

Reproducibility of a Noisy Limit-Cycle Oscillator Induced by a Fluctuating Input

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(Received June 30, 2005)

Reproducibility of a noisy limit-cycle oscillator driven by a random piecewise constant signal is analyzed. By reducing the model to random phase maps, it is shown that the reproducibility of the limit cycle generally improves when the phase maps are monotonically increasing.

§1. Introduction

When a spiking neuron receives a randomly fluctuating input, its reproducibility of spike generation improves compared with the case of a constant input.¹⁾ This phenomenon can be interpreted as phase synchronization between uncoupled nonlinear oscillators that receive a common fluctuating input, because repeated measurements on a single oscillator using the same input is equivalent to a single measurement on an ensemble of uncoupled identical oscillators. In our previous studies, we analyzed the cases where the fluctuating input is given by a random telegraphic signal²⁾ or by a random impulsive signal.³⁾ In this proceeding, we analyze the case where the fluctuating input is a slowly varying, piecewise constant random signal using the phase reduction technique,^{4),5)} as a generalization towards a full treatment of realistic continuous random signals.

§2. Fluctuation-induced phase synchronization

We consider an ensemble of N identical uncoupled limit-cycle oscillators subject to a common fluctuating input:

$$\dot{\mathbf{X}}_i(t) = \mathbf{G}(\mathbf{X}_i(t)) + \mathbf{I}(t) \quad (2.1)$$

for $i = 1, \dots, N$, where $\mathbf{X}_i(t)$ represents the internal state of the i -th oscillator at time t , $\mathbf{G}(\mathbf{X})$ the intrinsic dynamics of each oscillator, and $\mathbf{I}(t)$ a fluctuating input common to all the oscillators. The fluctuating input $\mathbf{I}(t)$ is a piecewise constant random signal that takes one of M values $\mathbf{I}_m \in \{\mathbf{I}_1, \dots, \mathbf{I}_M\}$ with equal probability. The changes of $\mathbf{I}(t)$ occur at time $\{t_1, t_2, \dots\}$ following a Poisson process of mean interval τ . We assume τ to be sufficiently larger than the period of the oscillator. At each t_n , $\mathbf{I}(t)$ changes its value in a stepwise manner. Namely, if $\mathbf{I}(t_n - 0) = \mathbf{I}_m$, its new value $\mathbf{I}(t_n + 0)$ after the change is either of $\mathbf{I}_{m+1}$ or $\mathbf{I}_{m-1}$ with equal probability. The probability density function (PDF) of the interval $T_n = t_{n+1} - t_n$ between changes obeys an exponential distribution $P(T) = \exp(-T/\tau)/\tau$. For each value of $\mathbf{I}_m$, Eq. (2.1) is assumed to have a stable limit-cycle solution, whose basin of attraction is the entire phase space except some unstable fixed points.

Though our theory itself is a general one, we use the FitzHugh-Nagumo model as an example, where $\mathbf{X} = \{u, v\}$, $\mathbf{G}(\mathbf{X}) = \{\epsilon(v + a - bu), v - v^3/3 - u\}$ and $\mathbf{I}(t) = \{0, I(t)\}$. The parameters are fixed at $a = 0.7$, $b = 0.8$ and $\epsilon = 0.08$. $I(t)$ takes one of $M = 7$ values $I_m \in \{0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2\}$. We set the mean interval between changes at $\tau = 40$ and consider $N = 25$ oscillators. In the numerical simulation, small Gaussian-white noise of zero-mean and intensity $D = 10^{-5}$ is independently applied to each variables of the oscillators to incorporate the effect of external disturbances. Figure 1(a) displays a typical realization of the piecewise-constant random signal, Fig. 1(b) zero-crossing events of the v -component from $v < 0$ to $v > 0$ under the constant input, and Fig. 1(c) zero-crossing events under the fluctuating input, sufficiently after initial transients. Due to the independent Gaussian-white noises, the zero-crossing events occur randomly under the constant input as shown in Fig. 1(b), whereas phase synchronization induced by fluctuating input can clearly be seen in Fig. 1(c).

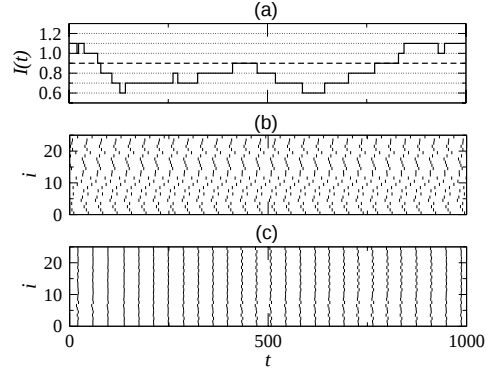


Fig. 1. (a) Typical realization of a piecewise-constant random signal. (b) Zero-crossing events under a constant input. (c) Zero-crossing events under a fluctuating input.

