

AN UNSATURATED GENERIC STRUCTURE

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ABSTRACT. We construct an *ab initio* generic structure for a predimension function with a positive rational coefficient strictly less than 1 which is unsaturated and has a non- ω -stable theory. Superstability of the theory will be discussed in a sequel paper.

1. INTRODUCTION

We consider graph structures. A graph structure has one binary relation as a first order structure. $X \subseteq_{\text{fin}} Y$ means that X is a finite subset of Y .

For a graph structure A , let

$$\delta_\alpha(A) = |A| - \alpha e(A).$$

Here, α is a rational number such that $0 < \alpha < 1$, $|A|$ the number of points in A , and $e(A)$ the number of edges in A . $\delta_\alpha(A)$ is called a *predimension function*.

Suppose $A \subseteq_{\text{fin}} B$ (substructure = induced subgraph).

$A \leq B$ (A is a *strong substructure* of B or A is *closed* in B) if

$$A \subseteq X \subseteq_{\text{fin}} B \Rightarrow \delta_\alpha(A) \leq \delta_\alpha(X).$$

In this case, if $A = \{a\}$ (a singleton) then a is called a *closed point* in B .

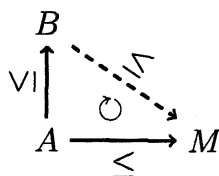
We say that $A \leq B$ is *minimal* if $A \leq B$, $A \neq B$, and $A \leq X \leq B$ implies $X = A$ or $X = B$.

With this notation, let

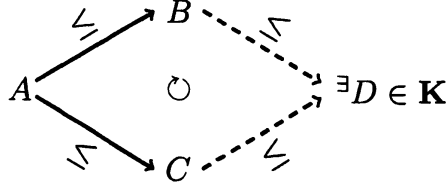
$$\mathbf{K}_\alpha = \{A : \text{finite} : A \geq \emptyset\}.$$

Definition 1.1. Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$. A countable graph M is a *generic structure* of $\mathbf{K}$ if

- $A \subseteq_{\text{fin}} M \Rightarrow$ there exists B such that $A \subseteq B \subseteq_{\text{fin}} M$ and $B \leq M$;
- $A \subset_{\text{fin}} M \Rightarrow A \in \mathbf{K}$;
- for any $A, B \in \mathbf{K}$,



Definition 1.2. A class $\mathbf{K}$ has the *amalgamation property* (AP, in short) if for any $A, B, C \in \mathbf{K}$,



Fact 1.3. Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$,

- (1) $\emptyset \in \mathbf{K}$,
- (2) $\mathbf{K}$ has the AP, and
- (3) $A \subset B \in \mathbf{K}$ implies $A \in \mathbf{K}$.

Then $\mathbf{K}$ has a generic structure.

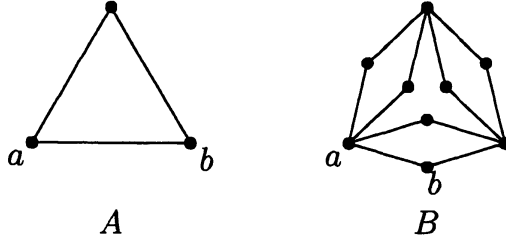
Definition 1.4. Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$. $\mathbf{K}$ has *thrifty amalgamation* if whenever $A \leq B$ is minimal, $A \leq C$ with $A, B, C \in \mathbf{K}$ then either $B \oplus_A C \in \mathbf{K}$ or there is a strong embedding of B into C over A .

2. AN AMALGAMATION CLASS

Definition 2.1. A graph A is a *minimal 1-component* (in $\mathbf{K}_\alpha$) if $|A| \geq 2$, $\delta_\alpha(A) = 1$, and $\delta_\alpha(X) > 1$ for any $X \subset A$ such that $1 < |X| < |A|$.

The following are examples of a minimal 1-component in the case $\alpha = 2/3$.

In the rest of the paper, we fix $\alpha = 2/3$ and δ_α will be written δ .



Let S_A be the set of connected substructures of (A, a, b) , i.e., the connected substructures of A containing a and b . Let S_B be the set of connected substructures of (B, a, b) . Let $S_0 = S_A \cup S_B$.

Let S_1 be the smallest class with thrifty amalgamation containing S_0 .

Lemma 2.2. (1) If $(X, a, b) \in S_0$, then (X, a, b) is (A, a, b) , (B, a, b) , or $(Y, a, b) \leq (X, a, b)$ for some proper substructure (Y, a, b) of (B, a, b) .
 (2) If $(X, a, b) \in S_0$ with $1 < \delta(X) < 2$ then $\delta(X) = 4/3$ or $5/3$ and there is $(Y, a, b) \in S_0$ such that $X \leq Y$ and $\delta(Y) \geq 2$.

