

On Weierstrass 7-semigroups

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§1. Introduction.

Let $\mathbb{N}$ be the additive semigroup of non-negative integers. A subsemigroup H of $\mathbb{N}$ is called a *numerical semigroup* if the complement $\mathbb{N} \setminus H$ of H in $\mathbb{N}$ is a finite set. For any positive integer n a numerical semigroup H is called an *n -semigroup* if H starts with n , i.e., the minimum positive integer in H is n . For a non-singular complete irreducible curve C over an algebraically closed field k of characteristic 0 (which is called a *curve* in this paper) and its point P we set

$$H(P) = \{n \in \mathbb{N} \mid \exists \text{ a rational function } f \text{ on } C \text{ with } (f)_\infty = nP\}.$$

A numerical semigroup is *Weierstrass* if there exists a curve C with its point P such that $H(P) = H$. We are interested in the following problem:

Problem 1. Is every n -semigroup Weierstrass ?

We have the following positive results:

Fact 2. For $n \leq 5$ every n -semigroup is Weierstrass. (For $n = 2$, classical, for $n = 3$, see [8] and for $n = 4, 5$, see [4], [5] respectively.)

But we know the negative result as follows:

Fact 3. For any $n \geq 13$, there exists a non-Weierstrass n -semigroup. (For $n = 13$, [1] and for $n \geq 14$ see, for example, [7].)

Thus, we have the following problem:

Problem 4. For $6 \leq n \leq 12$, is every n -semigroup Weierstrass or is there a non-Weierstrass n -semigroup ?

In this paper we are devoted to the study of 7-semigroups. In Section 2 we determine the 7-semigroups which are the semigroups $H(P)$ of ramification points P on cyclic coverings of the projective line $\mathbb{P}^1$ with degree 7. In Section 3 we divide the Weierstrass 7-semigroups generated by 4 elements into 31 cases and investigate whether such a 7-semigroup is of toric type in each case where a numerical semigroup is said to be *of toric type* if roughly speaking, the monomial curve associated to the numerical semigroup is a specialization of some affine toric variety, because we know that a numerical semigroup of toric type is Weierstrass ([4]).

§2. Cyclic 7-semigroups.

An n -semigroup is said to be *cyclic* if it is the semigroup $H(P)$ for some totally ramification point P on a cyclic covering of the projective line $\mathbb{P}^1$ with degree n . In this section we describe a necessary and sufficient condition on a 7-semigroup to be cyclic. Moreover, some non-cyclic Weierstrass 7-semigroups are given. We use the following notation: For an n -semigroup H we set

$$S(H) = \{n, s_1, \dots, s_{n-1}\}$$

where $s_i = \text{Min}\{h \in H \mid h \equiv i \pmod{n}\}$. We have the following necessary condition on an n -semigroup to be cyclic if n is prime.

Fact 5 ([9]). *Let p be a prime number. If H is a cyclic p -semigroup with*

$$S(H) = \{p, s_1, \dots, s_{p-1}\},$$

then

$$s_i + s_{p-i} = s_j + s_{p-j}, \text{ all } i, j.$$

We had already obtained an answer to the converse problem of the above statement.

Fact 6. i) *For a prime number $p \leq 7$, the converse of Fact 5 is true (See [9]).*
ii) *For any prime number $p \geq 11$, the converse of Fact 5 is false (See [3]).*

By Fact 6 i) we get the following:

Proposition 7. *Let H be a 7-semigroup with*

$$S(H) = \{7, s_1, \dots, s_6\}.$$

Assume that

$$s_1 + s_6 = s_2 + s_5 = s_3 + s_4.$$

Then H is cyclic, hence Weierstrass.

For any positive integers $b_0, \dots, b_m$, $\langle b_0, \dots, b_m \rangle$ denotes the semigroup generated by $b_0, \dots, b_m$. We give examples of cyclic 7-semigroups.

Example 8. (1) Let $H = \langle 7, 8, 10, 12 \rangle$. Then $S(H) = \{7, 8, 10, 12, 16, 18, 20\}$. Since $8 + 20 = 16 + 12 = 10 + 18$, H is cyclic, hence Weierstrass.

(2) Let $H = \langle 7, 15, 16, 17, 25, 26, 27 \rangle$. Then $S(H) = \{7, 15, 16, 17, 25, 26, 27\}$. Since $15 + 27 = 16 + 26 = 17 + 25$, H is cyclic, hence Weierstrass.

