in $\mathbf{H}^3$. Transform it by the unique orientation-preserving isometry of $\mathbf{H}^3$ which sends the first three vertices of the simplex to the points 0, 1 and ∞ in the upper half space model of $\mathbf{H}^3$. Its congruence class is determined by the position of the last vertex, which is at some point $z \in \mathbf{C} - \{0, 1\}$. This complex number z is a parameter of isometry classes of ideal tetrahedra. Then we write $S = S(z)$. The Euclidean triangle cut out of any vertex of $S(z)$ by a horosphere section is similar to the triangle in $\mathbf{C}$ with vertices 0, 1 and z . To each edge of $S(z)$ is associated one of the three numbers z , $(z-1)/z$ and $1/(1-z)$. Call this number the modulus of the edge. If z is on the real line, then the ideal tetrahedron $S(z)$ is flattened in a 2-dimensional hyperbolic subspace in $\mathbf{H}^3$. If $\text{Im}(z) > 0$, the map of the simplex preserves orientation. Then the ideal tetrahedron $S(z)$ with $\text{Im}(z) > 0$ is said to be positive. If $\text{Im}(z) < 0$, the map of the simplex reverses orientation. Then the ideal tetrahedron $S(z)$ with $\text{Im}(z) < 0$ is said to be negative. When we construct hyperbolic 3-manifolds by gluing the faces of ideal tetrahedra, we usually use positive ones. Note that $S(z)$, $S((z-1)/z)$, $S(1/(1-z))$, $S(1/z)$, $S(z/(z-1))$ and $S(1-z)$ are transformed to each other by orientation-preserving isometries. These orientation-preserving isometries change the order of the vertices of the simplex. On the other hand, an ideal tetrahedra $S(\bar{z})$ is transformed to $S(z)$ by the orientation-reversing isometry that is given by a composition of the inverse of the map of the simplex to $S(\bar{z})$ and the map of the simplex to $S(z)$.

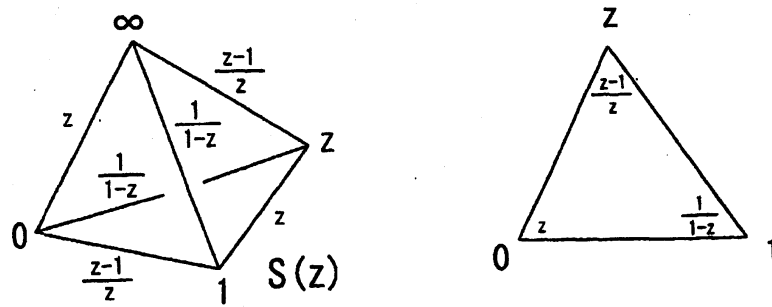


Fig. 1

In this paper, we use K to denote the figure eight knot in the 3-dimensional sphere S^3 . It is well known that its complement $S^3 - K$ admits a complete, volume finite hyperbolic structure σ_∞ . The hyperbolic 3-manifold $(S^3 - K, \sigma_\infty)$ is obtained by gluing the faces of two ideal tetrahedra $S(z_\infty)$ and $S(w_\infty)$ by orientation-reversing isometries according to the diagram in Fig. 2 (see [11]), where $z_\infty = w_\infty = -\omega^2$ and $\omega = e^{2\pi i/3}$. Then we write $(S^3 - K, \sigma_\infty) = S(z_\infty) \cup S(w_\infty)$. Since all the faces of any ideal tetrahedron are congruent, a complex $S(z) \cup S(w)$ ($\approx S^3 - K$) can be obtained by identifying their faces by orientation-reversing isometries with the same gluing pattern. The necessary and sufficient condition that $S(z) \cup S(w)$ is a smooth hyperbolic manifold is described by the algebraic equation $z(z-1)w(w-1) = 1$, since all the products of the corresponding moduli of the edges of $S(z)$ and $S(w)$ are equal to $e^{2\pi i}$ for each solution (z, w) of this equation. Then, denote by C the affine plane curve defined by this equation:

$$C = \{(z, w) \in \mathbb{C}^2 \mid z(z-1)w(w-1) = 1\}. \quad (1)$$

The complete hyperbolic structure σ_∞ corresponds to the point (z_∞, w_∞) on the affine plane curve C . Define

$$U := \{(z, w) \in \mathbb{C}^2 \mid z(z-1)w(w-1) = 1, \operatorname{Im}(z) > 0, \operatorname{Im}(w) > 0\}.$$

Then U is an open neighborhood of (z_∞, w_∞) in C and biholomorphic to the region

$$\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\} - \{z \in \mathbb{C} \mid z = \frac{1}{2} + yi \ (y \geq \frac{\sqrt{15}}{2})\}.$$

If $(z, w) \in U - \{(z_\infty, w_\infty)\}$, then $S(z) \cup S(w)$ gives a hyperbolic structure σ on $S^3 - K$ which is no longer complete. Let us denote the hyperbolic 3-manifold $(S^3 - K, \sigma)$ by $(S^3 - K)(z, w)$, and call it a deformation of $(S^3 - K, \sigma_\infty)$. Call the affine plane curve C the deformation curve of hyperbolic structures on the figure eight knot complement $S^3 - K$.

