

The Cauchy problem for a class of hyperbolic operators with double characteristics

KUNIHICO KAJITANI (梶谷 邦彦)
SEIICHIRO WAKABAYASHI (若林誠一郎)*

1. Introduction

In [10] we proved that the Cauchy problem for hyperbolic operators is C^∞ well-posed if the operators satisfy some microlocal *a priori* estimates. So, in the studies of C^∞ well-posedness of the hyperbolic Cauchy problem the problems are reduced to obtaining the microlocal *a priori* estimates. In [11] we investigated a class of hyperbolic operators with double characteristics, which contains effectively hyperbolic operators, applying results in [10]. In [11] we imposed some extra conditions on hyperbolic operators. In this article we shall show that the Cauchy problem for hyperbolic operators with double characteristics is C^∞ well-posed under reasonable assumptions. In doing so, we shall use ideas in Kajitani-Wakabayashi-Nishitani [13]. One of chief distinctions of our treatment is the use of ‘time functions’. Using ‘time functions’ we can consider effectively hyperbolic operators and a wide class of non effectively hyperbolic operators with a unified treatment. C^∞ well-podeness of the Cauchy problem for effectively hyperbolic operators was proved by Iwasaki [5] (see, also, [6], [14], [15], [16]). Ivrii [7] studied C^∞ well-posedness of the Cauchy problem for a class of non effectively hyperbolic operators (see, also, [2]).

Let $P(x, \xi)$ be a polynomial of $\xi = (\xi_1, \xi') = (\xi_1, \dots, \xi_n)$ of degree m whose coefficients are C^∞ functions of $x = (x_1, x') = (x_1, \dots, x_n) \in \mathbb{R}^n$. We define the operator $P^w(x, D)$ with Weyl symbol $P(x, \xi)$ by

$$P^w(x, D)u(x) = (2\pi)^{-n} \int \left\{ \int e^{i(x-y) \cdot \xi} P\left(\frac{x+y}{2}, \xi\right) u(y) dy \right\} d\xi$$

*Institute of Mathematics, University of Tsukuba

for $u \in C_0^\infty(\mathbf{R}^n)$. We consider the Cauchy problem

$$(CP) \quad \begin{cases} P^w(x, D)u = f & \text{in } \Omega, \\ \text{supp } u \subset \{x_1 \geq 0\} \end{cases}$$

in the C^∞ (or $\mathcal{D}'$) category, where Ω is an open subset of $\mathbf{R}^n$ and contains the origin, and $\text{supp } f \subset \{x_1 \geq 0\}$. Let $p(x, \xi)$ be the principal part of $P(x, \xi)$. We assume that

(P-1) $p(x, \xi)$ is hyperbolic with respect to $\vartheta = (1, 0, \dots, 0) \in \mathbf{R}^n$ for each $x \in \mathbf{R}^n$, i.e., $p(x, \xi - i\vartheta) \neq 0$ for $x \in \mathbf{R}^n$ and $\xi \in \mathbf{R}^n$.

To state our assumptions and results we need the following

Definition 1.1. Let $z^0 = (x^0, \xi^0) \in T^*\mathbf{R}^n \setminus 0$ and assume that (P-1) is satisfied. (i) The localization polynomial $p_{z^0}(\delta z)$ of $p(x, \xi)$ at z^0 is defined by

$$p(z^0 + s\delta z) = s^\mu(p_{z^0}(\delta z) + o(1)) \quad \text{as } s \rightarrow 0,$$