We also have non-cyclic Weierstrass 7-semigroups.

Fact 9. For integers g and s with $7 \leq g \leq s \leq 12$, let $H_{s,g}$ be a 7-semigroup with

$$\mathbb{N}_0 \setminus H_{s,g} = \{1, \dots, 6, 8 + s - g, \dots, s + 1\}.$$

Then we have the following:

- i) There exists a covering $C \rightarrow \mathbb{P}^1$ of degree 3 with non-ramification point $P \in C$ such that $H(P) = H_{s,g}$. Hence, $H_{s,g}$ is a Weierstrass 7-semigroup (See [2]).
- ii) If $(s, g) \neq (9, 9), (12, 9), (12, 12)$, then $H_{s,g}$ is non-cyclic. For example, $H_{11,9} = \langle 7, 8, 9, 13, 19 \rangle$ and $H_{12,10} = \langle 7, 8, 9, 19, 20 \rangle$ are non-cyclic Weierstrass 7-semigroups.

Fact 10. Let H be the 7-semigroup $\langle 7, 9, 10, 11, 12, 13 \rangle$. Then there is a cyclic covering of an elliptic curve of degree 8 with only two ramification points P_1 and P_2 , which are totally ramified, such that $H(P_1) = H(P_2) = H$. Hence $\langle 7, 9, 10, 11, 12, 13 \rangle$ is a non-cyclic Weierstrass 7-semigroup (See [6]).

§3. 7-semigroups of toric type.

For a numerical semigroup H we denote by $M(H)$ the minimal set of generators for H . In this section we are interested in 7-semigroups H with $M(H) = \{7, a_1, a_2, a_3\}$ which satisfy the following condition:

Definition 11. Let H be a numerical semigroup with $\sharp M(H) = m + 1$. The semigroup H is said to be of *toric type* if

- $\exists l$: a positive integer,
- $\exists S$: a saturated subsemigroup of $\mathbb{Z}^l$ generated by $b_1, \dots, b_{l+m}$ which generates $\mathbb{Z}^l$ as a group and
- $\exists g_j$'s ($j = 1, \dots, l + m$): monomials in $k[X_0, X_1, \dots, X_m]$ such that

$$\begin{array}{ccc} \text{Spec } k[H] & \hookrightarrow & \text{Spec } k[S][X_0, X_1, \dots, X_m] \\ \downarrow & \square & \downarrow \\ \text{Spec } k & \hookrightarrow & \text{Spec } k[Y_1, \dots, Y_{l+m}] \\ (0) & \mapsto & \text{the origin} \end{array}$$

where the right vertical map is induced by the k -algebra homomorphism

$$\eta_S : k[Y_1, \dots, Y_{l+m}] \longrightarrow k[S][X_0, X_1, \dots, X_m]$$

which sends Y_j to $T^{b_j} - g_j$, that is to say,

$$\begin{array}{ccc} \text{Spec } k[H] & \hookrightarrow & \text{Spec } k[X_0, X_1, \dots, X_m] \\ \downarrow & \square & \downarrow \\ \text{Spec } k[S] & \hookrightarrow & \text{Spec } k[Y_1, \dots, Y_{l+m}] \end{array}$$

where the horizontal maps are the embeddings through the generators and the right vertical map is induced by the k -algebra morphism from $k[Y_1, \dots, Y_{l+m}]$ to $k[X_0, X_1, \dots, X_m]$ sending Y_j to g_j .

We explain how to find a subsemigroup S of $\mathbb{Z}^l$ as in Definition 11 below.

Remark 12. Let H be a numerical semigroup with $M(H) = \{a_0, a_1, \dots, a_m\}$.

- i) Determine a generating system of relations among $a_0, a_1, \dots, a_m$, i.e., a set of generators for the ideal of the monomial curve $\text{Spec } k[H]$.
- ii) Determine a fundamental system of relations among $a_0, a_1, \dots, a_m$, i.e., a basis of the relation $\mathbb{Z}$ -module among $a_0, a_1, \dots, a_m$.
- iii) We construct a subsemigroup S of $\mathbb{Z}^l$ from the fundamental system. In this case, S is generated by $l + m$ elements b_j 's and generates $\mathbb{Z}^l$ as a group naturally. Moreover, we associate the generators b_j 's for S to monomials g_j 's in $k[X_0, \dots, X_m]$ such that we have the fiber products in Definition 11.
- iv) The remaining problem is whether the semigroup S is *saturated* or not. We note that S is saturated if and only if the semigroup ring $k[S]$ is normal, i.e., $\text{Spec } k[S]$ is an *affine toric variety*. If S is saturated, the numerical semigroup H become of *toric type*.

