

Hyperbolic nonwandering sets

without dense periodic points

by

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Let $f:M \rightarrow M$ be a C^∞ diffeomorphism of a closed C^∞ manifold M , and let $\Omega(f)$ be the nonwandering set of f . $\Omega(f)$ is hyperbolic if $\Omega(f)$ is compact and the restriction $T_{\Omega(f)}M$ of the tangent bundle TM of M on $\Omega(f)$ splits into the Whitney sum of Tf -invariant subbundles

$$T_{\Omega(f)}M = E^s \oplus E^u,$$

such that given a Riemannian metric on TM there are positive numbers c and $\lambda < 1$ such that $|Tf^n v| < c\lambda^n |v|$, for $v \in E^s$ and $n > 0$, and $|Tf^{-n} v| < c\lambda^n |v|$, for $v \in E^u$ and $n > 0$. The following problem was suggested in [3].

Problem. If a nonwandering set $\Omega(f)$ is hyperbolic, are the periodic points dense in $\Omega(f)$?

Newhouse and Palis proved that the answer is affirmative when M is a two dimensional closed manifold ([1], [2]).

In this paper we give the following.

Theorem. Suppose $\dim M \geq 4$. Then there is a diffeomorphism $F:M \rightarrow M$ such that the nonwandering set $\Omega(F)$ is hyperbolic but its periodic points are not dense in $\Omega(F)$.

Construction.

To simplify the construction, we assume $\dim M = 4$.

1. Denote $D = [-2, 6] \times [-1, 3] \subset \mathbb{R}^2$. Let an embedding $f:D \rightarrow D$ satisfy the followings (figure 1). Suppose that real numbers $a_{-1}, \dots, a_6$ satisfy