and $p_{z^0}(\delta z) \neq 0$ in $\delta z \in T_{z^0}(T^*\mathbf{R}^n) (\simeq \mathbf{R}^{2n})$. We denote by $\Gamma(p_{z^0}, (0, \vartheta))$ the connected component of the set $\{\delta z \in T_{z^0}(T^*\mathbf{R}^n); p_{z^0}(\delta z) \neq 0\}$ which contains $(0, \vartheta)$. (ii) Let $t(x, \xi)$ be a real-valued function in $C(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ which is positively homogeneous of degree 0. We say that $t(x, \xi)$ is a time function for p with respect to $(0, \vartheta) (\in \mathbf{R}^{2n})$ at z^0 if $t(z^0) = 0$ and if there are a neighborhood $\mathcal{U}$ of z^0 and $K \subset \subset \Gamma(p_{z^0}, (0, \vartheta))$ such that $t(x, \xi)$ is Lipschitz continuous in $\mathcal{U}$ and $-(|\xi| \nabla_\xi t(x, \xi), -\nabla_x t(x, \xi)) \in K$ for a.e. $(x, \xi) \in \mathcal{U}$. (iii) We denote by $F_p(z^0)$ the Hamilton map corresponding to $Hess p/2$ at z^0 , i.e., $F_p(z^0) = \frac{1}{2} \begin{pmatrix} p_{\xi\xi}(z^0) & p_{\xi x}(z^0) \\ -p_{x\xi}(z^0) & -p_{xx}(z^0) \end{pmatrix}$. We define $Tr^+ F_p(z^0) = \sum \lambda_j$, where $\lambda_j > 0$ and the $i\lambda_j$ are the eigenvalues of $F_p(z^0)$ on the positive imaginary axis. (iv) We denote by $K_{x^0}^\pm$ the sets $\{x(t); \pm t \geq 0\}$, and $\{x(t)\}$ is a Lipschitz continuous curve in $\mathbf{R}^n$ satisfying $\frac{d}{dt}x(t) \in \Gamma(p(x(t), \cdot), \vartheta)^*$ (a.e. t) and $x(0) = x^0$, where $\Gamma^* = \{x \in \mathbf{R}^n; x \cdot \xi \geq 0 \text{ for any } \xi \in \Gamma\}$.

Remark. (i) It can be proved that $p_{z^0}(\delta z)$ is hyperbolic with respect to $(0, \vartheta) \in \mathbf{R}^{2n}$ under (P-1) (see, e.g., [3]). (ii) We can also define ‘time functions’ for microhyperbolic functions (symbols) (see [8] and [17]). (iii) When $t(x, \xi)$ is a real-valued function in $C^1(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ and positively homogeneous of degree 0, $t(x, \xi)$ is a time function for p with respect to $(0, \vartheta)$ at z^0 if and only if $-H_t(z^0) \in \Gamma(p_{z^0}, (0, \vartheta))$, where $H_t(z) = \sum_{j=1}^n ((\partial t / \partial \xi_j)(z)(\partial / \partial x_j) - (\partial t / \partial x_j)(z)(\partial / \partial \xi_j))$.

In addition to (P-1) we impose the following condition on $p(x, \xi)$ for every $z^0 = (x^0, \xi^0) \in \mathbf{R}^n \times S^{n-1}$ with $dp(z^0) = 0$, where $S^{n-1} = \{\xi \in \mathbf{R}^n; |\xi| = 1\}$.

(P-2)_{z⁰} There are conic neighborhoods $\mathcal{C}$ and $\tilde{\mathcal{C}}$ of z^0 and (y^0, η^0) , respectively, a homogeneous canonical transformation $\chi: \tilde{\mathcal{C}} \xrightarrow{\sim} \mathcal{C}$, time functions $t_j(y, \eta)$ ($1 \leq j \leq d$) for $p \circ \chi$ with respect to $(0, \vartheta)$ at (y^0, η^0) , a real-valued symbol $\lambda(y, \eta')$ of positively homogeneous of degree 1, a non-negative symbol $\alpha(y, \eta')$ of positively homogeneous of degree 2, an elliptic symbol $e(y, \eta')$ and $C > 0$ such that $z^0 = \chi(y^0, \eta^0)$, $d\chi_{(y^0, \eta^0)}(0, \vartheta) \in \Gamma(p_{z^0}, (0, \vartheta))$,

$$(1.1) \quad p(\chi(y, \eta)) = e(y, \eta)\{\eta_1(\eta_1 - \lambda(y, \eta')) - \alpha(y, \eta')\} \quad \text{in } \tilde{\mathcal{C}},$$

$$T(y, \eta') \frac{\partial \alpha}{\partial y_1}(y, \eta') \leq C\alpha(y, \eta') \quad \text{for } (y, \eta') \in \tilde{\mathcal{C}}',$$

where $T(y, \eta') = \min_{1 \leq j \leq d} |t_j(y, 0, \eta')|$ and $\tilde{\mathcal{C}}' = \{(y, \eta'); (y, \eta) \in \tilde{\mathcal{C}} \text{ for some } \eta_1\}$.

