

Coefficient conditions for certain univalent functions

Toshio Hayami and Shigeyoshi Owa

Abstract

For functions $f(z)$ which belong to $\mathcal{T}(\alpha)$, $\mathcal{U}(\alpha)$, and $\mathcal{CC}_\lambda(\alpha; g(z))$ in the open unit disk $\mathbb{U}$, some interesting sufficient conditions for coefficient inequalities of $f(z)$ are discussed.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Furthermore, let $\mathcal{P}$ be the class of functions $p(z)$ of the form

$$(1.2) \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

which are analytic in $\mathbb{U}$.

If $f(z) \in \mathcal{A}$ satisfies

$$(1.3) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U}),$$

then $f(z)$ is said to be starlike in $\mathbb{U}$.

We denote by $\mathcal{S}^*$ all functions $f(z)$ which are starlike in $\mathbb{U}$. Also, $\mathcal{K}$ is said to be the class of convex functions $f(z)$ if $f(z) \in \mathcal{A}$ satisfy

$$(1.4) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

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We begin with the definitions for the subclasses $\mathcal{T}(\alpha)$, $\mathcal{U}(\alpha)$ and $\mathcal{CC}_\lambda(\alpha; g(z))$ of $\mathcal{A}$.

Definition 1.1 A function $f(z) \in \mathcal{A}$ belongs to $\mathcal{T}(\alpha)$ if and only if it satisfies

$$(1.5) \quad \operatorname{Re} \frac{f(z)}{z} > \alpha \quad (z \in \mathbb{U})$$

for some α ($0 \leq \alpha < 1$).

Definition 1.2 A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{U}(\alpha)$ if and only if it satisfies

$$(1.6) \quad \operatorname{Re} f'(z) > \alpha \quad (z \in \mathbb{U})$$

for some α ($0 \leq \alpha < 1$).

Definition 1.3 (see, for details, [2]) If $f(z) \in \mathcal{A}$ satisfies

$$(1.7) \quad \operatorname{Re} e^{i\lambda} \left(\frac{zf'(z)}{g(z)} - \alpha \right) > 0 \quad (z \in \mathbb{U})$$

for some α ($0 \leq \alpha < 1$), λ ($-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$) and starlike function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then $f(z)$ is said to be close-to-convex of order α with respect to a fixed starlike function $g(z)$, and let $\mathcal{CC}_\lambda(\alpha; g(z))$ denote the class of functions $f(z)$ satisfying this condition.

Remark Replacing $g(z)$ by $f(z)$ in (1.7), we say that $f(z)$ is said to be λ -spiral of order α in $\mathbb{U}$, and write $\mathcal{SP}(\lambda, \alpha)$ defined by

$$\mathcal{SP}(\lambda, \alpha) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re} e^{i\lambda} \left(\frac{zf'(z)}{f(z)} - \alpha \right) > 0 \right\}.$$

We need the following lemmas to prove our results.

Lemma 1.1 (see, [1], [3]) A function $p(z) \in \mathcal{P}$ satisfies $\operatorname{Re} p(z) > 0$ ($z \in \mathbb{U}$) if and only if

$$p(z) \neq \frac{x-1}{x+1} \quad (z \in \mathbb{U})$$

for all $|x| = 1$.

Lemma 1.2 A function $f(z) \in \mathcal{A}$ is in $\mathcal{T}(\alpha)$ if and only if

$$(1.8) \quad 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0$$

where

$$A_n = \frac{x+1}{2(1-\alpha)} a_n \quad (n \geq 2).$$

Proof. Putting $p(z) = \frac{f(z) - \alpha}{1 - \alpha}$ for $f(z) \in \mathcal{T}(\alpha)$, we obtain that $p(z) \in \mathcal{P}$, and $\operatorname{Re} p(z) > 0$. Using Lemma 1.1, we have that

$$\frac{f(z) - \alpha}{1 - \alpha} \neq \frac{x - 1}{x + 1} \quad (\text{for all } |x| = 1, z \in \mathbb{U}).$$