§3. Reduction to random phase maps

The phase synchronization is the result of the stabilization of each limit-cycle oscillator against phase disturbances due to the fluctuating input. To analyze its mechanism, we reduce our model to random phase maps. We consider the single-oscillator problem, because the stability is a property of individual oscillators.

Corresponding to the M values of $\mathbf{I}(t)$, the orbit of our model moves among M limit cycles. Since τ is assumed to be large, the orbit is on one of those limit cycles most of the time, except for short transients between limit cycles after the changes of the input, as shown in Fig. 2. Following the standard procedure,^{4),5)} we define a phase variable $\theta_m(\mathbf{X}) \in [0, 1]$ using the limit cycle m corresponding to the input $\mathbf{I}_m$ for each $m = 1, \dots, M$, where 0 and 1 represent the same phase. We specify the value of $\mathbf{I}(t)$ by m hereafter. When the input is m , i.e., $\mathbf{I}(t) = \mathbf{I}_m$, the dynamics of the orbit can simply be described as $\dot{\theta}_m(t) = \omega_m$ by using the corresponding phase variable θ_m , where ω_m is the angular velocity of the limit cycle m .

When the input changes from m to m' , the orbit of our model originally at phase θ_m on the limit cycle m will be mapped to new phase $\theta_{m'}$ on the limit cycle m' . We describe this mapping by $\theta_{m'} = F_{m \rightarrow m'}(\theta_m)$, which we call a “phase map”. It is a periodic function on $[0, 1]$ satisfying $F_{m \rightarrow m'}(\theta_m + 1) = F_{m \rightarrow m'}(\theta_m) + 1 = F_{m \rightarrow m'}(\theta_m)$, where 0 and 1 should be interpreted as the same phase. Figure 3 displays the phase maps of the FitzHugh-Nagumo model obtained for all contiguous pairs of (m, m') . The curves are appropriately shifted to adjust their origins.

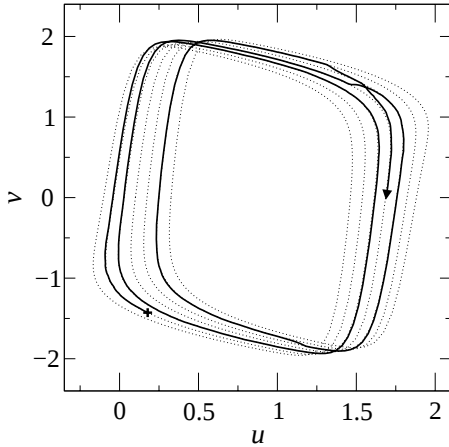


Fig. 2. Typical trajectory of the model (bold arrow). M limit-cycle orbits corresponding to the M input values are shown in the background.

We use the time step n rather than the real time t in the following discussion, which is the number of changes in $\mathbf{I}(t)$ from the beginning. Since we consider a Poisson process, the time step n roughly corresponds to the real time t as $n \simeq t/\tau$, because the mean inter-impulse interval is τ . Let us represent the temporal sequence of $\mathbf{I}(t)$ by $m(n)$, and consider a situation where the input changes from $m(n)$ to a new value $m(n+1)$ at $t = t_n$ and keeps this value until $t = t_{n+1}$ for an interval of $T_n = t_{n+1} - t_n$. The corresponding dynamics of the orbit from phase $\theta_{m(n)}(n)$ on the limit cycle $m(n)$ to the new phase $\theta_{m(n+1)}(n+1)$ on the limit cycle $m(n+1)$ can be described using the phase map F as

$$\theta_{m(n+1)}(n+1) = \omega_{m(n+1)}T_n + F_{m(n) \rightarrow m(n+1)}(\theta_{m(n)}(n)), \quad (3.1)$$

where $\omega_{m(n+1)}T_n$ represents constant increase of the phase on the limit cycle $m(n+1)$. Since T_n is a random variable, this equation describes random phase maps.