From now on we treat only 7-semigroups generated by 4 elements.

Lemma 13. Let H be a 7-semigroup generated by 4 elements, i.e., $M(H) = \{7, a_1, a_2, a_3\}$. Renumbering a_1, a_2 and a_3 it satisfies one of the following:

- (I) $a_1 + a_2 + a_3 \equiv 0 \pmod{7}$,
- (II) $a_1 + a_2 \equiv 0 \pmod{7}$,
- (III) $2a_1 + a_2 \equiv 0 \pmod{7}$ and $2a_2 + a_3 \equiv 0 \pmod{7}$.

We give the construction of a saturated subsemigroup S of $\mathbb{Z}^l$ as in Definition 11 in (I) and some cases of (II).

Case (I) $a_1 + a_2 + a_3 \equiv 0 \pmod{7}$. A fundamental system of relations consists of

$$\frac{a_1 + a_2 + a_3}{7}a_0 = a_1 + a_2 + a_3, \quad 2a_1 = \frac{2a_1 - a_2}{7}a_0 + a_2, \quad 2a_2 = \frac{2a_2 - a_3}{7}a_0 + a_3.$$

For example, the relation

$$2a_3 = \frac{2a_3 - a_1}{7}a_0 + a_1$$

is derived from the addition of the three relations. The determinant of the matrix consisting of the coefficients of the three relations is

$$\begin{vmatrix} (a_1 + a_2 + a_3)/7 & -1 & -1 \\ -(2a_1 - a_2)/7 & 2 & -1 \\ -(2a_2 - a_3)/7 & 0 & 2 \end{vmatrix} = a_3.$$

A numerical semigroup H with $M(H) = \{a_0, a_1, a_2, a_3\}$ satisfying the above condition is said to be *1-neat*. Under the above condition we get a saturated subsemigroup S of $\mathbb{Z}^6$ as in Definition 11 from the fundamental system.

Case (II-1) $a_1 + a_2 \equiv 0 \pmod{7}$ and $2a_1 \equiv a_3 \pmod{7}$.

Case (II-1-i) $2a_2 < a_1 + 2a_3$ and $2a_3 < 3a_2$. A generating system for relations consists of

$$\begin{aligned} \frac{a_1 + a_2}{7}a_0 &= a_1 + a_2, \quad 2a_1 = \frac{2a_1 - a_3}{7}a_0 + a_3, \quad 3a_2 = \frac{3a_2 - 2a_3}{7}a_0 + 2a_3, \\ 3a_3 &= \frac{3a_3 - a_2}{7}a_0 + a_2, \quad \frac{a_2 + a_3 - a_1}{7}a_0 + a_1 = a_2 + a_3, \\ \frac{a_1 + 2a_3 - 2a_2}{7}a_0 + 2a_2 &= a_1 + 2a_3. \end{aligned}$$

i.e., the kernel of

$$\begin{array}{ccc} \varphi_H : k[X_0, X_1, X_2, X_3] & \longrightarrow & k[t^{a_0}, t^{a_1}, t^{a_2}, t^{a_3}] \\ X_i & \longmapsto & t^{a_i} \end{array}$$

is generated by

$$\begin{aligned} X_0^{\frac{a_1 + a_2}{7}} - X_1X_2, \quad X_1^2 - X_0^{\frac{2a_1 - a_3}{7}}X_3, \quad X_2^3 - X_0^{\frac{3a_2 - 2a_3}{7}}X_3, \\ X_3 - X_0^{\frac{3a_3 - a_2}{7}}X_2, \quad X_0^{\frac{a_2 + a_3 - a_1}{7}}X_1 - X_2X_3, \quad X_0^{\frac{a_1 + 2a_3 - 2a_2}{7}}X_2^2 - X_1X_3^2. \end{aligned}$$

A fundamental system of relations is the following:

$$\frac{a_1 + a_2}{7}a_0 = a_1 + a_2, \quad 2a_1 = \frac{2a_1 - a_3}{7}a_0 + a_3, \quad 3a_2 = \frac{3a_2 - 2a_3}{7}a_0 + 2a_3.$$

For example, the addition of the first and second relations

$$\frac{a_1 + a_2}{7}a_0 + 2a_1 = \left(a_1 + a_2\right) + \left(\frac{2a_1 - a_3}{7}a_0 + a_3\right)$$

induces the fifth relation. To get a subsemigroup S of $\mathbb{Z}^l$ we divide this case into three cases again.