Then, we need not consider Lemma 1.1 for $z = 0$, because it follows that

$$p(0) = 1 \neq \frac{x - 1}{x + 1}.$$

This implies that

$$(1.9) \quad (x + 1)f(z) + (1 - 2\alpha - x)z \neq 0.$$

It follows that (1.9) is equivalent to

$$(x + 1) \left(z + \sum_{n=2}^{\infty} a_n z^n \right) + (1 - 2\alpha - x)z \neq 0$$

or

$$(1.10) \quad 2(1 - \alpha)z \left\{ 1 + \sum_{n=2}^{\infty} \frac{x + 1}{2(1 - \alpha)} a_n z^{n-1} \right\} \neq 0.$$

Dividing the both sides of (1.10) by $2(1 - \alpha)z$ ($z \neq 0$), we know that

$$1 + \sum_{n=2}^{\infty} \frac{x + 1}{2(1 - \alpha)} a_n z^{n-1} \neq 0.$$

This completes the proof of lemma. □

2 Coefficient conditions for functions in the classes $\mathcal{T}(\alpha)$ and $\mathcal{CC}_\lambda(\alpha; g(z))$

Our result for $f(z)$ to be in $\mathcal{T}(\alpha)$ is contained in

Theorem 2.1 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \leq 1 - \alpha$$

for some α ($0 \leq \alpha < 1$), $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{T}(\alpha)$.

Proof. Note that

$$(1 - z)^\beta \neq 0, (1 + z)^\gamma \neq 0 \quad (\beta, \gamma \in \mathbb{R}; z \in \mathbb{U}).$$

Hence if the following inequality

$$(2.1) \quad \left(1 + \sum_{n=2}^{\infty} A_n z^{n-1}\right) (1 - z)^\beta (1 + z)^\gamma \neq 0$$

holds true, then we have

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,$$

which is the relation (1.8) of Lemma 1.2. We know that (2.1) is equivalent to

$$1 + \sum_{n=2}^{\infty} \left[\sum_{k=1}^n \left\{ \sum_{j=1}^k A_j (-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right] z^{n-1} \neq 0.$$

Therefore, if $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k A_j (-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right| \leq 1,$$

that is, that

$$\begin{aligned} & \frac{1}{2(1-\alpha)} \sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (x+1) a_j (-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right| \\ & \leq \frac{1}{2(1-\alpha)} \sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right. \\ & \quad \left. + |x| \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right] \\ & = \frac{1}{1-\alpha} \sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \leq 1, \end{aligned}$$

then $f(z) \in \mathcal{T}(\alpha)$. This completes the proof of Theorem 2.1. □

Putting $\beta = \gamma = 0$ in Theorem 2.1, we see the following corollary.

Corollary 2.1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} |a_n| \leq 1 - \alpha$$

for some α ($0 \leq \alpha < 1$), then $f(z) \in \mathcal{T}(\alpha)$.

Next, we derive the coefficient condition for $f(z)$ to be in the class $\mathcal{U}(\alpha)$.

Theorem 2.2 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (-1)^{k-j} \binom{\beta}{k-j} j a_j \right\} \binom{\gamma}{n-k} \right| \leq 1 - \alpha$$

for some α ($0 \leq \alpha < 1$), $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{U}(\alpha)$.

Proof. Since $f(z) \in \mathcal{U}(\alpha) \Leftrightarrow z f'(z) \in \mathcal{T}(\alpha)$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z f'(z) = z + \sum_{n=2}^{\infty} n a_n z^n,$$

replacing a_j of Theorem 2.1 with $j a_j$, we prove the theorem. □

Taking $\beta = \gamma = 0$ in Theorem 2.2, we obtain

Corollary 2.2 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n |a_n| \leq 1 - \alpha$$

for some α ($0 \leq \alpha < 1$), then $f(z) \in \mathcal{U}(\alpha)$.