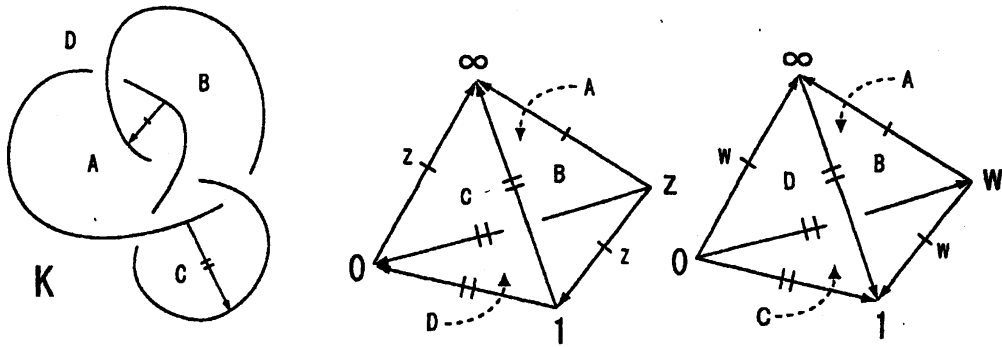


Fig. 2

Associated with each point $(z, w) \in U$ is a representation $\rho : \pi_1(S^3 - K) \rightarrow \text{PSL}_2(\mathbb{C})$. The image of ρ is not necessarily discrete subgroup of $\text{PSL}_2(\mathbb{C})$. Let ρ_∞ denote a representation which corresponds to (z_∞, w_∞) . Then the image $\rho_\infty(\pi_1(S^3 - K))$ is a torsion-free discrete subgroup of $\text{PSL}_2(\mathbb{C})$ and $\mathbb{H}^3/\rho_\infty(\pi_1(S^3 - K))$ is exactly the complete, volume finite hyperbolic 3-manifold $(S^3 - K, \sigma_\infty)$.

Let T be a boundary torus of a tubular neighborhood of K in S^3 and m and l be the standard meridian and longitude of K on T , respectively. Then for each coprime pair of integers (p, q) , one can obtain a closed 3-manifold $(S^3 - K)_{(p, q)}$ by performing Dehn filling along K with killing the homotopy class of the simple closed curve $pm + ql$. First, we remark the following.

- If $(p, q) = (\pm 1, 0)$, then $(S^3 - K)_{(p, q)}$ admits a spherical structure. In fact, both of $(S^3 - K)_{(\pm 1, 0)}$ are homeomorphic to S^3 .

Let $(z, w) \in U$. Then, associated to a representation $\rho : \pi_1(S^3 - K) \rightarrow \text{PSL}_2(\mathbb{C})$ corresponding to (z, w) is a representation $\mu : \pi_1(T) \rightarrow \mathbb{C}^*$. It can be checked that $\mu(m) = (1 - z)w$ and $\mu(l) = z^2(1 - z)^2$. Hence, if $(z, w) \neq (z_\infty, w_\infty)$, we define the generalized Dehn filling coefficient $(p, q) \in \mathbb{R}^2$ by the equation

$$p \log \{(1 - z)w\} + q \log \{z^2(1 - z)^2\} = 2\pi i, \quad (2)$$

where $\log$ is taken with $-\pi < \arg \leq \pi$. Note that there is a unique solution (p, q) to this equation, if $(z, w) \in U - \{(z_\infty, w_\infty)\}$, because $\log \{(1 - z)w\}$ is not a real multiple of $\log \{z^2(1 - z)^2\}$ if $(z, w) \in U - \{(z_\infty, w_\infty)\}$. If $(z, w) = (z_\infty, w_\infty)$, we take $(p, q) = \infty$. Then define a map

$$HDF : U \rightarrow \mathbb{R}^2 \cup \{\infty\}$$

by

$$HDF((z, w)) = (p, q),$$

where (p, q) is given as above.

The image of the map HDF includes every coprime pairs of integers except the following twenty pairs : $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 1, \pm 1)$, $(\pm 2, \pm 1)$, $(\pm 3, \pm 1)$, $(\pm 4, \pm 1)$ (see Sections 4.6 and 4.7 in [11]). Then, if (p, q) is a coprime pair of integers and is not any exception above, the closed 3-manifold $(S^3 - K)_{(p, q)}$ admits a hyperbolic structure $\bar{\sigma}$. This hyperbolic structure $\bar{\sigma}$ on $(S^3 - K)_{(p, q)}$ is the completion of the hyperbolic structure σ of $(S^3 - K)(z, w)$, where (z, w) is an inverse image of (p, q) by HDF . The twenty exceptions above do not have hyperbolic structures. If $(p, q) = (\pm 1, 0)$, then $(S^3 - K)_{(p, q)}$ admits a spherical structure as described above. In each of the other eighteen exceptional cases, there is a point (z_0, w_0)

on ∂U which satisfies the equation (2) with the coprime pair of integers in question. Then a degeneration of hyperbolic structures happens and another geometric structure appears if (z, w) tends to the point (z_0, w_0) (see Sect. 4.9 in [11] and Sect. 5.7 in [1]):

- If $(p, q) = (0, \pm 1)$, then $(S^3 - K)_{(p,q)}$ admits a Sol-structure. The image of the corresponding representation $\rho_{(p,q)} : \pi_1(S^3 - K) \rightarrow \mathrm{PSL}_2(\mathbb{C})$ is included in $\mathrm{PSL}_2(\mathbb{R})$.
- If $(p, q) = (\pm 1, \pm 1), (\pm 2, \pm 1)$ or $(\pm 3, \pm 1)$, then $(S^3 - K)_{(p,q)}$ admits a $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ -structure. The image of the corresponding representation $\rho_{(p,q)} : \pi_1(S^3 - K) \rightarrow \mathrm{PSL}_2(\mathbb{C})$ is included in $\mathrm{PSL}_2(\mathbb{R})$.
- If $(p, q) = (\pm 4, \pm 1)$, then $(S^3 - K)_{(p,q)}$ contains an incompressible torus which splits the manifold into the union of the trefoil knot complement and the non-trivial I -bundle over the Klein bottle, where I denotes the closed interval $[0, 1]$. The trefoil knot complement has a $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ -structure and the non-trivial I -bundle over the Klein bottle has a Euclidian structure. The images of the representations corresponding to them are included in $\mathrm{PSL}_2(\mathbb{R})$ or $SO(3)$ respectively.