Case (II-1-i-A) $a_1 + 2a_2 > 3a_3$. We divide the coefficients in the fundamental system of relations into the following:

$$\begin{aligned} (\alpha'_0 + \alpha''_0 + \alpha'''_0)a_0 &= \alpha_{01}a_1 + \alpha_{02}a_2, \quad 2\alpha_{01}a_1 = (\alpha'_0 + \alpha''_0)a_0 + \alpha_{13}a_3, \\ (2\alpha_{02} + \alpha'_2)a_2 &= (\alpha'_0 + \alpha'''_0)a_0 + \alpha_{23}a_3. \end{aligned}$$

We associate elements of $\mathbb{Z}^5$ to the components of the above system as follows:

$$\begin{aligned} \alpha'_0a_0 &\mapsto \mathbf{b}_1 = \mathbf{e}_1, \quad \alpha''_0a_0 \mapsto \mathbf{b}_2 = \mathbf{e}_2, \quad \alpha'''_0a_0 \mapsto \mathbf{b}_3 = \mathbf{e}_3, \quad \alpha_{01}a_1 \mapsto \mathbf{b}_4 = \mathbf{e}_4, \\ \alpha'_2a_2 &\mapsto \mathbf{b}_5 = \mathbf{e}_5, \quad \alpha_{02}a_2 \mapsto \mathbf{b}_6 = (1, 1, 1, -1, 0), \end{aligned}$$

$$\alpha_{13}a_3 \mapsto \mathbf{b}_7 = (-1, -1, 0, 2, 0), \alpha_{23}a_3 \mapsto \mathbf{b}_8 = (1, 2, 1, -2, 1).$$

where $\mathbf{e}_i$ denotes the vector whose i -th component is 1 and j -th component is 0 if $j \neq i$. Let S be the subsemigroup of $\mathbb{Z}^5$ generated by $\mathbf{b}_1, \dots, \mathbf{b}_8$. We can show that

$$\sum_{i=1}^8 \mathbb{R}_+ \mathbf{b}_i \cap \mathbb{Z}^5 \subseteq S$$

where $\mathbb{R}_+$ denotes the set of non-negative real numbers. Hence, S is saturated.

Case (II-1-i-B) $a_1 + 2a_2 < 3a_3$. We divide the coefficients in the fundamental system of relations into the following:

$$(\alpha'_0 + \alpha_{10} + \alpha_{20})a_0 = \alpha_{01}a_1 + \alpha_{02}a_2, 2\alpha_{01}a_1 = \alpha_{10}a_0 + \alpha_{13}a_3,$$

$$(2\alpha_{02} + \alpha'_2)a_2 = \alpha_{20}a_0 + \alpha_{23}a_3.$$

We associate elements of $\mathbb{Z}^5$ to the components of the above system as follows:

$$\alpha'_0 a_0 \mapsto \mathbf{b}_1 = \mathbf{e}_1, \alpha_{10}a_0 \mapsto \mathbf{b}_2 = \mathbf{e}_2, \alpha_{20}a_0 \mapsto \mathbf{b}_3 = \mathbf{e}_3, \alpha_{01}a_1 \mapsto \mathbf{b}_4 = \mathbf{e}_4,$$

$$\alpha'_2 a_2 \mapsto \mathbf{b}_5 = \mathbf{e}_5, \alpha_{02}a_2 \mapsto \mathbf{b}_6 = (1, 1, 1, -1, 0),$$

$$\alpha_{13}a_3 \mapsto \mathbf{b}_7 = (0, -1, 0, 2, 0), \alpha_{23}a_3 \mapsto \mathbf{b}_8 = (2, 2, 1, -2, 1).$$

Let S be the subsemigroup of $\mathbb{Z}^5$ generated by $\mathbf{b}_1, \dots, \mathbf{b}_8$. Then S is saturated.