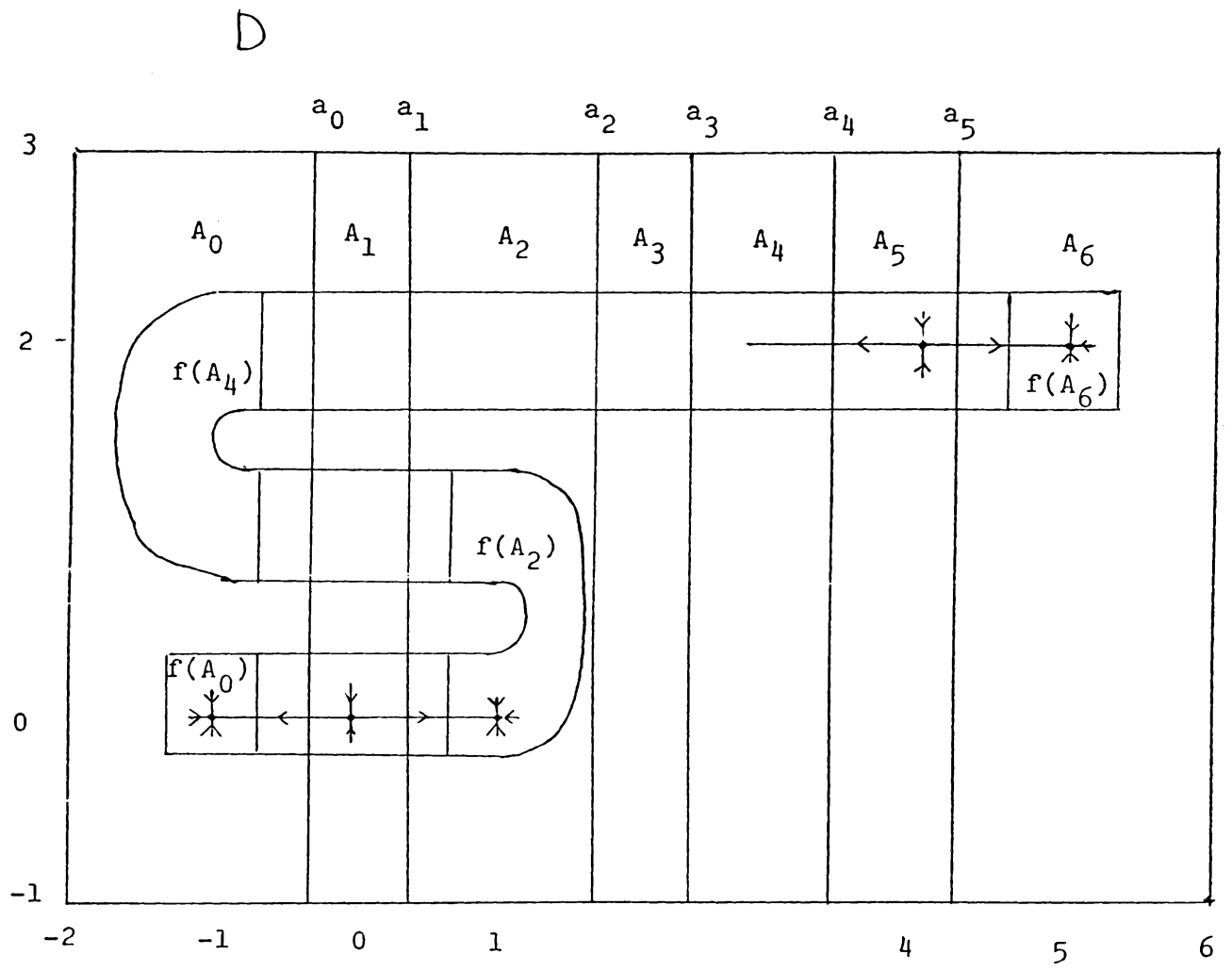


figure 1

$$\begin{aligned} \underline{1.1} \quad a_{-1} = -2 < -1 < a_0 = -a_1 < 0 < a_1 < 1 < a_2 < a_3 \\ < a_4 < 4 < a_5 < 5 < a_6 = 6, \end{aligned}$$

and the rectangle A_i ($i = 0, \dots, 6$) is given by

$$A_i = \{ (x, y) \in D \mid a_{i-1} \leq x \leq a_i \}.$$

Then f satisfies $\underline{1.2} \sim \underline{1.5}$.

1.2 $f|_{A_0}$, $f|_{A_2}$ and $f|_{A_6}$ are contractions with three sinks $(-1, 0)$, $(1, 0)$ and $(5, 2)$,

$$\underline{1.3} \quad f(A_4) \subset \text{int}A_0,$$

1.4 $f|_{A_i} : A_i \rightarrow f(A_i)$ ($i = 1, 3, 5$) maps A_i linearly onto a rectangle $f(A_i)$, expanding horizontally and contracting vertically. There are two hyperbolic fixed points, $(0, 0)$ and $(4, 2)$.

1.5 There are numbers $\alpha > 1$ and $0 < \beta < 1$ such that

$$f(x, y) = \begin{cases} (\alpha x, \beta y) & \text{for } (x, y) \in A_1 \\ (\alpha(x-4)+4, \beta(y-2)+2) & \text{for } (x, y) \in A_5. \end{cases}$$

2. Let $D' \subset \mathbb{R}^2$ satisfy the followings (figure 2).

D' is a neighbourhood of $(\{0\} \times [-1, 1]) \cup ([-2, 0] \times \{0\})$ which is diffeomorphic to a 2-dimensional disk, and there is a sufficiently small positive number ε such that

$$\{(x, y) \in D' \mid |y+1| \leq \varepsilon\} = [-\varepsilon, \varepsilon] \times [-1-\varepsilon, -1+\varepsilon]$$

and

$$\{(x, y) \in D' \mid |x+1| \leq \varepsilon\} = [-1-\varepsilon, -1+\varepsilon] \times [-\varepsilon, \varepsilon].$$

Let an embedding $g: D' \rightarrow D'$ satisfy $\underline{2.1} \sim \underline{2.9}$.