Thurston showed in Sect. 4.11 of his lecture notes [11] that for each coprime pair of integers (p, q) except the cases $(\pm 4, \pm 1)$ and $(0, \pm 1)$ the 3-manifold $(S^3 - K)_{(p,q)}$ is not Haken.

If the generalized Dehn filling coefficient $(p, q) \in \mathbb{R}^2 - \{(0, 0)\}$ satisfies the condition that $p \neq 0$ and q/p is rational, take a coprime pair of integers (α, β) so that $q/p = \beta/\alpha$. Then a closed 3-manifold $(S^3 - K)_{(\alpha, \beta)}$ obtained by Dehn filling of type (α, β) along K could have a hyperbolic 3-cone-structure with a simple closed curve as its singular locus of cone angle $2\pi|\alpha/p|$. Now consider specially the case as $q = 0$. In this case, we take $(\alpha, \beta) = (1, 0)$ or $(\alpha, \beta) = (-1, 0)$ depending on whether p is positive or negative. Note that both of $(S^3 - K)_{(\pm 1, 0)}$ are homeomorphic to S^3 and singularities of cone-structures could appear along K . First, let us consider the case where p is positive. Thurston analyzed that the image of HDF includes the open interval $(\frac{2\pi}{\sin^{-1}(\sqrt{15}/4)}, +\infty)$ in the p -axis. (Note that $2\pi/\sin^{-1}(\sqrt{15}/4) \approx 4.76679$.) Then S^3 admits a hyperbolic 3-cone-structure with singular locus K of cone angle $2\pi/p$, where $p > \frac{2\pi}{\sin^{-1}(\sqrt{15}/4)}$. After his works, without using ideal tetrahedral decompositions, Hilden, Lozano and Montesinos-Amilibia [4] constructed a one-parameter family of 3-cone-manifolds $\{(S^3, \tau(p))\}_{p \in [3, +\infty]}$ whose sectional curvature increases monotonely from -1 at $p = +\infty$ to 0 at $p = 3$ and whose singular locus is the figure eight knot K with cone angle $2\pi/p$. The 3-cone-manifold $(S^3, \tau(+\infty))$ coincides with the hyperbolic 3-manifold (S^3, σ_∞) . The 3-cone-manifold $(S^3, \tau(3))$ is a Euclidean 3-orbifold with K as the singular locus of cone angle $2\pi/3$. For each $p \in (3, +\infty)$, rescaling the Riemannian metric of $(S^3, \tau(p))$ adequately, we obtain a hyperbolic 3-cone-manifold

$(S^3, \hat{\tau}(p))$ whose singular locus is K with cone angle $2\pi/p$. This collapses to a point as $p \searrow 3$. Then, on the contrary, it is considered that the hyperbolic 3-cone-manifold $(S^3 - K, \hat{\tau}(p))$ can be rescaled to converge to the Euclidean 3-orbifold $(S^3, \tau(3))$ as $p \rightarrow 3$ (see also Sect. 5.7 in Cooper-Hodgson-Kerckhoff [1]). If p is negative, p moves in the open interval $(-\infty, -\frac{2\pi}{\sin^{-1}(\sqrt{15}/4)})$ and the same argument holds. Denote by $(S^3, \tau(-3))$ a Euclidean 3-orbifold which is obtained in this case. For each $p \in [-\infty, -3] \cup [3, +\infty]$, there corresponds a representation $\rho_{\hat{\tau}(p)} : \pi_1(S^3 - K) \rightarrow \text{PSL}_2(\mathbb{C})$. With the Euclidean 3-orbifolds $(S^3, \tau(\pm 3))$ are associated representations $\rho_{\tau(\pm 3)} : \pi_1(S^3 - K) \rightarrow \text{SO}(3)$. By taking conjugation, the representation $\rho_{\hat{\tau}(p)}$ converges to $\rho_{\tau(\pm 3)}$ when $p \rightarrow \pm 3$.

3. A birational map from the deformation curve C to an elliptic curve in Weierstrass form.

Let V be the projective completion of the complex affine plane curve C . Then $V = \{(x_0 : x_1 : x_2) \in \mathbb{CP}^2 \mid x_1(x_1 - x_0)x_2(x_2 - x_0) = x_0^4\}$. Let A and B denote points at infinity of C which correspond to the points $(0 : 0 : 1)$ and $(0 : 1 : 0)$ on V , respectively. These points $(0 : 0 : 1)$ and $(0 : 1 : 0)$ are ordinary double points. Then the genus of C is one.

