

ON CERTAIN MEROMORPHIC P-VALENT FUNCTIONS

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ABSTRACT. A certain differential operator D^n is introduced for functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k$$

which are analytic in $E^* = \{z : 0 < |z| < 1\}$. The object of the present paper is to give an application of the above operator D^n to the differential inequalities.

Keywords. Analytic, p-valent, meromorphic.

1. INTRODUCTION

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \dots\})$$

which are analytic in $E^* = \{z : 0 < |z| < 1\}$. Define

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = \frac{1}{z^p} + (p+1)a_0 + (p+2)a_1 z + (p+3)a_2 z^2 + \dots$$

$$D^2 f(z) = D(D^1 f(z))$$

and for $n = 1, 2, \dots$

$$D^n f(z) = D(D^{n-1} f(z)) = \frac{1}{z^p} + \sum_{m=1}^{\infty} (p+m)^n a_{m-1} z^{m-1}.$$

Recently Uralegaddi and Somanatha [1] and Aouf and Hossen [2] have studied certain class of meromorphic multivalent functions defined by the operator $D^n f(z)$. The object of the present paper is to investigate some new properties of meromorphic p-valent functions defined by the above operator.

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Definition. Let H be the set of complex valued functions $h(r, s, t) : \mathbb{C}^3 \rightarrow \mathbb{C}$ ($\mathbb{C}$ is the complex plane) such that

$$(1.1) \quad h(r, s, t) \text{ is continuous in a domain } D \subset \mathbb{C}^3;$$

$$(1.2) \quad (1, 1, 1) \in D \text{ and } |h(1, 1, 1)| < 1;$$

$$(1.3) \quad \left| h(e^{i\theta}, m + e^{i\theta}, \frac{m + L + 3me^{i\theta} + e^{2i\theta}}{m + e^{i\theta}}) \right| \geq 1,$$

whenever

$$(e^{i\theta}, m + e^{i\theta}, \frac{m + L + 3me^{i\theta} + e^{2i\theta}}{m + e^{i\theta}}) \in D$$

with $\operatorname{Re} L \geq m(m - 1)$ for real θ and for real $m \geq 1$.

2. MAIN RESULT

In proving our main result, we shall need the following lemma due to Miller and Mocanu [3].

Lemma. Let $w(z) = a + w_k z^k + \dots$ be analytic in $E = \{z : |z| < 1\}$ with $w(z) \neq a$ and $k \geq 1$. If $z_0 = r_0 e^{i\theta}$ ($0 < r_0 < 1$) and $|w(z_0)| = \max_{|z| \leq r_0} |w(z)|$, then

$$(2.1) \quad z_0 w'(z_0) = m w(z_0) \quad \text{and}$$

$$(2.2) \quad \operatorname{Re} \left\{ 1 + \frac{z_0 w''(z_0)}{w'(z_0)} \right\} \geq m,$$

where m is real and

$$m \geq k \frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \geq k \frac{|w(z_0)| - |a|}{|w(z_0)| + |a|}.$$

Theorem. Let $h(r, s, t) \in H$ and let $f(z)$ belonging to $\Sigma(p)$ satisfy

$$(2.3) \quad \left(\frac{D^n f(z)}{D^{n-1} f(z)}, \frac{D^{n+1} f(z)}{D^n f(z)}, \frac{D^{n+2} f(z)}{D^{n+1} f(z)} \right) \in D \subset \mathbb{C}^3 \quad \text{and}$$

$$(2.4) \quad \left| h\left(\frac{D^n f(z)}{D^{n-1} f(z)}, \frac{D^{n+1} f(z)}{D^n f(z)}, \frac{D^{n+2} f(z)}{D^{n+1} f(z)} \right) \right| < 1$$

for all $z \in E$. Then we have

$$\left| \frac{D^n f(z)}{D^{n-1} f(z)} \right| < 1 \quad (z \in E).$$

Proof. Let

$$\frac{D^n f(z)}{D^{n-1} f(z)} = w(z),$$

then it follows that $w(z)$ is either analytic or meromorphic in E , $w(0) = 1$ and $w(z) \neq 1$. With the aid of the identity (easy to verify)

$$z(D^n f(z))' = D^{n+1} f(z) - (p+1)D^n f(z),$$

we obtain

$$\begin{aligned} \frac{D^{n+1} f(z)}{D^n f(z)} &= w(z) + \frac{zw'(z)}{w(z)} \\ \frac{D^{n+2} f(z)}{D^{n+1} f(z)} &= w(z) + \frac{zw'(z)}{w(z)} + \frac{zw'(z) + \frac{zw'(z)}{w(z)} + \frac{z^2 w''(z)}{w(z)} - (\frac{zw'(z)}{w(z)})^2}{w(z) + \frac{zw'(z)}{w(z)}} \end{aligned}$$

we claim that $|w(z)| < 1$ for $z \in E$. Otherwise there exists a point $z_0 \in E$ such that $\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$. Letting $w(z_0) = e^{i\theta}$ and using lemma with $a = 1$ and $k = 1$, we see that

$$\begin{aligned} \frac{D^n f(z_0)}{D^{n-1} f(z_0)} &= e^{i\theta}, \\ \frac{D^{n+1} f(z_0)}{D^n f(z_0)} &= m + e^{i\theta}, \\ \frac{D^{n+2} f(z_0)}{D^{n+1} f(z_0)} &= \frac{m + L + 3me^{i\theta} + e^{2i\theta}}{m + e^{i\theta}}, \end{aligned}$$

where $L = \frac{z_0^2 w''(z_0)}{w(z_0)}$ and $m \geq 1$.

Further, an application of (2.2) in lemma gives

$$\operatorname{Re} L \geq m(m-1).$$

Since $h(r, s, t) \in H$, we have

$$\begin{aligned} &\left| h\left(\frac{D^n f(z_0)}{D^{n-1} f(z_0)}, \frac{D^{n+1} f(z_0)}{D^n f(z_0)}, \frac{D^{n+2} f(z_0)}{D^{n+1} f(z_0)}\right) \right| \\ &= \left| h(e^{i\theta}, m + e^{i\theta}, \frac{m + L + 3me^{i\theta} + e^{2i\theta}}{m + e^{i\theta}}) \right| \\ &\geq 1. \end{aligned}$$

which contradicts the condition (2.4) of the theorem. Therefore, we conclude that

$$\left| \frac{D^n f(z)}{D^{n-1} f(z)} \right| < 1 \quad (z \in E).$$

This completes the proof of Theorem.

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