§4. Stability against phase disturbances

The stability against phase disturbances can be characterized by the average Lyapunov exponent of the random phase maps, Eq. (3.1). Let us consider a small phase deviation $\Delta\theta_{m(n)}(n)$ from $\theta_{m(n)}(n)$. Its linearized evolution equation is

$$\Delta\theta_{m(n+1)}(n+1) = F'_{m(n) \rightarrow m(n+1)}(\theta_{m(n)}(n))\Delta\theta_{m(n)}(n), \quad (4.1)$$

where $F'_{m \rightarrow m'}(\theta_m) = dF_{m \rightarrow m'}(\theta_m)/d\theta_m$. Therefore, the phase deviation grows as

$$|\Delta\theta_{m(n)}(n)/\Delta\theta_{m(0)}(0)| = \prod_{n'=0}^{n-1} |F'_{m(n') \rightarrow m(n'+1)}(\theta_{m(n')}(n'))| \simeq \exp(\lambda n), \quad (4.2)$$

where we defined the average Lyapunov exponent as $\lambda = \langle \log |F'_{m \rightarrow m'}(\theta_m)| \rangle$. The average should be taken over all possibilities of (m, m') and over the phase distributions on all limit cycles.

When the mean interval τ is sufficiently large, the phase distribution on each limit cycle tends to be uniform, because the jumps between the limit cycles occur irrespectively of where the orbit is, leading to complete randomization of the phase. Under this condition, we can make a general statement on the sufficient condition for the phase synchronization: *when all phase maps $F_{m \rightarrow m'}(\theta_m)$ are monotonically increasing non-identity functions, the Lyapunov exponent λ is negative, leading to fluctuation-induced phase synchronization.*

Actually, when $F'_{m \rightarrow m'}(\theta_m) > 0$ holds for all m , we can bound the Lyapunov exponent λ from above as

$$\begin{aligned} \lambda &= \frac{1}{\#(m, m')} \sum_{(m, m')} \int_0^1 d\theta_m \log F'_{m \rightarrow m'}(\theta_m) \\ &\leq \frac{1}{\#(m, m')} \sum_{(m, m')} \int_0^1 d\theta_m \{F'_{m \rightarrow m'}(\theta_m) - 1\} \\ &= \frac{1}{\#(m, m')} \sum_{(m, m')} \{[F_{m \rightarrow m'}(\theta_m)]_0^1 - 1\} = 0, \end{aligned} \quad (4.3)$$

where the summation is taken over all combinations of m and m' , and $\#(m, m')$ represents the number of them. In the above inequalities, we utilized the fact that $\log F' \leq F' - 1$, and that $F_{m \rightarrow m'}(1) - F_{m \rightarrow m'}(0) = 1$ because $F_{m \rightarrow m'}(\theta_m)$ is a phase map. The equality holds only when $F'_{m \rightarrow m'}(\theta_m) \equiv 1$ for all m , namely, when the phase maps are trivial identity maps. For the FitzHugh-Nagumo model with the parameter values assumed here, all the phase maps $F_{m \rightarrow m'}(\theta_m)$ are monotonically increasing as can immediately be seen from Fig. 3. Therefore, by applying a piecewise-constant random signal with large mean interval τ , fluctuation-induced synchronization occurs as demonstrated in Fig. 2(c). In general, as long as the separation between neighboring values of $I(t)$ are small, the phase maps should be monotonic, and fluctuation-induced synchronization should occur.

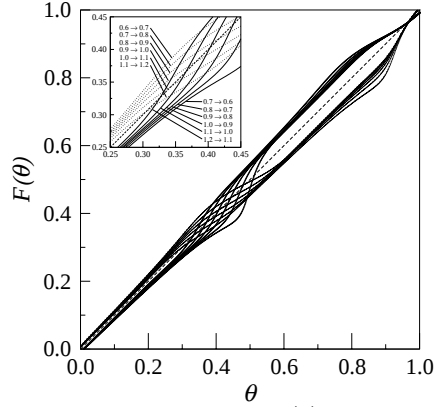


Fig. 3. Phase maps $F_{m \rightarrow m'}(\theta)$ between two successive values of the input values m and m' . The inset is an enlargement showing the correspondence between the curve and the pair of input values (m, m') .

§5. Summary

We analyzed fluctuation-induced phase synchronization among uncoupled noisy oscillators for the case of a slowly varying, piecewise-constant random input. By reducing the model to random phase maps, we gave a general sufficient condition for the phase synchronization. Extension of our current analysis to a realistic continuous random signal will be tackled in the future.

Acknowledgements

H. N. is deeply indebted to Professor Yoshiki Kuramoto for his continuous support. We also thank Y. Tsubo, D. Tanaka and J. Teramae for useful comments.

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