The curve C is transformed to a non-singular cubic curve in Weierstrass form E by the following birational map. The curve E is the elliptic curve denoted by 15A8 in the table of Cremona [2].

$$\begin{aligned}
 C : z(z-1)w(w-1) &= 1 \\
 \Downarrow z=u, w=\frac{1}{v} \\
 u^2v + v^2 - uv - u^2 + u &= 0 \\
 \Downarrow u=\frac{x_1}{x_0}, v=\frac{x_2}{x_0} \\
 X_1^2X_2 + X_0X_2^2 - X_0X_1X_2 - X_0X_1^2 + X_0^2X_1 &= 0 \\
 \Downarrow X_0=-x_0-x_1, X_1=x_2, X_2=-x_1 \\
 -x_0x_1^2 - x_1^3 + x_0x_1x_2 + x_0x_2^2 + x_0^2x_2 &= 0 \\
 \Downarrow x=\frac{x_1}{x_0}, y=\frac{x_2}{x_0} \\
 E : y^2 + xy + y &= x^3 + x^2
 \end{aligned} \tag{3}$$

Any elliptic curve allows an addition law $+$ and the points on it form an abelian group under this operation $+$, with a point $\mathcal{O}$ as identity element. On our elliptic curve $E : y^2 + xy + y = x^3 + x^2$, the addition law is described explicitly (see Appendix). We choose the point at infinity of E as the identity element. Using the addition law on E , we consider additions of points of C . Note that this addition law on C depends on choice of birational transformations from C to E and is determined up to translation on E .

Let $E(\mathbf{Q})$ be the Mordell-Weil group over $\mathbf{Q}$ that is the subset consisting of all rational points of E . It is a subgroup of E . On our elliptic curve E , it is isomorphic to $\mathbf{Z}/4\mathbf{Z}$, the cyclic group of order 4 (see [2]). By the birational map, the singular point A (resp. B) of the deformation curve C is transformed to two non-singular points A_1, A_3 (resp. B_2, B_4) on E . These four points A_1, B_2, A_3 and B_4 on E form the Mordell-Weil group $E(\mathbf{Q})$. In fact, we have $A_1 = (0, -1)$, $2A_1 = B_2 = (-1, 0)$, $3A_1 = A_3 = (0, 0)$ and $4A_1 = B_4 = \mathcal{O}$.

4. Points of finite order on the elliptic curve E and Dehn filling on the figure eight knot.

4.1. Rational points on E and Haken manifolds.

Culler and Shalen [3] investigated the character varieties of representations of fundamental groups of hyperbolic 3-manifolds. They showed that there corresponds an incompressible surface in a hyperbolic 3-manifold to each ideal point of a complex affine algebraic curve in its character variety. On the other hand, Yoshida [15] gave an explicit understanding of the restriction of the general theory given by Culler-Shalen to the special case of hyperbolic 3-manifolds with one cusp which have ideal tetrahedral decompositions. Instead of the character varieties, he used deformation curves of hyperbolic structures constructed by making use of ideal tetrahedral decompositions and defined ideal points of deformation curves. Our complex affine plane curve C is an example of such a deformation curve. He showed that there are four ideal points on our deformation curve C and constructed incompressible surfaces explicitly to these four ideal point. The generalized Dehn filling coefficients corresponding to these four ideal points are $(\pm 4, \pm 1)$. In fact, these incompressible surfaces are capped with disks to be the ones discussed in Sect. 2. Let b_1, b_2, b_3 and b_4 be the ideal points of C corresponding to $(4, -1)$, $(-4, -1)$, $(4, 1)$ and $(-4, 1)$, respectively.

It can be checked that the four rational points A_1, B_2, A_3 and B_4 on E correspond to the four ideal points b_1, b_2, b_3 and b_4 , respectively. In fact, these points A_1, B_2, A_3 and B_4 are said to be ideal points in the sense of Culler and Shalen if we apply their setting for an algebraic curve in the character variety to our deformation curve C . Then on hyperbolic structures of the figure eight knot complement $S^3 - K$ the rational points of its deformation curve correspond exactly to the closed Haken 3-manifolds $(S^3 - K)_{(\pm 4, \pm 1)}$ which contain incompressible surfaces each of that comes from a non-trivial splitting of $\pi_1(S^3 - K)$.

4.2. Points of finite order on E and Sol-manifolds.

The point $(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ on C corresponds to the point $T := (\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ on E . Al-

though this point $(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ is not contained in U , there is a sequence of points in U which converges to this point $(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$. Then this point gives a degeneration of hyperbolic structures on $S^3 - K$ as follows. Let us consider the equation (2). Take the path L_T on C connecting the two points $(-\omega^2, -\omega^2)$ and $(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ defined as follows:

$$L_T = (z, w) = (z(t), w(t)),$$

$$\text{where } \begin{cases} z(t) = t + i \sqrt{-(t - \frac{1}{2})^2 - \frac{1}{4} + \sqrt{(t - \frac{1}{2})^2 + 1}}, \\ w(t) = -t + 1 + i \sqrt{-(t - \frac{1}{2})^2 - \frac{1}{4} + \sqrt{(t - \frac{1}{2})^2 + 1}}, \end{cases} \quad (\frac{1}{2} \geq t \geq \frac{1-\sqrt{5}}{2}).$$