Case (II-1-i-C) $a_1 + 2a_2 = 3a_3$. In the Case (II-1-i-A) let $\alpha'_0 = 0$. We get a subsemigroup S of $\mathbb{Z}^4$ generated by 7 elements. Then S is saturated.

But our method does not work well in the following case.

Case (III-2-i) $2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 \leq a_2 + a_3, 2a_2 > 3a_1$. We have the following generating system of relations

$$\frac{2a_1 + a_2}{7}a_0 = 2a_1 + a_2, \quad (1)$$

$$4a_1 = \frac{4a_1 - a_3}{7}a_0 + a_3, \quad (2)$$

$$2a_2 = \frac{2a_2 - 3a_1}{7}a_0 + 3a_1, \quad (3)$$

$$2a_3 = \frac{2a_3 - a_1}{7}a_0 + a_1, \quad (4)$$

$$\frac{a_2 + a_3 - 2a_1}{7}a_0 + 2a_1 = a_2 + a_3, \quad (5)$$

$$\frac{a_1 + a_3 - a_2}{7}a_0 + a_2 = a_1 + a_3. \quad (6)$$

The three equations (1), (2) and (6) in the generating system of relations form a fundamental system. In fact,

$$(1) + (2) = (5), {}^t(1) + {}^t(2) + (6) = (3) \text{ and } {}^t(1) + {}^t(2) + {}^t(6) = (4).$$

We divide the coefficients in the fundamental system of relations into the following:

$$(\alpha_{10} + \alpha_{20} + \alpha'_0)a_0 = \alpha_{01}a_1 + \alpha'_2a_2, (\alpha_{01} + \alpha'_1 + \alpha_{31})a_1 = \alpha_{10}a_0 + \alpha_{13}a_3,$$

$$\alpha'_0a_0 + \alpha'_2a_2 = \alpha'_1a_1 + \alpha_{13}a_3.$$

We associate elements of $\mathbb{Z}^5$ to the components of the above system as follows:

$$\alpha_{10}a_0 \mapsto \mathbf{b}_1 = \mathbf{e}_1, \alpha_{20}a_0 \mapsto \mathbf{b}_2 = \mathbf{e}_2, \alpha'_0a_0 \mapsto \mathbf{b}_3 = \mathbf{e}_3, \alpha_{01}a_1 \mapsto \mathbf{b}_4 = \mathbf{e}_4,$$

$$\alpha'_1a_1 \mapsto \mathbf{b}_5 = \mathbf{e}_5, \alpha'_2a_2 \mapsto \mathbf{b}_6 = (1, 1, 1, -1, 0),$$

$$\alpha_{31}a_1 \mapsto \mathbf{b}_7 = (2, 1, 2, -2, -2), \alpha_{13}a_3 \mapsto \mathbf{b}_8 = (1, 1, 2, -1, -1).$$

Let S be the subsemigroup of $\mathbb{Z}^5$ generated by $\mathbf{b}_1, \dots, \mathbf{b}_8$. Then S is not saturated. In fact,

$$2(1, 1, 1, -1, -1) = (2, 2, 2, -2, -2) = \mathbf{b}_2 + \mathbf{b}_7 \in S,$$

but

$$(1, 1, 1, -1, -1) \notin S \text{ and } (1, 1, 1, -1, -1) \in \mathbb{Z}^5.$$

Hence, $\text{Spec } k[S]$ is *not a toric variety*.

To check whether a 7-semigroup generated by 4 elements is of toric type we divide them into the 31 cases in the following table. But this problem is still open in the last three cases. The right-hand side of column in the table means the dimension of the affine toric variety which is constructed from a numerical semigroup of given type in our way.