$$\underline{2.1} \quad g(D') \subset \text{int}D',$$

$$\underline{2.2} \quad g \text{ is isotopic to the identity,}$$

$$\underline{2.3} \quad \bigcap_{n>0} g^n(D') = (\{0\} \times [-1, 1]) \cup ([-2, 0] \times \{0\}),$$

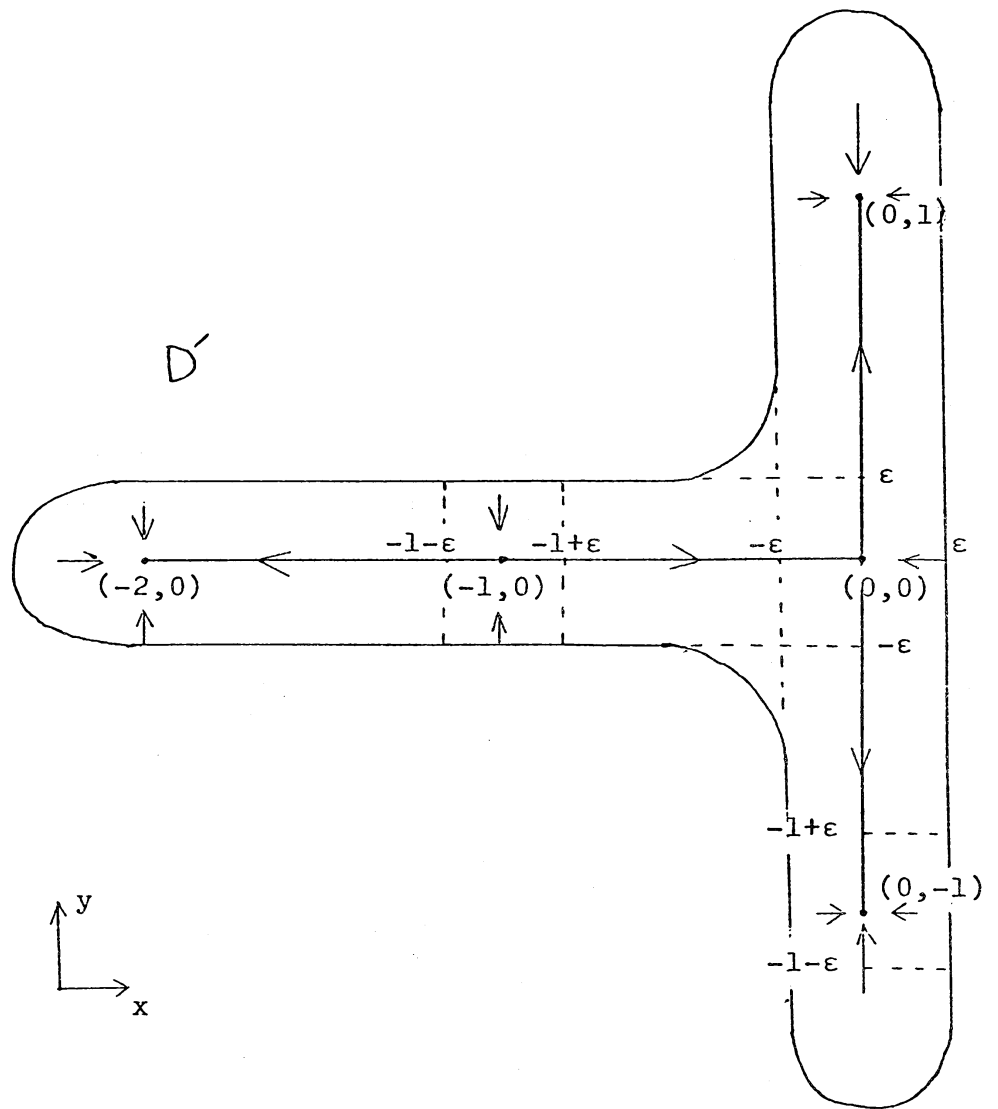


figure 2

2.4 There are five fixed points: three sinks $(-2,0)$, $(0,1)$, $(0,-1)$, and two saddle points $(0,0)$, $(-1,0)$.

$$\underline{2.5} \quad W^u((0,0)) = \{0\} \times (-1,1),$$

$$\underline{2.6} \quad W^u((-1,0)) = (-2,0) \times \{0\},$$

$$\underline{2.7} \quad W^s((0,0)) \cap D' = \{(x,0) \in D' \mid x \geq -1\},$$

where $W^s(p)$ (resp. $W^u(p)$) is the stable (resp. unstable) manifold through p . $(-1,1)$ and $(-2,0)$ denote open intervals.