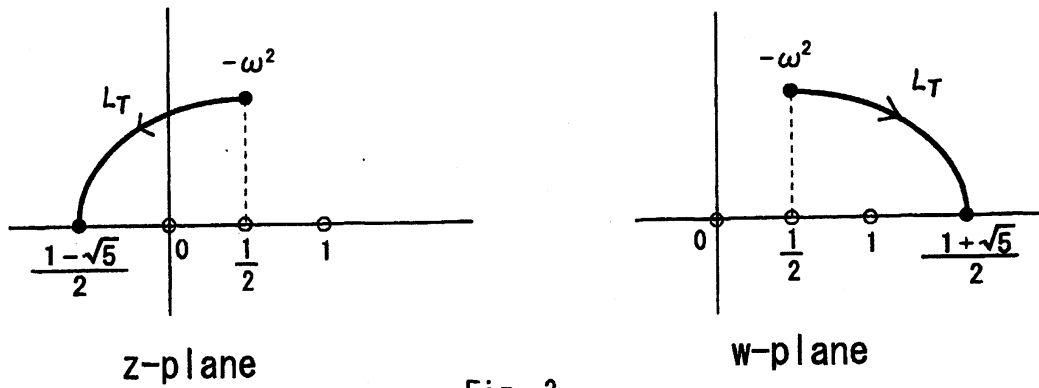


Fig. 3

Then, along the path L_T , the corresponding generalized Dehn filling coefficients are of type $(0, q)$. The value of q decreases monotonely from ∞ at $t = \frac{1}{2}$ to 1 at $t = \frac{1-\sqrt{5}}{2}$. Especially, if (z, w) goes to $(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ along L_T , then $(0, q)$ goes to $(0, 1)$. Denote the incomplete hyperbolic structure of $(S^3 - K)(z(t), w(t))$ by $\sigma_{(0,q)}$. The incomplete hyperbolic 3-manifold $(S^3 - K, \sigma_{(0,q)})$ is completed to be a hyperbolic 3-cone-manifold $((S^3 - K)_{(0,1)}, \bar{\sigma}_{(0,q)})$. Its underlying space $(S^3 - K)_{(0,1)}$ is homeomorphic to a torus bundle over a circle with the Anosov monodromy $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Its singular locus is a section of this torus bundle and its cone angle is equal to $2\pi/q$. When $q \rightarrow 1$, the hyperbolic 3-cone-manifold $((S^3 - K)_{(0,1)}, \bar{\sigma}_{(0,q)})$ collapses to the circle that forms the base of the torus bundle. On the other hand, the 3-manifold $(S^3 - K)_{(0,1)}$ admits a Sol-structure. After rescaling the hyperbolic metric $\bar{\sigma}_{(0,q)}$, this converges to the Sol-structure as $q \rightarrow 1$. Then the point T can be considered to correspond to the 3-manifold with the Sol-structure that occurs by the degeneration of the hyperbolic structures. Note that the developing images of the ideal tetrahedra $S(z(t))$ and $S(w(t))$ are flattening out in a 2-dimensional subspace $\mathbb{H}^2$ in $\mathbb{H}^3$ as $q \rightarrow 1$ and then the

developing map of the hyperbolic 3-cone-manifold $((S^3 - K)_{(0,1)}, \bar{\sigma}_{(0,q)})$ converges to that of a transversely codimension 2 hyperbolic foliation on $(S^3 - K)_{(0,1)}$ when $q \rightarrow 1$. Then the corresponding representation converges as $q \rightarrow 1$ to a (non-discrete) representation into $\mathrm{PSL}_2(\mathbf{R})$ by taking a suitable conjugation. (See Sect. 4.9 in [11].)

We can check that $2T = A_3$, $3T = (\frac{1-\sqrt{5}}{2}, -2+\sqrt{5})$, $4T = B_2$, $5T = (\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$, $6T = A_1$, $7T = (\frac{1+\sqrt{5}}{2}, -2-\sqrt{5})$ and $8T = \mathcal{O}$. The three points $(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$, $(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$ and $(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ on C correspond to $3T$, $5T$ and $7T$, respectively, by the birational map. Then, by computations similar to the above, it is checked that $3T$, $5T$ and $7T$ correspond to the generalized Dehn filling coefficients $(0, -1)$, $(0, -1)$ and $(0, 1)$, respectively. As in the case of T , the point $5T$ can be considered to be a limit of points in U and gives a Sol-structure on $(S^3 - K)_{(0,-1)}$ that is a limit of hyperbolic structures on $S^3 - K$. On the other hand, the points $3T$ and $7T$ are not limits of points in U , since both paths connecting the original point (z_∞, w_∞) to $(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$ and to $(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ should cross the region $C - U$. So far, there is no idea to construct hyperbolic structures along a path connecting the points $(-\omega^2, -\omega^2)$ and $(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$ or $(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ especially in the outside of U . Therefore we consider the points $3T$ and $7T$ do not correspond to 3-manifolds obtained by Dehn filling of the hyperbolic 3-manifold $(S^3 - K, \sigma_\infty)$.

4.3. Points of finite order on E and Euclidean orbifolds.

The point $(z_\infty, w_\infty) = (-\omega^2, -\omega^2)$ on the deformation curve C corresponds to the point $R := (\omega^2, -1)$ on E . We can check that $2R = A_1$ and that the point $5R$ also corresponds to $(\bar{z}_\infty, \bar{w}_\infty) = (-\omega, -\omega)$. Since $\mathrm{Im}(-\omega) < 0$, the ideal tetrahedron $S(-\omega)$ is negative and we can consider that $S(-\omega) \cup S(-\omega)$ is a hyperbolic 3-manifold which is overlapped to $(S^3 - K, \sigma_\infty) = S(-\omega^2) \cup S(-\omega^2)$ by orientation-reversing isometry. So far, there is no idea to construct deformations of the hyperbolic structure σ_∞ connecting $(S^3 - K)(z_\infty, w_\infty)$ and $(S^3 - K)(\bar{z}_\infty, \bar{w}_\infty)$. Then we have to consider now that the hyperbolic 3-manifold $(S^3 - K)(\bar{z}_\infty, \bar{w}_\infty)$ is not obtained by hyperbolic Dehn filling of the original hyperbolic 3-manifold $(S^3 - K)(z_\infty, w_\infty)$.