Let $z^0 = (x^0, \xi^0) \in \mathbf{R}^n \times S^{n-1}$ satisfy $dp(z^0) = 0$, and let F_1 and F_2 be classical Fourier integral operators corresponding to χ and χ^{-1} which are elliptic at (y^0, η^0) and z^0 , respectively. Under the assumption (P-2)_{z⁰} we have

$$\sigma(F_2 P^w(x, D) F_1)(y, \eta) = \tilde{e}(y, \eta)\{\eta_1(\eta_1 - \lambda(y, \eta')) - \alpha(y, \eta') + \beta(y, \eta)\}$$

in a conic neighborhood $\tilde{\mathcal{C}}_0$ of (y^0, η^0) if $|\eta| \geq 1$, where $\sigma(a^w(y, D))(y, \eta) = a(y, \eta)$, $\tilde{e}(y, \eta)$ is an elliptic classical symbol in $\tilde{\mathcal{C}}_0$ and $\beta(y, \eta)$ is a classical symbol in $S_{1,0}^1$. For the imaginary part of the subprincipal symbol of $P^w(x, D)$ we assume that for every $z^0 \in \mathbf{R}^n \times S^{n-1}$ with $dp(z^0) = 0$

(P-3)_{z⁰} There are $A(y, \eta') \in S_{1,0}^0$ and $C > 0$ such that

$$T(y, \eta') |\operatorname{Im} \beta(y, 0, \eta') + \frac{1}{2} \frac{\partial \lambda}{\partial y_1}(y, \eta') - A(y, \eta') \lambda(y, \eta')| \leq C(\sqrt{\alpha(y, \eta')} + 1) \quad \text{in } \tilde{\mathcal{C}}'_0.$$

To control $\operatorname{Re} \beta(x, \xi)$ we assume that at least one of the following conditions (P-4-1)_{z⁰} and (P-4-2)_{z⁰} is satisfied for every $z^0 \in \mathbf{R}^n \times S^{n-1}$ with $dp(z^0) = 0$:

(P-4-1)_{z⁰} There are $B(y, \eta') \in S_{1,0}^0$ and $C > 0$ such that

$$T(y, \eta') |\operatorname{Re} \beta(y, 0, \eta') - B(y, \eta') \lambda(y, \eta')| \leq C(\sqrt{\alpha(y, \eta')} + 1) \quad \text{in } \tilde{\mathcal{C}}'_0.$$

(P-4-2) $_{z^0}$ $\text{Re } P_{m-1}(z^0) < \text{Tr}^+ F_p(z^0)$, where $P_{m-1}(x, \xi)$ denotes the homogeneous part of degree $(m-1)$ of $P(x, \xi)$.

Now we can state our main result.

Theorem 1.2. *Assume that Ω is bounded. Under the above assumptions, For any $f \in \mathcal{D}'$ with $\text{supp } f \subset \{x_1 \geq 0\}$ there is $u \in \mathcal{D}'$ satisfying (CP). Moreover, if $x^0 \in \Omega$, $K_{x^0}^- \cap \{x_1 \geq 0\} \subset \subset \Omega$, $\text{supp } u \subset \{x_1 \geq 0\}$ and $P^w(x, D)u = 0$ (resp. $P^w(x, D)u \in C^\infty$) near $K_{x^0}^-$, then $x^0 \notin \text{supp } u$ (resp. $x^0 \notin \text{sing supp } u$).*

Remark. If the hypotheses of Theorem 1.2 are fulfilled, taking $\Omega = \mathbf{R}^n$ (CP) is well-posed in $\mathcal{D}'$ and C^∞ , and $\text{supp } u \subset \{x \in \mathbf{R}^n; x \in K_y^+ \text{ for some } y \in \text{supp } f\}$.