$$\underline{2.8} \quad g(x,y) = \left(\frac{1}{2}x, \frac{1}{2}(y+1)-1 \right) \quad \text{if } |y+1| \leq \epsilon,$$

$$\underline{2.9} \quad g(x,y) = \left(2(x+1)-1, \frac{1}{2}y \right) \quad \text{if } |x+1| \leq \epsilon.$$

3. Define

$$N = D \times D' \cup_{\psi} D^3(\delta) \times [0, 1],$$

where

$$D^3(\delta) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid \sqrt{y_1^2 + y_2^2 + y_3^2} \leq \delta\}$$

and

$$0 < \delta < \frac{1}{4}\epsilon.$$

The attaching map

$$\psi : D^3(\delta) \times ([0, \epsilon] \cup [1-\epsilon, 1]) \longrightarrow D \times D'$$

is given by

$$\psi(y_1, y_2, y_3, t) = \begin{cases} (y_1, y_2, t, y_3-1) & \text{if } 0 \leq t \leq \epsilon \\ (y_1+4, y_2+2, y_3-1, 1-t) & \text{if } 1-\epsilon \leq t \leq 1 \end{cases}$$

(figure 3).

In 4 ~ 10, we will construct an embedding $F: N \longrightarrow N$.

After this, (x_1, x_2, x_3, x_4) (resp. (y_1, y_2, y_3, t)) denotes a point of $D \times D' \subset N$ (resp. $D^3(\delta) \times [0, 1] \subset N$).