	Condition	Toric	dim
I	$a_1 + a_2 + a_3 \equiv 0$	$\bigcirc$	6
II-1-i-A	$a_1 + a_2 \equiv 0, 2a_1 \equiv a_3, 2a_2 < a_1 + 2a_3, 2a_3 < 3a_2, a_1 + 2a_2 > 3a_3$	$\bigcirc$	5
II-1-i-B	$a_1 + a_2 \equiv 0, 2a_1 \equiv a_3, 2a_2 < a_1 + 2a_3, 2a_3 < 3a_2, a_1 + 2a_2 < 3a_3$	$\bigcirc$	5
II-1-i-C	$a_1 + a_2 \equiv 0, 2a_1 \equiv a_3, 2a_2 < a_1 + 2a_3, 2a_3 < 3a_2, a_1 + 2a_2 = 3a_3$	$\bigcirc$	4
II-1-ii-A	$a_1 + a_2 \equiv 0, 2a_1 \equiv a_3, 2a_3 > 3a_2, 4a_2 > a_1 + a_3$	$\bigcirc$	6
II-1-ii-B	$a_1 + a_2 \equiv 0, 2a_1 \equiv a_3, 2a_3 > 3a_2, 4a_2 = a_1 + a_3$	$\bigcirc$	5
II-1-ii-C	$a_1 + a_2 \equiv 0, 2a_1 \equiv a_3, 2a_3 > 3a_2, 4a_2 < a_1 + a_3$	$\bigcirc$	5
II-1-iii-A	$a_1 + a_2 \equiv 0, 2a_1 \equiv a_3, 2a_2 > a_1 + 2a_3$	$\bigcirc$	6
II-1-iii-B	$a_1 + a_2 \equiv 0, 2a_1 \equiv a_3, 2a_2 = a_1 + 2a_3$	$\bigcirc$	5
II-1-iv	$a_1 + a_2 \equiv 0, 2a_1 \equiv a_3, 2a_3 = 3a_2$	$\bigcirc$	4
II-2-i-A	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 > 2a_1 + a_3$	$\bigcirc$	6
II-2-i-B	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 = 2a_1 + a_3$	$\bigcirc$	5
II-2-ii-A	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 > a_2 + a_3, 3a_2 > a_1 + a_3$	$\bigcirc$	7
II-2-ii-B	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 > a_2 + a_3, 3a_2 = a_1 + a_3$	$\bigcirc$	6
II-2-ii-C	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 > a_2 + a_3, 3a_2 < a_1 + a_3, a_1 + 2a_2 > 2a_3$	$\bigcirc$	6
II-2-ii-D	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 > a_2 + a_3, 3a_2 < a_1 + a_3, a_1 + 2a_2 = 2a_3$	$\bigcirc$	5
II-2-ii-E	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 > a_2 + a_3, 3a_2 < a_1 + a_3, a_1 + 2a_2 < 2a_3$	$\bigcirc$	6
II-2-iii-A	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 < a_2 + a_3, 3a_2 \geq a_1 + a_3, a_1 + 2a_2 > 2a_3$	$\bigcirc$	6
II-2-iii-B	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 < a_2 + a_3, 3a_2 \geq a_1 + a_3, a_1 + 2a_2 = 2a_3$	$\bigcirc$	5
II-2-iii-C	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 < a_2 + a_3, 3a_2 > a_1 + a_3, a_1 + 2a_2 < 2a_3$	$\bigcirc$	6
II-2-iii-D	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 < a_2 + a_3, 3a_2 = a_1 + a_3, a_1 + 2a_2 < 2a_3$	$\bigcirc$	5
II-2-iii-E	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 < a_2 + a_3, 3a_2 < a_1 + a_3, 2a_1 + 3a_2 < 2a_3$	$\bigcirc$	5
II-2-iii-F	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 < a_2 + a_3, 3a_2 < a_1 + a_3, 2a_1 + 3a_2 = 2a_3$	$\bigcirc$	4
II-2-iii-G	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 < a_2 + a_3, 3a_2 < a_1 + a_3, 2a_1 + 3a_2 > 2a_3$	$\bigcirc$	5
II-2-iv-A	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 = a_2 + a_3, 3a_2 > a_1 + a_3$	$\bigcirc$	6
II-2-iv-B	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 = a_2 + a_3, 3a_2 < a_1 + a_3$	$\bigcirc$	6
II-2-iv-C	$a_1 + a_2 \equiv 0, 3a_1 \equiv a_3, 2a_2 < 2a_1 + a_3, 2a_1 = a_2 + a_3, 3a_2 = a_1 + a_3$	$\bigcirc$	5
III-1	$2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 > a_2 + a_3$	$\bigcirc$	6
III-2-i	$2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 \leq a_2 + a_3, 2a_2 > 3a_1$	?	(5)
III-2-ii	$2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 \leq a_2 + a_3, 2a_2 < 3a_1$	?	(5)
III-2-iii	$2a_1 + a_2 \equiv 0, 2a_2 + a_3 \equiv 0, 2a_1 \leq a_2 + a_3, 2a_2 = 3a_1$	?	(4)

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