In [11] we assumed that all time functions $t_j(y, \eta)$ ($1 \leq j \leq d$) in (P-2) $_{z^0}$ do not depend on η under suitable choice of canonical coordinates and belong to $\mathcal{B}^\infty(\mathbf{R}^n)$. Then we could use usual symbol calculus in $S_{1,1/2}^\infty$. Under the assumption (P-2) $_{z^0}$ we need symbol calculus with large parameters in a subclass of $S_{1/2,1/2}^\infty$ which is not included in $S_{\rho,1/2}^\infty$ for $\rho > 1/2$.

2. Outline of the proof of Theorem 1.2

We assume that (P-1) is satisfied and that (P-2) $_{z^0}$, (P-3) $_{z^0}$ and at least one of the conditions (P-4-1) $_{z^0}$ and (P-4-2) $_{z^0}$ are satisfied for every $z^0 \in \mathbf{R}^n \times S^{n-1}$ with $dp(z^0) = 0$. Fix $z^0 = (x^0, \xi^0) \in \mathbf{R}^n \times S^{n-1}$ so that $dp(z^0) = 0$, and let $t_j(x, \xi)$ ($1 \leq j \leq d$) be the time functions in (P-2) $_{z^0}$. Let $\chi(t)$ be a function in $C^\infty(\mathbf{R})$ such that $\chi(t) = 0$ for $|t| \leq 1/2$, $\chi(t) = 1$ for $|t| \geq 1$ and $0 \leq \chi(t) \leq 1$. Let $N \geq 1$, and put

$$W(x, \xi) = \sum_{j=1}^d \langle \xi \rangle_N^{1/2} (t_j(x, \xi)^2 \chi(|\xi|/N)^2 \langle \xi \rangle_N + N)^{-1/2},$$

where $\langle \xi \rangle_N = (N^2 + |\xi|^2)^{1/2}$. We define a metric g in $\mathbf{R}^n \times \mathbf{R}^n$ by

$$g_{x,\xi} = W(x, \xi)^2 (|dx|^2 + \langle \xi \rangle_N^{-2} |d\xi|^2).$$

Then g is σ temperate in the sense of Hörmander (see [4]). Here and after we use notations and terminologies in [4]. Define

$$\Phi(x, \xi) = \prod_{j=1}^d ((t_j(x, \xi)^2 \chi(|\xi|/N)^2 \langle \xi \rangle_N + t_j(x, \xi) \chi(|\xi|/N) \langle \xi \rangle_N^{1/2}).$$

We can prove that $\Phi(x, \xi)$ is σ, g temperate in the sense of Hörmander (see [4]). Choose $\rho(x, \xi) \in C^\infty(\mathbf{R}^{2n})$ such that $\text{supp } \rho \subset \{(x, \xi); |x|^2 + |\xi|^2 < c(\rho)\}$, $\int \rho(x, \xi) dx d\xi = 1$, $\rho(x, \xi) \geq 0$ and

$$|\rho_{(\beta)}^{(\alpha)}(x, \xi)| \leq C(\rho)A(\rho)^{|\alpha|+|\beta|}|\alpha + \beta|^\kappa$$

for any $(x, \xi) \in \mathbf{R}^{2n}$ and any multi-indices α and β , where $\rho_{(\beta)}^{(\alpha)}(x, \xi) = D_x^\beta \partial_\xi^\alpha \rho(x, \xi)$, $c(\rho)$, $C(\rho)$ and $A(\rho)$ are positive constants and $\kappa > 1$ will be specified later. Taking $c(\rho)$ to be small enough, we put

$$\begin{aligned} \widetilde{W}(x, \xi) &= \int \rho(W(y, \eta)(x - y), \langle \eta \rangle_N^{-1} W(y, \eta)(\xi - \eta)) \\ &\quad \times \langle \eta \rangle_N^{-n} W(y, \eta)^{2n+1} dy d\eta, \\ \widetilde{\Phi}(x, \xi) &= \int \rho(\widetilde{W}(x, \xi)(x - y), \langle \xi \rangle_N^{-1} \widetilde{W}(x, \xi)(\xi - \eta)) \\ &\quad \times \langle \xi \rangle_N^{-n} \widetilde{W}(x, \xi)^{2n} \Phi(y, \eta) dy d\eta. \end{aligned}$$