The point on C corresponding to the point $7R = (\omega^2, -\omega^2)$ (resp. $3R = (\omega, -\omega)$) on E is $(-\omega, -\omega^2)$ (resp. $(-\omega^2, -\omega)$). Since $\mathrm{Im}(-\omega) < 0$, these points $(-\omega, -\omega^2)$ and $(-\omega^2, -\omega)$ are not contained in U . Now let us consider the equation (2) on the Riemannian surface of C . Take the path L_{7R} on this Riemannian surface connecting the two points $(-\omega^2, -\omega^2)$ and $(-\omega, -\omega^2)$ defined as follows:

$$L_{7R} = (z, w) = (z(t), w(t)), \text{ where } z(t) = \frac{1}{2} + it, \quad w(t) = \frac{1}{2} + \frac{i}{2} \sqrt{\frac{-4t^2 + 15}{4t^2 + 1}}, \quad \left(\frac{\sqrt{3}}{2} \geq t \geq -\frac{\sqrt{3}}{2}\right).$$

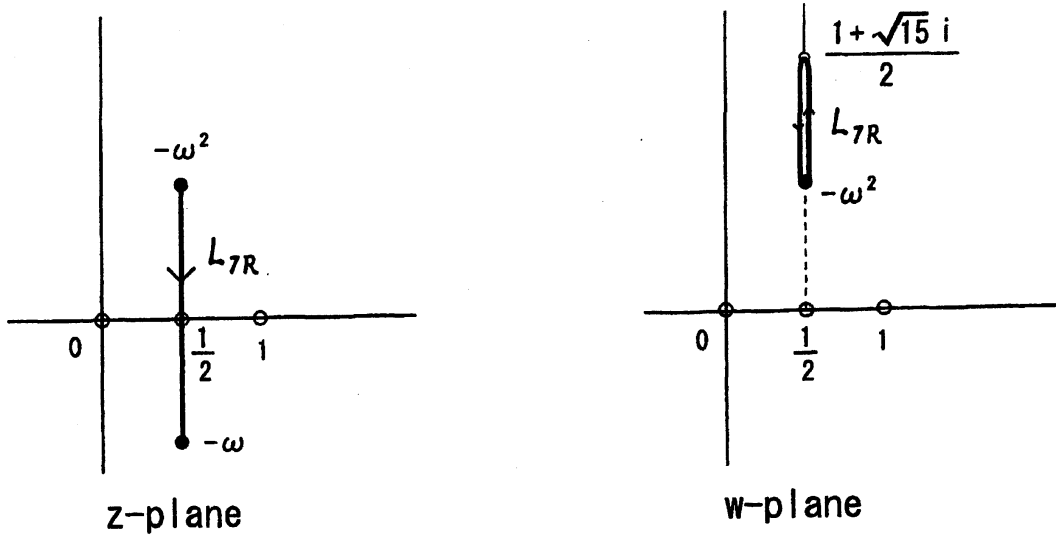


Fig. 4

Then, along this line L_{7R} , the corresponding Dehn filling coefficients are of type $(p, 0)$. The value of p decreases monotonely from $+\infty$ at $t = \sqrt{3}/2$ to 3 at $t = -\sqrt{3}/2$. Especially, if (z, w) goes to $(-\omega, -\omega^2)$ along L_{7R} , then $(p, 0)$ goes to $(3, 0)$. Unfortunately however, the paths L_{7R} gets into the region $C - U$ at $(z', w') = (1/2, (1 + \sqrt{15}i)/2)$. Let $(p', 0)$ be the Dehn surgery coefficient corresponding to (z', w') . As described above, the value of p' is near 4.76679. If $p > p'$, the ideal tetrahedra $S(z(t))$ and $S(w(t))$ are positive and a hyperbolic 3-manifold $(S^3 - K)(z(t), w(t))$ can be constructed by gluing the faces of $S(z(t))$ and $S(w(t))$ with the same pattern as $(S^3 - K, \sigma_\infty)$. Denote its hyperbolic structure by $\sigma_{(p, 0)}$. This gives rise to a hyperbolic 3-cone-manifold $((S^3 - K)_{(1, 0)}, \bar{\sigma}_{(p, 0)})$ by its completion. The underlying space $(S^3 - K)_{(1, 0)}$ is homeomorphic to S^3 and K forms the singular locus with cone angle $2\pi/p$. Beyond the point (z', w') , so far it is difficult to argue deformations of $S(z(t)) \cup S(w(t))$. But the hyperbolic Dehn filling does not break down at $(z', w') \in \partial U$. In fact, as mentioned in Sect. 2, the hyperbolic 3-cone-structures $\hat{\tau}(p)$ on S^3 ($\approx (S^3 - K)_{(1, 0)}$) are obtained beyond the point $(p', 0)$ (and until the point $(3, 0)$). If $p > p'$, the cone angles of the two hyperbolic 3-cone-manifolds $(S^3, \bar{\sigma}_{(p, 0)})$ and $(S^3, \hat{\tau}(p))$ are the same and are smaller than $2\pi/p'$. Then, by Kojima's global rigidity theorem on hyperbolic 3-cone-manifolds (see [7]), $(S^3, \hat{\tau}(p))$ is isometric to $(S^3, \bar{\sigma}_{(p, 0)})$ if $p > p'$. Hence, $(S^3, \hat{\tau}(p))$ ($p' \geq p > 3$) can be considered to realizations of the completions of $(S^3 - K)(z, w)$ along the paths L_{7R} beyond the point (z', w') . Moreover, when $p \searrow 3$, $(S^3, \hat{\tau}(p))$ is rescaled to converge to the Euclidean 3-orbifold $(S^3, \tau(3))$. Then the point $(p, q) = (3, 0)$ can be regarded to correspond to the