4. For $(x_1, x_2, x_3, x_4) \in D \times D'$ with $|x_3+1| \geq \epsilon$ and $|x_4+1| \geq \epsilon$, define

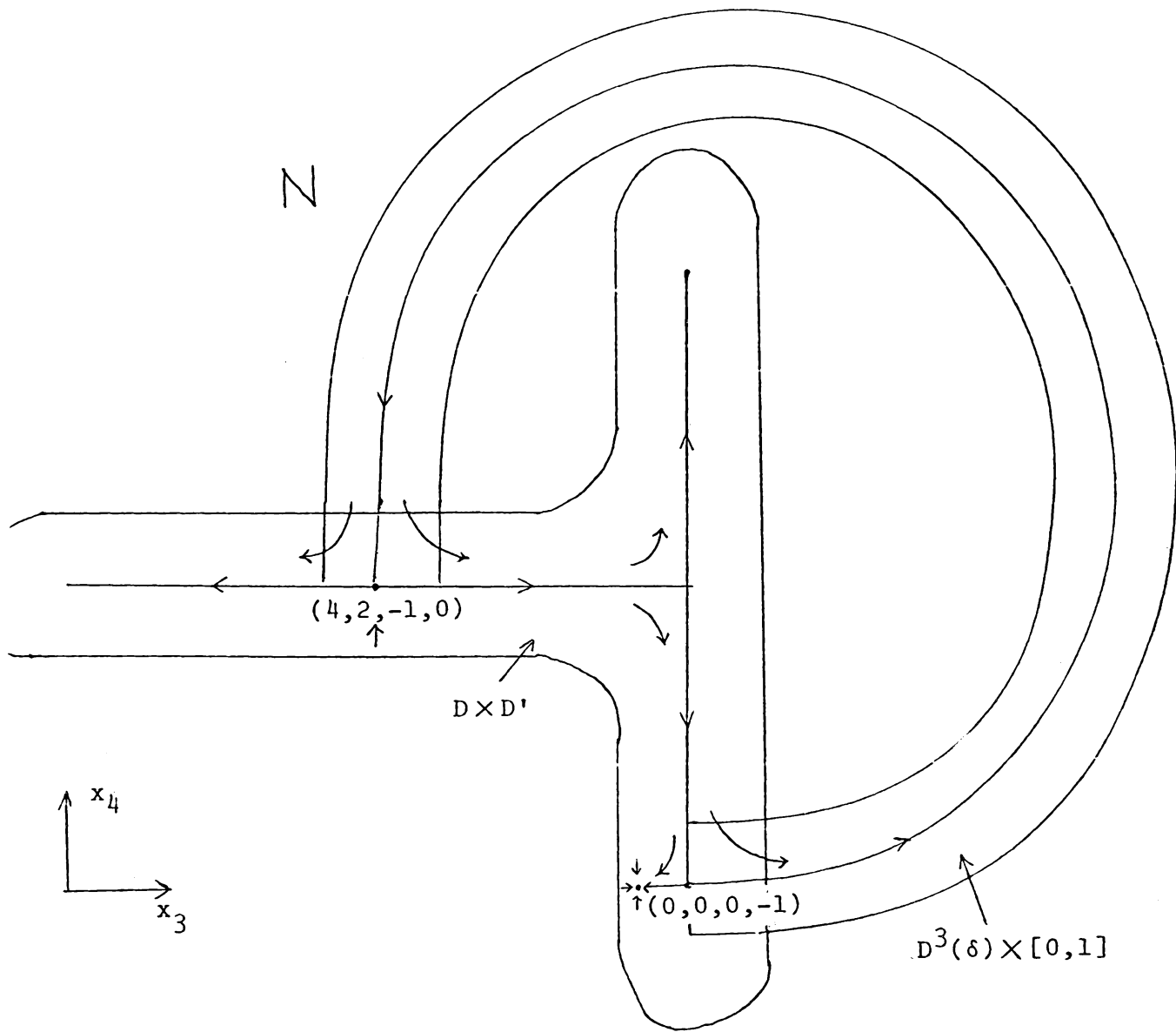


figure 3

$$\underline{4.1} \quad F(x_1, x_2, x_3, x_4) = (f(x_1, x_2), g(x_3, x_4)).$$

5. For $(x_1, x_2, x_3, x_4) \in D \times D'$ with $\frac{1}{4}\epsilon \leq |x_4+1| \leq \epsilon$,

define

$$\underline{5.1} \quad F(x_1, x_2, x_3, x_4) = (f|_{|x_4+1|}(x_1, x_2), g(x_3, x_4)), \text{ where}$$

$f_t: D \rightarrow D$ ($0 \leq t \leq \epsilon$) is an isotopy satisfying $\underline{5.2} \sim \underline{5.6}$.

Suppose that positive numbers $b_1, \dots, b_4$ satisfy

$$\underline{5.2} \quad 0 < b_1 < b_2 < \delta < b_3 < b_4 < a_1,$$

$$ab_1 < b_2,$$

and

$$b_4 < \min\{4-a_4, a_5-4\}.$$

Then

$$\underline{5.3} \quad f_t(x_1, x_2) = f(x_1, x_2) \quad \text{if} \quad |x_1| < b_1 \quad \text{or} \quad |x_1| > b_4,$$

$$\underline{5.4} \quad f_t = f \quad \text{for} \quad \frac{1}{2}\epsilon \leq t \leq \epsilon,$$

$$\underline{5.5} \quad f_t = f_0 \quad \text{for} \quad 0 \leq t \leq \frac{1}{4}\epsilon,$$

and

$$\underline{5.6} \quad f_t(x_1, x_2) = (\bar{f}_t(x_1), \beta x_2) \quad \text{for} \quad |x_1| \leq b_4,$$

where $\bar{f}_t$ is an isotopy of a neighbourhood of 0 in $\mathbb{R}^1$ and

$\bar{f}_0$ has five fixed points: three sources 0, $\pm b_3$, and two sinks $\pm b_2$.