Definition 2.3. Let S be a class of structures (X, a, b) where X is a graph and a, b are two distinguished points in X .

Suppose that there are graphs $A_1, A_2, \dots, A_n$ and points $a_{i-1}, a_i \in A_i$ such that (A_i, a_{i-1}, a_i) is isomorphic to some element of S for each i , and

$$Y = A_1 \oplus_{a_1} A_2 \oplus_{a_2} \cdots \oplus_{a_{n-1}} A_n.$$

We call Y a S -chain. n is called the length of the S -chain Y . Each A_i is called an *amalgamand* of Y . With such Y , if we can write

$$X = Y/(a_0 = a_n)$$

then we call X a S -cycle. n is called the length of the S -cycle X . Each amalgamand of Y is also called an *amalgamand* of X .

If S consists of one graph with two points and one edge, we simply call an S -chain a chain, and an S -cycle a cycle.

Let K_0 be the set of S_1 -cycles of length greater than $|B|$.

Proposition 2.4. *Suppose $X \in K_0$.*

- (1) $\delta(X) = 0$ if and only if every amalgamand of X is isomorphic to (A, a, b) or (B, a, b) .
- (2) $\delta(X) = 1/3$ if and only if exactly one amalgamand of X is isomorphic to a proper substructure of (A, a, b) or (B, a, b) with $\delta = 4/3$ and each of the remaining amalgamands is isomorphic to (A, a, b) or (B, a, b) .
- (3) $\delta(X) = 2/3$ if and only if either exactly one amalgamand of X is isomorphic to a proper substructure of (A, a, b) or (B, a, b) with $\delta = 5/3$ or exactly two amalgamands of X are isomorphic to a proper substructure of (A, a, b) or (B, a, b) with $\delta = 4/3$, and each of the remaining amalgamands is isomorphic to (A, a, b) or (B, a, b) .
- (4) $0 < \delta(X) < 1$ if and only if $\delta(X) = 1/3$ or $\delta(X) = 2/3$.

Proposition 2.5. *Suppose $X \in K_0$.*

- (1) If $\delta(X) = 0$ then there is no proper substructure of X closed in X .
- (2) If $\delta(C) \geq 2$ for exactly one amalgamand C of X , and each of the remaining amalgamands of X is isomorphic to (A, a, b) or (B, a, b) , then there is a closed point of X in C , and all the closed points of X are in C .
- (3) If $\delta(C), \delta(D) \geq 2$ for exactly two amalgamands C, D of X , and each of the remaining amalgamands of X is isomorphic to (A, a, b) or (B, a, b) , then there is a closed point of X in C , and also in D , and all the closed points of X are in C or D .

Let K_1 be the set of S_1 -chains and its substructures.

Let K_2 be the smallest set with thrifty amalgamation containing K_0 and K_1 .

Proposition 2.6. *Suppose $X \in K_2$ and X is connected. If $\delta(X) < 1$ then $X \in K_0$.*

Proposition 2.7. *Suppose c_1 and c_2 are two closed points in $X \in K_2$. Then there is $Y \in K_2$ such that $X \leq Y$ and c_1 and c_2 are connected in Y .*

Proof. If c_1 and c_2 are connected then there is nothing to prove. Suppose c_1 and c_2 are not connected in $X \in K_2$. Let X_1 be the connected component of X containing c_1 and X_2 the connected component of X containing c_2 . If $c_1, c_2 \in U \subset X$, then

$$\delta(U) \geq \delta(U \cap X_1) + \delta(U \cap X_2) \geq 1 + 1 = 2$$

since $c_i \leq U \cap X_i$ for $i = 1, 2$. Hence, $\{c_1, c_2\} \leq X$. Consider a chain C_3 of length 3 with end points c_1 and c_2 . then $\{c_1, c_2\} \leq C_3 \in K_2$. Hence there is $Y \in K_2$ such that X and C_3 are strongly embedded in Y over $\{c_1, c_2\}$. $\square$

3. AN UNSATURATED GENERIC STRUCTURE

Let M be the generic structure of $\mathbf{K}_2$.

Proposition 3.1. *M has only one connected component with closed points.
The other connected components are exactly $\{A, B\}$ -cycles.*

Proposition 3.2. *$Th(M)$ is not ω -stable.*

Proof. In a saturated model of $Th(M)$, we have all $\{A, B\}$ -chains of countable length by compactness. Therefore, there are continuum many types over $\emptyset$. $\square$

We will discuss superstability of $Th(M)$ in a sequel paper.

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