

Stability, Bifurcation and Classification of Minimal Sets in Random Complex Dynamics

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Since nature has many random terms, it is natural and important to investigate random dynamical systems. Many physicists are investigating “noise-induced phenomena” (new phenomena caused by noise and randomness, e.g. [1]) in random dynamical systems. Regarding the dynamics of a rational map h with $\deg(h) \geq 2$ on the Riemann sphere $\hat{\mathbb{C}}$, we always have the **chaotic part** in $\hat{\mathbb{C}}$. **However**, we show that in the (i.i.d.) random dynamics of polynomials on $\hat{\mathbb{C}}$, **generically**, (1) **the chaos of the averaged system disappears**, due to the automatic cooperation of many kinds of maps in the system (**cooperation principle**), and (2) the limit states are **stable** under perturbations of the system.

Moreover, we investigate the **bifurcation** of 1-parameter families of random complex dynamical systems.

Definition 1.

- (1) We denote by $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ the Riemann sphere and denote by d the spherical distance on $\hat{\mathbb{C}}$.
- (2) We set $\text{Rat} := \{h : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \mid h \text{ is a non-const. rational map}\}$ endowed with the distance η defined by $\eta(f, g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$. We set $\text{Rat}_+ := \{h \in \text{Rat} \mid \deg(h) \geq 2\}$.
- (3) We set $\mathcal{P} := \{h : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \mid h \text{ is a polynomial map, } \deg(h) \geq 2\}$ endowed with the relative topology from Rat .
- (4) For a metric space X , we denote by $\mathfrak{M}_1(X)$ the space of all Borel probability measures on X . We set

$$\mathfrak{M}_{1,c}(X) := \{\tau \in \mathfrak{M}_1(X) \mid \text{supp } \tau \text{ is compact}\},$$

where $\text{supp } \tau$ denotes the topological support of τ .

From now on, we take a $\tau \in \mathfrak{M}_1(\text{Rat})$ and we consider the **(i.i.d.) random dynamics** on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in \text{Rat}$ according to τ . This determines a time-discrete Markov process with time-homogeneous transition probabilities on the phase space $\hat{\mathbb{C}}$ such that for each $x \in \hat{\mathbb{C}}$ and for each Borel measurable subset A of $\hat{\mathbb{C}}$, the transition probability $p(x, A)$ from x to A is defined as

$$p(x, A) = \tau(\{h \in \text{Rat} \mid h(x) \in A\}).$$

- (5) Note that Rat and $\mathcal{P}$ are **semigroups** where the semigroup operation is functional composition. A subsemigroup of Rat is called a **rational semigroup**. A subsemigroup G of $\mathcal{P}$ is called a **polynomial semigroup**.

- (6) For a rational semigroup G , we set

$$F(G) := \{z \in \hat{\mathbb{C}} \mid \exists \text{ nbd } U \text{ of } z \text{ s.t. } G \text{ is equicontinuous on } U\}.$$

This $F(G)$ is called the **Fatou set** of G . Moreover, we set

$$J(G) := \hat{\mathbb{C}} \setminus F(G).$$

This $J(G)$ is called the **Julia set** of G .

- (7) **(Key)** For a rational semigroup G , we set

$$J_{\text{ker}}(G) := \bigcap_{h \in G} h^{-1}(J(G)).$$

This is called the **kernel Julia set** of G .

- (8) For a $\tau \in \mathfrak{M}_1(\text{Rat})$, let G_τ be the rational semigroup generated by $\text{supp } \tau$. Thus G_τ is the set of all finite compositions of elements in $\text{supp } \tau$.

Remark: Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. If there exists an $f_0 \in \mathcal{P}$ and a non-empty open subset U of $\mathbb{C}$ s.t. $\{f_0 + c \mid c \in U\} \subset \text{supp } \tau$, then $J_{\text{ker}}(G_\tau) = \emptyset$. Thus, for **most** $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$, $J_{\text{ker}}(G_\tau) = \emptyset$.

Theorem 0.1 (Theorem A, Cooperation Principle and Disappearance of Chaos). *Let $\tau \in \mathfrak{M}_{1,c}(\text{Rat}_+)$. Suppose $J_{\ker}(G_\tau) = \emptyset$. Then, we have all of the following (1)(2)(3).*

- (1) *We say that a non-empty compact subset L of $\hat{\mathbb{C}}$ is a **minimal set** of G_τ if L is minimal in*

$$\{K \subset \hat{\mathbb{C}} \mid \emptyset \neq K \text{ is compact, } \forall h \in G_\tau, h(K) \subset K\}$$

with respect to the inclusion. Moreover, we set

$$\text{Min}(G_\tau) := \{L \mid L \text{ is a minimal set of } G_\tau\}.$$

Then, $1 \leq \#\text{Min}(G_\tau) < \infty$.