Lemma 2.1 *A function $f(z) \in \mathcal{A}$ is in $\mathcal{CC}_\lambda(\alpha; g(z))$ if and only if*

$$(2.2) \quad 1 + \sum_{n=2}^{\infty} B_n z^{n-1} \neq 0$$

where

$$(2.3) \quad B_n = \frac{n a_n + (2(1 - \alpha) e^{-i\lambda} \cos \lambda - 1) b_n + x(n a_n - b_n)}{2(1 - \alpha) e^{-i\lambda} \cos \lambda}.$$

Proof. Letting $p(z) = \frac{e^{i\lambda} \left(\frac{z f'(z)}{g(z)} - \alpha \right) - i(1 - \alpha) \sin \lambda}{(1 - \alpha) \cos \lambda}$, we see that $p(z) \in \mathcal{P}$ and $\operatorname{Re} p(z) > 0$ ($z \in \mathbb{U}$). It follows from Lemma 1.1 that

$$(2.4) \quad \frac{e^{i\lambda} \left(\frac{z f'(z)}{g(z)} - \alpha \right) - i(1 - \alpha) \sin \lambda}{(1 - \alpha) \cos \lambda} \neq \frac{x - 1}{x + 1} \quad (\text{for all } |x| = 1, z \in \mathbb{U}).$$

Then, we need not consider Lemma 1.1 for $z = 0$, because it follows that

$$p(0) = 1 \neq \frac{x - 1}{x + 1}.$$

Since (2.4) implies that

$$(x+1) \{e^{i\lambda}(zf'(z) - \alpha g(z)) - i(1-\alpha)g(z) \sin \lambda\} \neq (x-1)(1-\alpha)g(z) \cos \lambda,$$

we obtain that

$$(2.5) \quad (x+1)e^{i\lambda}zf'(z) - \alpha e^{i\lambda}g(z) - x\alpha e^{i\lambda}g(z) - i(1-\alpha)g(z) \sin \lambda - ix(1-\alpha)g(z) \sin \lambda \\ \neq x(1-\alpha)g(z) \cos \lambda - (1-\alpha)g(z) \cos \lambda.$$

The relation (2.5) is equivalent to

$$(x+1)e^{i\lambda}zf'(z) - \alpha e^{i\lambda}g(z) - x\alpha e^{i\lambda}g(z) - x(1-\alpha)e^{i\lambda}g(z) + (1-\alpha)e^{-i\lambda}g(z) \neq 0$$

that is,

$$(1+x)e^{i\lambda}zf'(z) + (e^{-i\lambda} - xe^{i\lambda} - 2\alpha \cos \lambda)g(z) \neq 0.$$

Note that the above relation can be weitten with

$$(x+1)e^{i\lambda} \left(z + \sum_{n=2}^{\infty} na_n z^n \right) + (e^{-i\lambda} - xe^{i\lambda} - 2\alpha \cos \lambda) \left(z + \sum_{n=2}^{\infty} b_n z^n \right) \neq 0$$

or

$$(2.6) \quad 2(1-\alpha) \cos \lambda z \left\{ 1 + \sum_{n=2}^{\infty} \frac{n(x+1)a_n + (e^{-2i\lambda} - x - 2\alpha e^{-i\lambda} \cos \lambda)b_n}{2(1-\alpha)e^{-i\lambda} \cos \lambda} z^{n-1} \right\} \neq 0.$$

Dividing the both sides of (2.6) by $2(1-\alpha) \cos \lambda z$ ($z \neq 0$) and noting

$$(2.7) \quad e^{-2i\lambda} = -1 + 2e^{-i\lambda} \cos \lambda,$$

we know that

$$1 + \sum_{n=2}^{\infty} \frac{na_n + (2(1-\alpha)e^{-i\lambda} - 1)b_n + x(na_n - b_n)}{2(1-\alpha)e^{-i\lambda} \cos \lambda} z^{n-1} \neq 0.$$

This completes the proof of the lemma. □

Applying Lemma 2.1, we obtain

Theorem 2.3 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (ja_j + ((1-\alpha)e^{-2i\lambda} - \alpha)b_j) (-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right| \right. \\ \left. + \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (ja_j - b_j) (-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right| \right] \leq 2(1-\alpha) \cos \lambda$$

for some α ($0 \leq \alpha < 1$), λ ($-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$), $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*$, then $f(z) \in \mathcal{CC}_\lambda(\alpha; g(z))$.