Euclidean 3-orbifold $(S^3, \tau(3))$. Therefore we might consider that $7R$ corresponds to the Euclidean 3-orbifold $(S^3, \tau(3))$ that occurs by the degeneration of the hyperbolic structures. By taking another path which connects $(-\omega^2, -\omega^2)$ and $(-\omega^2, -\omega)$ and performing the same process, we also might consider that $3R$ corresponds to the Euclidean 3-orbifold $(S^3, \tau(-3))$ given in Sect. 2.

4.4. Theorem.

By summerizing the above, we have the following theorem.

Theorem. *Let K be the figure eight knot in the 3-dimensional sphere S^3 . Let*

$$C : z(z-1)w(w-1) = 1$$

be the deformation curve of hyperbolic structures on the figure eight knot complement $S^3 - K$. For each coprime pair of integers (p, q) , let us denote by $(S^3 - K)_{(p,q)}$ a closed 3-manifold obtained from the figure eight knot complement $S^3 - K$ by a Dehn filling along K which kills the homotopy class of the simple closed curve $pm + ql$, where m and l are the standard meridian and longitude of K in S^3 .

(1) *The deformation curve C is birationally equivalent to the elliptic curve*

$$E : y^2 + xy + y = x^3 + x^2$$

whose conductor is 15 and Mordell-Weil group $E(\mathbf{Q})$ is isomorphic to $\mathbf{Z}/4\mathbf{Z}$, the cyclic group of order four.

(2) *The four points $A_1 := (0, -1)$, $B_2 := (-1, 0)$, $A_3 := (0, 0)$ and $B_4 := \mathcal{O}$ on E form the Mordell-Weil group $E(\mathbf{Q})$, where $\mathcal{O}$ denotes the point at infinity of E .*

(3) *The two points $R := (\omega^2, -1)$ and $T := (\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$ on E are points of order eight and satisfy that $2R = 6T = A_1$, $4R = 4T = B_2$, $6R = 2T = A_3$ and $8R = 8T = B_4 = \mathcal{O}$. Here ω denotes the complex number $e^{2\pi i/3}$.*

(4) *There is the following correspondence between the points nR , nT ($n = 1, \dots, 8$) on the elliptic curve E except the three points $5R$, $3T$ and $7T$ and the 3-manifolds $(S^3 - K)_{(p,q)}$ $((p, q) = \infty, (\pm 4, \pm 1), (\pm 1, 0), (0, \pm 1))$ with geometric structures, where $(S^3 - K)_\infty$ means the unsurgered manifold $S^3 - K$.*

R	$\iff S^3 - K$ with a complete volume finite hyperbolic structure
$2R = 6T = A_1$	$\iff (S^3 - K)_{(4,-1)}$ contains an incompressible torus that gives rise to a splitting into a Euclidean manifold and a $\widetilde{PSL}_2(\mathbf{R})$ -manifold
$3R$	$\iff (S^3 - K)_{(-1,0)}$ with a Euclidean orbifold structure of cone angle $2\pi/3$ along the singular locus K
$4R = 4T = B_2$	$\iff (S^3 - K)_{(-4,-1)}$ contains an incompressible torus that gives rise to a splitting into a Euclidean manifold and a $\widetilde{PSL}_2(\mathbf{R})$ -manifold
$6R = 2T = A_3$	$\iff (S^3 - K)_{(4,1)}$ contains an incompressible torus that gives rise to a splitting into a Euclidean manifold and a $\widetilde{PSL}_2(\mathbf{R})$ -manifold
$7R$	$\iff (S^3 - K)_{(1,0)}$ with a Euclidean orbifold structure of cone angle $2\pi/3$ along the singular locus K
$8R = 8T = B_4 = \mathcal{O}$	$\iff (S^3 - K)_{(-4,1)}$ contains an incompressible torus that gives rise to a splitting into a Euclidean manifold and a $\widetilde{PSL}_2(\mathbf{R})$ -manifold
T	$\iff (S^3 - K)_{(0,1)}$ with a Sol-structure
$5T$	$\iff (S^3 - K)_{(0,-1)}$ with a Sol-structure