- (2) *For each $z \in \hat{\mathbb{C}}$, there exists a Borel subset $\mathcal{A}_z$ of $(\text{Rat})^\mathbb{N}$ with $(\prod_{j=1}^\infty \tau)(\mathcal{A}_z) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \dots) \in \mathcal{A}_z$, the following (a) and (b) hold.*

- (a) *There exists a $\delta = \delta(z, \gamma) > 0$ such that $\text{diam} \gamma_n \cdots \gamma_1(B(z, \delta)) \rightarrow 0$ as $n \rightarrow \infty$.*

- (b) *$d(\gamma_n, \cdots \gamma_1(z), \bigcup_{L \in \text{Min}(G_\tau)} L) \rightarrow 0$ as $n \rightarrow \infty$.*

- (3) *We set $C(\hat{\mathbb{C}}) := \{\varphi : \hat{\mathbb{C}} \rightarrow \mathbb{C} \mid \varphi \text{ is conti.}\}$ endowed with the sup. norm $\|\cdot\|_\infty$. Let $M_\tau : C(\hat{\mathbb{C}}) \rightarrow C(\hat{\mathbb{C}})$ be the operator defined by*

$$M_\tau(\varphi)(z) := \int_{\text{Rat}} \varphi(h(z)) d\tau(h), \quad \forall \varphi \in C(\hat{\mathbb{C}}), \forall z \in \hat{\mathbb{C}}.$$

*Let $\mathcal{U}_\tau$ be the space of all finite linear combinations of unitary eigenvectors of $M_\tau : C(\hat{\mathbb{C}}) \rightarrow C(\hat{\mathbb{C}})$, where an eigenvector is said to be **unitary** if the absolute value of the corresponding eigenvalue is 1.*

Then, $1 \leq \dim_{\mathbb{C}} \mathcal{U}_\tau < \infty$ and

$$C(\hat{\mathbb{C}}) = \mathcal{U}_\tau \oplus \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \rightarrow 0 \text{ as } n \rightarrow \infty\}.$$

Moreover, each $\varphi \in \mathcal{U}_\tau$ is locally constant on $F(G_\tau)$ and is Hölder continuous on $\hat{\mathbb{C}}$.

Remark: Theorem A describes **new phenomena** which **cannot hold in the usual iteration dynamics** of a single $h \in \text{Rat}$ with $\deg(h) \geq 2$.

When $J_{\ker}(G_\tau) = \emptyset$?

Definition 2. Let $\tau \in \mathfrak{M}_{1,c}(\text{Rat}_+)$. We say that τ is **mean stable** if there exist non-empty open subsets U, V of $F(G_\tau)$ and a number $n \in \mathbb{N}$ such that all of the following (1)(2)(3) hold.

- (1) $\bar{V} \subset U \subset F(G_\tau)$.
- (2) For all $\gamma = (\gamma_1, \gamma_2, \dots) \in (\text{supp } \tau)^\mathbb{N}$, $(\gamma_n \circ \dots \circ \gamma_1)(U) \subset V$.
- (3) For all $z \in \hat{\mathbb{C}}$, there exists an $h \in G_\tau$ such that $h(z) \in U$.

Remark: If τ is mean stable, then $J_{\ker}(G_\tau) = \emptyset$. Note that **the converse is NOT true** in general.

When is a $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ mean stable?

Definition 3. Let $\mathcal{Y}$ be a closed subset of Rat . Let $\mathcal{O}$ be the topology in $\mathfrak{M}_{1,c}(\mathcal{Y})$ such that $\tau_n \rightarrow \tau$ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ if and only if

- (1) $\int \varphi d\tau_n \rightarrow \int \varphi d\tau$ for each bounded continuous function $\varphi : \mathcal{Y} \rightarrow \mathbb{R}$, and
- (2) $\text{supp } \tau_n \rightarrow \text{supp } \tau$ with respect to the Hausdorff metric in the space of all non-empty compact subsets of $\mathcal{Y}$.

Theorem 0.2 (Theorem B). (Density of Mean Stable Systems) The set $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid \tau \text{ is mean stable}\}$ is open and dense in $(\mathfrak{M}_{1,c}(\mathcal{P}), \mathcal{O})$.

Theorem 0.3 (Theorem C, Stability). Suppose $\tau \in \mathfrak{M}_{1,c}(\text{Rat}_+)$ is mean stable. Then there exists a neighborhood Ω of τ in $(\mathfrak{M}_{1,c}(\text{Rat}_+), \mathcal{O})$ such that all of the following (1)(2)(3) hold.

- (1) For each $\nu \in \Omega$, ν is mean stable, $J_{\ker}(G_\nu) = \emptyset$ and thus Theorem A for ν holds.
- (2) The map $\nu \mapsto \mathcal{U}_\nu$ is continuous on Ω .
- (3) The map $\nu \mapsto \#\text{Min}(G_\nu)$ is constant on Ω .

Theorem 0.4 (Theorem D, Bifurcation). For each $t \in [0, 1]$, let μ_t be an element of $\mathfrak{M}_{1,c}(\text{Rat}_+)$. Suppose that all of the following (1)–(4) hold.