Then we have the following

Lemma 2.1. *There are positive constants C_1 , C_2 and A such that*

$$\begin{aligned} C_1^{-1} W(x, \xi) &\leq \widetilde{W}(x, \xi) \leq C_1 W(x, \xi), \\ C_1^{-1} \Phi(x, \xi) &\leq \widetilde{\Phi}(x, \xi) \leq C_1 \Phi(x, \xi), \\ |\widetilde{W}_{(\beta)}^{(\alpha)}(x, \xi)| &\leq C_2 A^{|\alpha|+|\beta|} |\alpha + \beta|^\kappa W(x, \xi)^{1+|\alpha|+|\beta|} \langle \xi \rangle_N^{-|\alpha|} \\ |\widetilde{\Phi}_{(\beta)}^{(\alpha)}(x, \xi)| &\leq C_2 A^{|\alpha|+|\beta|} |\alpha + \beta|^\kappa \Phi(x, \xi) W(x, \xi)^{|\alpha|+|\beta|} \langle \xi \rangle_N^{-|\alpha|} \end{aligned}$$

Moreover, there are a conic neighborhood $\mathcal{C}_1$ of z^0 , a closed convex cone Γ in $T^*\mathbf{R}^n \setminus 0$, $c > 0$ and $\gamma(N) > 0$ such that $\Gamma \subset \subset \Gamma(p_z, (0, \vartheta))$ for $z = (x, \xi) \in \mathcal{C}_1$ with $|\xi| = 1$ and

$$\begin{aligned} (-\langle \xi \rangle_N \nabla_\xi \widetilde{\Phi}(x, \xi), \nabla_x \widetilde{\Phi}(x, \xi)) &\in \Gamma, \\ |(-\langle \xi \rangle_N \nabla_\xi \widetilde{\Phi}(x, \xi), \nabla_x \widetilde{\Phi}(x, \xi))| &\geq c W(x, \xi) \Phi(x, \xi) \end{aligned}$$

if $(x, \xi) \in \mathcal{C}_1$ and $|\xi| \geq 2\gamma(N)$.

Define a metric g_0 by

$$g_{0,x,\xi} = |dx|^2 + \langle \xi \rangle_h^{-2} |d\xi|^2,$$

where $h \geq N$. Following the arguments in §18.4 of [4] and in [13], we can prove the following

Proposition 2.2. Let $m(x, \xi)$ be σ, g_0 temperate and $a(x, \xi) \in S(m, g_0)$. Fix k, κ and δ so that $k \geq 2$, $1 < \kappa < 3/2$ and $0 < \delta < 3 - 2\kappa$. Then there are $C_{\alpha, \beta} > 0$, $M_0 > 1$, $s_{\alpha, \beta}(x, \xi)$, $\tilde{s}_{\alpha, \beta}(x, \xi)$ and $r_k(x, \xi)$ such that $\tilde{s}_{\alpha, \beta}(x, \xi)$ is real-valued,