6. For $(x_1, x_2, x_3, x_4) \in D \times D'$ with $|x_4+1| < \frac{1}{4}\epsilon$, F is defined as follows. Let

$$\underline{6.1} \quad U = \{(x_1, x_2, x_3, x_4) \in D \times D' \mid \sqrt{x_1^2 + x_2^2 + (x_4+1)^2} \leq \delta\},$$

and

$$\underline{6.2} \quad U_1 = \{(x_1, x_2, x_3, x_4) \in D \times D' \mid \sqrt{x_1^2 + x_2^2 + (x_4+1)^2} \leq \delta_1\},$$

where $b_2 < \delta_1 < \delta$.

Then F is defined as follows.

$$\begin{aligned} \underline{6.3} \quad F(x_1, x_2, x_3, x_4) &= (f_0(x_1, x_2), g(x_3, x_4)) \\ &\text{if } (x_1, x_2, x_3, x_4) \in D \times D' - U \quad \text{and} \quad |x_4 + 1| < \frac{1}{4}\epsilon, \end{aligned}$$

$$\begin{aligned} \underline{6.4} \quad F(x_1, x_2, x_3, x_4) &= (f_0(x_1, x_2), \bar{g}(x_1, x_2, x_3, x_4), \frac{1}{2}(x_4 + 1) - 1) \\ &\text{if } (x_1, x_2, x_3, x_4) \in U \cap F^{-1}(U), \end{aligned}$$

where $\bar{g}$ satisfies $\underline{6.5} \sim \underline{6.7}$.

$$\underline{6.5} \quad \bar{g}(x_1, x_2, x_3, x_4) = \frac{1}{2}x_3 \quad \text{near the frontier of } U,$$

$$\begin{aligned} \underline{6.6} \quad \bar{g}(x_1, x_2, x_3, x_4) &= 2x_3 \\ &\text{if } (x_1, x_2, x_3, x_4) \in U_1 \quad \text{and} \quad -\frac{1}{4}\epsilon \leq x_3 \leq \frac{1}{2}\epsilon, \end{aligned}$$

and

$$\underline{6.7} \quad \bar{g}(x_1, x_2, x_3, x_4) \text{ does not depend on } x_1 \text{ if } |x_1| < b_1.$$

$$\begin{aligned} \underline{6.8} \quad F(\{(x_1, x_2, x_3, x_4) \in U \mid x_3 < 0\}) \\ \subset \{(x_1, x_2, x_3, x_4) \in U \mid x_3 < 0\}. \end{aligned}$$

In $\{(x_1, x_2, x_3, x_4) \in U \mid x_3 < 0\}$ there are only a finite number of nonwandering points, which are hyperbolic fixed points.

Furthermore F satisfies the conditions in 10.

7. On $D^3(\delta) \times [0, 1 - \epsilon]$, F is given as follows

$$\underline{7.1} \quad F(y_1, y_2, y_3, t) = (f_0(y_1, y_2), \frac{1}{2}y_3, \phi(y_1, y_2, y_3, t)) \in D^3(\delta) \times [0, 1],$$

where ϕ satisfies the followings.

$$\text{If } \sqrt{y_1^2 + y_2^2 + y_3^2} < \delta_1 \quad \text{or} \quad \frac{1}{2} < t,$$

$$\underline{7.2} \quad \phi(y_1, y_2, y_3, t) \text{ depends only on } t$$

and

$$\underline{7.3} \quad \frac{\partial \phi}{\partial t} > 0.$$

$$\underline{7.4} \quad \phi(y_1, y_2, y_3, t) = 1 - \frac{1}{2}(1 - t) \quad \text{for} \quad 1 - 2\epsilon \leq t \leq 1 - \epsilon.$$

$$\underline{7.5} \quad \phi(y_1, y_2, y_3, t) = \bar{g}(y_1, y_2, t, y_3 - 1) \quad \text{if} \quad 0 \leq t \leq \epsilon.$$