- (1) $t \mapsto \mu_t \in (\mathfrak{M}_{1,c}(\text{Rat}_+), \mathcal{O})$ is continuous on $[0, 1]$.

- (2) If $t_1, t_2 \in [0, 1]$ and $t_1 < t_2$, then $\text{supp } \mu_{t_1} \subset \text{int}(\text{supp } \mu_{t_2})$ with respect to the topology of Rat_+ .
- (3) $\text{int}(\text{supp } \mu_0) \neq \emptyset$ and $F(G_{\mu_1}) \neq \emptyset$.
- (4) $\sharp \text{Min}(G_{\mu_0}) \neq \sharp \text{Min}(G_{\mu_1})$.

Let $B := \{t \in [0, 1] \mid \mu_t \text{ is not mean stable}\}$.

Then, we have all of the following (a)(b)(c)(d).

- (a) For each $t \in [0, 1]$, we have $J_{\ker}(G_{\mu_t}) = \emptyset$ and all statements in Theorem A (with $\tau = \mu_t$) hold.
- (b) $1 \leq \sharp(B \cap [0, 1]) \leq \sharp \text{Min}(G_{\mu_0}) - \sharp \text{Min}(G_{\mu_1}) < \infty$.
- (c) For each $t \in [0, 1] \setminus B$ and for each $L \in \text{Min}(G_{\mu_t})$, L is **attracting** for G_{μ_t} , i.e. there exist non-empty open subsets U, V of $F(G_{\mu_t})$ and a number $n \in \mathbb{N}$ such that
 - (i) $L \subset V \subset \overline{V} \subset U \subset F(G_{\mu_t})$, and
 - (ii) for each $\gamma = (\gamma_1, \gamma_2, \dots) \in (\text{supp } \mu_t)^\mathbb{N}$, $\gamma_n \cdots \gamma_1(U) \subset V$.
- (d) For each $t \in B$, there exists an $L \in \text{Min}(G_{\mu_t})$ such that either
 - (i) L is **J-touching** for G_{μ_t} , i.e., $L \cap J(G_{\mu_t}) \neq \emptyset$, or
 - (ii) L is **sub-rotative** for G_{μ_t} , i.e., $L \subset F(G_{\mu_t})$ and L meets a Siegel disc or a Hermann ring of some element of G_{μ_t} .

Idea of proofs of results.

Lemma 1 (Classification of Minimal Sets). *Let $\tau \in \mathfrak{M}_{1,c}(\text{Rat}_+)$. Let $L \in \text{Min}(G_\tau)$. Then, exactly one of the following holds.*

- (a) L is attracting for G_τ .
- (b) L is J-touching for G_τ .
- (c) L is sub-rotative for G_τ .

(For the definitions of the terms “attracting”, “J-touching” and “sub-rotative”, see Theorem D with μ_t replaced by τ .)

Outline of the proof of Theorem B: Take any $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Enlarge $\text{supp } \tau$ just a little bit. Then any J -touching or sub-rotative minimal set of G_τ collapses. Now we observe that each minimal set of G_ν is attracting if and only if ν is mean stable.

Summary and Remarks: (1) Regarding the random dynamics of polynomials, **generically**, the chaos of the averaged system **disappears** and the limit states are **stable** under perturbations of the system. (2) In order to prove the above result, we need the **classification of minimal sets**. (3) We can investigate the **bifurcation** of the 1-parameter family of random complex dynamical systems. (4) There exist a lot of examples of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ such that $J_{\ker}(G_\tau) = \emptyset$ (thus the chaos disappears) but τ is **not mean stable**. At such a τ , a kind of **bifurcation** occurs. (5) There exists an example of means stable $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $\# \text{supp } \tau < \infty$ such that there exists a $\varphi \in \mathcal{U}_\tau$ whose Hölder exponent is strictly less than 1 (“**Devil’s Coliseum**”, which is the function of probability of tending to ∞ . To prove this result, we use ergodic theory and potential theory). Therefore, even if the chaos disappears in the “ C^0 ” sense, the chaos may remain in the “ C^1 ” sense (or in the space of Hölder continuous functions with some exponent $\alpha_0 < 1$). Thus, in random dynamics, we have a kind of gradation between non-chaos and chaos. It is interesting to investigate the pointwise Hölder exponent of the above φ . The above φ is a continuous function on $\hat{\mathbb{C}}$ which varies precisely on the Julia set $J(G_\tau)$, which is a thin fractal set. Thus it is important to estimate the Hausdorff dimension $\dim_H(J(G_\tau))$ of $J(G_\tau)$. By using the thermodynamical formalisms, we can show that $\dim_H(J(G_\tau))$ is equal to the zero of the pressure function, under certain conditions. Also, in order to investigate the pointwise Hölder exponent of this function φ in detail, we can sometimes apply the thermodynamical formalisms. We are very interested in studying the pointwise-Hölder-exponent spectrum of this function $\varphi \in \mathcal{U}_\tau$.

References

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Also available from <http://arxiv.org/abs/0812.4483>. **(Some of the contents of this presentation are included in this paper.)**

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