$$\begin{aligned} & \tilde{\Phi}^{\mp M} \# a \# \tilde{\Phi}^{\pm M} = \\ & \sum_{|\alpha|+|\beta| \leq k-1} (-1)^{|\alpha|} (\pm M \nabla_{\xi} \tilde{\Phi} / \tilde{\Phi})^{\alpha} (\mp i M \nabla_x \tilde{\Phi} / \tilde{\Phi})^{\beta} a_{(\alpha)}^{(\beta)}(x, \xi) / (\alpha! \beta!) \\ & + \sum_{|\alpha|+|\beta| \leq k-1} s_{\alpha, \beta}(x, \xi) a_{(\alpha)}^{(\beta)}(x, \xi) + \sum_{2 \leq |\alpha|+|\beta| \leq k-1} \tilde{s}_{\alpha, \beta}(x, \xi) a_{(\alpha)}^{(\beta)}(x, \xi) + r_k(x, \xi), \\ & |s_{\alpha, \beta}^{(\tilde{\alpha})}(x, \xi)| \leq C_{\alpha, \beta} M^{|\alpha|+|\beta|} W(x, \xi)^{2+|\alpha|+|\beta|+|\tilde{\alpha}|+|\tilde{\beta}|} \langle \xi \rangle_N^{-1-|\alpha|-|\tilde{\alpha}|}, \\ & |\tilde{s}_{\alpha, \beta}^{(\tilde{\alpha})}(x, \xi)| \leq C_{\alpha, \beta} M^{|\alpha|+|\beta|-1} W(x, \xi)^{|\alpha|+|\beta|+|\tilde{\alpha}|+|\tilde{\beta}|} \langle \xi \rangle_N^{-|\alpha|-|\tilde{\alpha}|}, \\ & r_k(x, \xi) \in S(m(W/\langle \xi \rangle_N)^k, g) \end{aligned}$$

if $N = M^{2-\delta}$ and $M \geq M_0$.

Remark. Proposition 2.2 was essentially proved in [13].

Let $t_0(x, \xi)$ be real-valued functions in $S_{1,0}^0(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ such that $t_0(x, \xi)$ are positively homogeneous of degree 0, $t_0(x, \xi) = x_1 - x_1^0 + |x - x^0|^2 + |\xi/\xi| - \xi^0|^2$ near z^0 . Put

$$\Lambda(x, \xi) = (at_0(x, \xi) - b) \log \langle \xi \rangle (1 - \Theta_{h/4}(\xi)) \psi(x, \xi),$$

where $\Theta(t) \in C_0^\infty(\mathbf{R})$ satisfies $\Theta(t) = 1$ if $|t| \leq 1$ and $\text{supp } \Theta \subset (-2, 2)$, $\Theta_h(\xi) = \Theta(|\xi|/h)$, $\psi(x, \xi) \in C^\infty(T^*\mathbf{R}^n \setminus 0)$ is positively homogeneous of degree 0, $\psi(x, \xi) = 1$ in a conic neighborhood of z^0 , $a \geq 1$, $b \in \Omega$ and $h \geq 1$. Roughly speaking, by virtue of results in [10], in order to prove Theorem 1.2 it suffices to obtain uniform microlocal *a priori* estimates in $\gamma \geq \gamma_0$ for $P_\Lambda^w(x, D - i\gamma\vartheta) \equiv (e^{-\Lambda})^w(x, D) P^w(x, D - i\gamma\vartheta) (e^\Lambda)^w(x, D)$, where $h = K\gamma$, and γ_0 and K are positive constants. In doing so, we put

$$Q_\Lambda^w(y, D; \gamma) = F_2 P_\Lambda^w(x, D - i\gamma\vartheta) F_1,$$

where F_1 and F_2 are the Fourier integral operators given in the assumptions. In order to get *a priori* estimates for $Q_\Lambda^w(y, D; \gamma)$ we use norms $\|(\tilde{\Phi}^{-M})^w(x, D)u\|_{L^2}$, i.e., we study $Q^w(y, D; \gamma) \equiv (\tilde{\Phi}^{-M})^w(x, D) Q_\Lambda^w(y, D; \gamma) (\tilde{\Phi}^M)^w(x, D)$. Then Proposition 2.2 admits us to calculate the symbol $Q(y, \eta; \gamma)$ of $Q^w(y, D; \gamma)$. After the calculation the procedure to prove Theorem 1.2 is almost the same as in [9] and [11]. However, the

symbols appearing in the proof are not so good as in [9] and [11]. To apply Fefferman-Phong's inequality [1] we need more complicate discussions. For a detail of the proof we refer to [12].