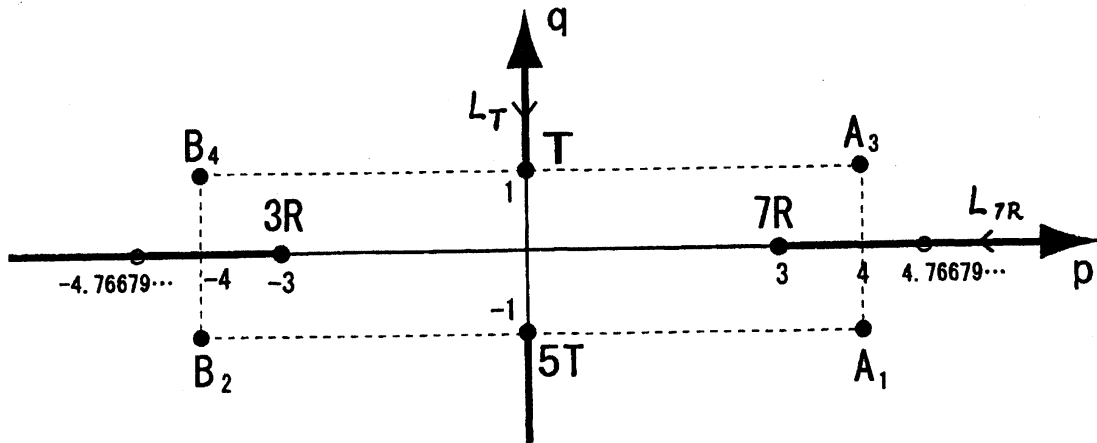


Fig. 5

Remark. The correspondence in (4) is not canonical, since it depends on choice of birational maps from C to E . It is determined up to translation of E .

The theorem above intimates that there are some relationship between information on arithemeticity of the deformation curves of hyperbolic 3-manifolds and one on the topology and the geometry of 3-manifolds obtained from the hyperbolic 3-manifolds by Dehn filling.

5. An automorphic form and the L -function for E .

Our deformation curve C is birational to the elliptic curve E . The condutor of this elliptic curve is 15. Let $\Gamma_0(15)$ be the Hecke congruence group of level 15:

$$\Gamma_0(15) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \mid c \equiv 0 \pmod{15} \right\}.$$

The modular curve $X_0(15) := \mathbf{H}^2 \cup \mathbf{Q} \cup \{\infty\} / \Gamma_0(15)$ is isomorphic to the elliptic curve $y^2 + xy + y = x^3 + x^2 - 10x - 10$. Let $S_2(\Gamma_0(15))$ denote the space of cusp forms of weight 2 and level 15. The complex dimension of the space $S_2(\Gamma_0(15))$ is 1. The η -product

$$\varphi(\tau) = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau) \quad (\tau \in \mathbf{H}^2)$$

is a newform and becomes a base of this space, where $\eta(\tau) = e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$ is the Dirichlet η -function (see Cor. in Section 3.1 of Ligozat [8]). Let

$$\varphi(\tau) = \sum_{n=1}^{\infty} c_n e^{2\pi i n \tau}$$

be a Fourier expansion of the newform. Then the L -function of the elliptic curve E is given by the Dirichlet series

$$L(E, s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}.$$

This L -function $L(E, s)$ has an Euler product expansion

$$L(E, s) = \left[\frac{1}{1 + 3^{-s}} \right] \left[\frac{1}{1 - 5^{-s}} \right] \prod_{\substack{p: \text{ prime} \\ p \neq 3, 5}}^{\infty} \left[\frac{1}{1 - c_p p^{-s} + p^{1-2s}} \right]$$

which converges for $\mathrm{Re} s > \frac{3}{2}$.

Appendix.

In this appendix, we describe the addition law on the elliptic curve $E : y^2 + xy + y = x^3 + x^2$ explicitly (see [5] [6] [10] [14]). The elliptic curve E has a unique point at infinity. Denote this point by $\mathcal{O}$. If we identify $(x, y) \in E$ with a point $(1 : x : y)$ on the corresponding projective variety of E , then $\mathcal{O} = (0 : 0 : 1)$. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be points on E . We give formulas to compute $P + Q$ as follows.

- If $x_1 \neq x_2$, then

$$(x_1, y_1) + (x_2, y_2) = (x_3, y_3),$$

where

$$x_3 = \left(\frac{y_2 - y_1}{x_2 - x_1} \right)^2 + \left(\frac{y_2 - y_1}{x_2 - x_1} \right) - 1 - x_1 - x_2, \quad y_3 = - \left(\frac{y_2 - y_1}{x_2 - x_1} + 1 \right) x_3 - \left(\frac{y_1 x_2 - y_2 x_1}{x_2 - x_1} \right) - 1.$$

- If $x_1 = x_2$ and $y_1 + y_2 + x_2 + 1 = 0$, then

$$(x_1, y_1) + (x_1, y_2) = \mathcal{O}.$$

- If $x_1 = x_2$, $y_1 = y_2$ and $y_1 + y_2 + x_2 + 1 = 0$, then

$$2(x_1, y_1) = \mathcal{O}.$$

- If $x_1 = x_2$, $y_1 = y_2$ and $y_1 + y_2 + x_2 + 1 \neq 0$, then

$$2(x_1, y_1) = (x_3, y_3),$$

where

$$\begin{aligned} x_3 &= \left(\frac{3x_1^2 + 2x_1 - y_1}{2y_1 + x_1 + 1} \right)^2 + \left(\frac{3x_1^2 + 2x_1 - y_1}{2y_1 + x_1 + 1} \right) - 1 - 2x_1, \\ y_3 &= - \left(\frac{3x_1^2 + 2x_1 - y_1}{2y_1 + x_1 + 1} + 1 \right) x_3 - \left(\frac{-x_1^3 - y_1}{2y_1 + x_1 + 1} \right) - 1. \end{aligned}$$

- $-(x_1, y_1) = (x_1, -x_1 - y_1 - 1).$

Then the points on E form an abelian group under this addition law. The point $\mathcal{O}$ is the identity element of this group.

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