Proof. Applying the same method of the proof in Theorem 2.1, we know that $f(z)$ belongs to $\mathcal{CC}_\lambda(\alpha; g(z))$ if $f(z) \in \mathcal{A}$ satisfies

$$1 + \sum_{n=2}^{\infty} \left[\sum_{k=1}^n \left\{ \sum_{j=1}^k B_j (-1)^{k-j} c_{k-j} \right\} d_{n-k} \right] \neq 0$$

where $c_n = \binom{\beta}{n}$, $d_n = \binom{\gamma}{n}$ and B_j is defined by (2.3).

Now, we consider that

$$\begin{aligned} & \sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k B_j (-1)^{k-j} c_{k-j} \right\} d_{n-k} \right| \\ &= \sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k \frac{ja_j + (2(1-\alpha)e^{-i\lambda} \cos \lambda - 1)b_j + x(ja_j - b_j)}{2(1-\alpha)e^{-i\lambda} \cos \lambda} (-1)^{k-j} c_{k-j} \right\} d_{n-k} \right| \\ &\leq \frac{1}{2(1-\alpha) \cos \lambda} \sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (ja_j + (2(1-\alpha)e^{-i\lambda} \cos \lambda - 1)b_j) (-1)^{k-j} c_{k-j} \right\} d_{n-k} \right| \right. \\ &\quad \left. + |x| \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (ja_j - b_j) (-1)^{k-j} c_{k-j} \right\} d_{n-k} \right| \right] \\ &\leq 1. \end{aligned}$$

This implies that if $f(z) \in \mathcal{A}$ satisfies

$$\begin{aligned} & \sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (ja_j + (2(1-\alpha)e^{-i\lambda} \cos \lambda - 1)b_j) (-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right| \right. \\ &\quad \left. + \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (ja_j - b_j) (-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right| \right] \leq 2(1-\alpha) \cos \lambda, \end{aligned}$$

then $f(z) \in \mathcal{CC}_\lambda(\alpha; g(z))$. This completes the proof of Theorem 2.3. $\square$

Considering $g(z) = f(z)$ in Theorem 2.3 and noting (2.7), we have the following corollary.

Corollary 2.3 (see, [1], Theorem 3) *If $f(z) \in \mathcal{A}$ satisfies the following inequality*

$$\begin{aligned} & \sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (j - \alpha + (1-\alpha)e^{-2i\lambda}) (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right. \\ &\quad \left. + \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (j-1) (-1)^{k-j} \binom{\beta}{k-j} a_j \right\} \binom{\gamma}{n-k} \right| \right] \leq 2(1-\alpha) \cos \lambda \end{aligned}$$

for some α ($0 \leq \alpha < 1$), λ ($-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$), $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{SP}(\lambda, \alpha)$.

Furthermore, setting $\lambda = 0$ in Theorem 2.3, we obtain the following condition for $\mathcal{CC}_0(\alpha; g(z))$.

Corollary 2.4 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (ja_j + (1-2\alpha)b_j) (-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right| \right. \\ \left. + \left| \sum_{k=1}^n \left\{ \sum_{j=1}^k (ja_j - b_j) (-1)^{k-j} \binom{\beta}{k-j} \right\} \binom{\gamma}{n-k} \right| \right] \leq 2(1-\alpha)$$

for some α ($0 \leq \alpha < 1$), $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*$, then $f(z) \in \mathcal{CC}_0(\alpha; g(z))$.

References

- [1] T. Hayami, S. Owa, and H. M. Srivastava, *Coefficient inequalities for certain classes of analytic and univalent functions*, preprint.
- [2] I. R. Nezhmetdinov and S. Ponnusamy, *New coefficient conditions for the starlikeness of analytic functions and their applications*, Houston J. Math. **31**, No. 2 (2005), 587-604.
- [3] H. Silverman, E. M. Silvia, and D. Telage, *Convolution conditions for convexity, starlikeness and spiral-likeness*, Math. Z., **162** (1978), 125-130.

Toshio Hayami
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
E-mail: ha_ya_to112@hotmail.com

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
E-mail: owa@math.kindai.ac.jp