Moreover F satisfies 10.

$$8. \quad \text{For } (x_1, x_2, x_3, x_4) \in D \times D' \quad \text{with} \quad |x_3 + 1| < \frac{1}{4}\epsilon,$$

F is given as follows. Let $h_t: D \rightarrow D$ ($0 \leq t \leq \epsilon$) be an isotopy such that

$$\underline{8.1} \quad h_t = f \quad \text{if} \quad \frac{1}{2}\epsilon \leq t \leq \epsilon,$$

$$\underline{8.2} \quad h_t(x_1, x_2) = f(x_1, x_2) \\ \text{if} \quad -2 \leq x_1 \leq 4-b_4 \quad \text{or} \quad 4+b_4 \leq x_1 \leq 6,$$

and

$$\underline{8.3} \quad h_t(x_1, x_2) = f(x_1-4, x_2-2) + (4, 2) \quad \text{if} \quad |x_1-4| < b_4.$$

Then

$$\underline{8.4} \quad F(x_1, x_2, x_3, x_4) = (h_0(x_1, x_2), \bar{h}(x_1, x_2, x_3, x_4), \frac{1}{2}x_4),$$

where $\bar{h}$ satisfies the followings.

$$\underline{8.5} \quad \bar{h}(x_1, x_2, x_3, x_4) = \frac{1}{2}(x_3+1)-1 \\ \text{if} \quad \sqrt{(x_1-4)^2 + (x_2-2)^2 + (x_3+1)^2} \leq \delta \quad \text{and} \quad x_4 > \frac{2}{3}\epsilon,$$

$$\underline{8.6} \quad \bar{h}(x_1, x_2, x_3, x_4) = 2(x_3+1)-1 \\ \text{if} \quad \sqrt{(x_1-4)^2 + (x_2-2)^2 + (x_3+1)^2} \geq \delta_2 \quad \text{or} \quad x_4 < \frac{1}{3}\epsilon,$$

where $\delta < \delta_2 < \frac{1}{4}\epsilon$.

$$\underline{8.7} \quad \bar{h}(x_1, x_2, x_3, x_4) \text{ does not depend on } x_1 \text{ if } |x_1-4| < b_1.$$

Furthermore F satisfies 10.

9. For $(x_1, x_2, x_3, x_4) \in D \times D'$ with $\frac{1}{4}\epsilon \leq |x_3+1| < \epsilon$, define

$$\underline{9.1} \quad F(x_1, x_2, x_3, x_4) = (h_{|x_3+1|}(x_1, x_2), 2(x_3+1)-1, \frac{1}{2}x_4).$$

10. F is an embedding of N such that

$$\underline{10.1} \quad F(N) \subset \text{int}N,$$

and

$$\underline{10.2} \quad F \text{ is isotopic to the identity.}$$

11. Straightening the corner (and modifying F near the

corner), we can regard N as a submanifold of M which is diffeomorphic to $D^3 \times S^1$. Extend F to a diffeomorphism of M such that the nonwandering sets of F in $M-N$ consists of a finite number of hyperbolic fixed points.

12. The nonwandering set of F consists of a finite number of hyperbolic fixed points and two non-periodic orbits

$\{(x_1, x_2, 0, 0) \in D \times D' \mid (x_1, x_2) \text{ satisfies 12.i} \} \quad (i = 1, 2),$

where

12.1 there is an integer n_0 such that

$$f^n(x_1, x_2) \in A_5 \quad \text{if} \quad n < n_0,$$

$$f^n(x_1, x_2) \in A_3 \quad \text{if} \quad n = n_0,$$

$$f_n(x_1, x_2) \in A_1 \quad \text{if} \quad n > n_0,$$

and

12.2 there is an integer n_0 such that

$$f^n(x_1, x_2) \in A_5 \quad \text{if} \quad n < n_0,$$

$$f^n(x_1, x_2) \in A_1 \quad \text{if} \quad n \geq n_0.$$

The details will be published elsewhere.

References

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