3. Some remarks

We remarked that C^∞ well-posedness of the Cauchy problem for hyperbolic operators can be proved if microlocal *a priori* estimates are proved (see [10]). So, one can also prove well-posedness of the Cauchy problem if one can prove microlocal *a priori* estimates under other microlocal assumptions. For example, the Cauchy problem for $P^w(x, D)$ is C^∞ well-posed if $P^w(x, D)$ satisfies at least one of the conditions given in [11], [13] and here for every $z^0 = (x^0, \xi^0) \in \mathbf{R}^n \times S^{n-1}$ with $dp(z^0) = 0$.

For every $z^0 = (x^0, \xi^0) \in \mathbf{R}^n \times S^{n-1}$ with $dp(z^0) = 0$, choosing a suitable homogeneous canonical transformation χ from a conic neighborhood $\tilde{\mathcal{C}}$ of $(y^0, \eta^0) = (0, 0, \dots, 0, 1)$ to a conic neighborhood $\mathcal{C}$ of z^0 and representing $p(\chi(y, \eta))$ in the form of (1.1), we shall give some examples which satisfy the condition $(P-2)_{z^0}$ when χ satisfies $d\chi_{(y^0, \eta^0)}(0, \vartheta) \in \Gamma(p_{z^0}, (0, \vartheta))$. We note that $dp(z^0) = 0$ implies that $\lambda(y^0, \eta^{0'}) = \alpha(y^0, \eta^{0'}) = 0$.

Example 3.1. Let $f(s)$ be a function in $C^\infty(\mathbf{R}^d)$ such that $f(0) = 0$ and $f(s) \geq 0$, $\partial f / \partial s_j(s) \geq 0$ and $\sum_{j=1}^d s_j \partial f / \partial s_j(s) \leq C f(s)$ if $0 \leq s_j \leq 1$ ($1 \leq j \leq d$), where $s = (s_1, s_2, \dots, s_d)$ and $C \geq 0$. If $f(s)$ is a polynomial of s with non-negative coefficient, then $f(s)$ satisfies the above conditions. Let $t_j(y, \eta)$ ($1 \leq j \leq d$) be real-valued functions in $C^\infty(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ which are positively homogeneous of degree 0 and satisfy $t_j(y^0, \eta^0) = 0$. Choose symbols $\alpha_j(y', \eta')$, $q_j(y, \eta')$ and $r_j(y, \eta')$ ($1 \leq j \leq d$) so that these are positively homogeneous of degree 0, $\alpha_j(y', \eta') \geq 0$, $q_j(y, \eta') > 0$, $r_j(y, \eta') \geq 0$ and $\alpha_j(y^{0'}, \eta^{0'}) r_j(y^0, \eta^{0'}) = 0$. Put

$$\begin{aligned} s_j(y, \eta') &= \alpha_j(y', \eta') (q_j(y, \eta') t_j(y, 0, \eta')^2 + r_j(y, \eta')), \\ \alpha(y, \eta') &= f(s_1(y, \eta'), \dots, s_d(y, \eta')) \eta_n^2. \end{aligned}$$

Then $(P-2)_{z^0}$ is satisfied if

$$(3.1) \quad \frac{\partial t_j}{\partial y_1}(y^0, \eta^0) > 0 \quad \text{and} \quad q_{(y^0, \eta^0)}(-H_{t_j}(y^0, \eta^0)) > 0 \quad (1 \leq j \leq d),$$

where

$$\begin{aligned}
q_{(y^0, \eta^0)}(\delta y, \delta \eta) &= \delta \eta_1 (\delta \eta_1 - \nabla_y \lambda(y^0, \eta^{0'}) \cdot \delta y - \nabla_{\eta'} \lambda(y^0, \eta^{0'}) \cdot \delta \eta') \\
&\quad - \sum_{j=1}^d \frac{\partial f}{\partial s_j}(0) \{ \alpha_j(y^{0'}, \eta^{0'}) q_j(y^0, \eta^{0'}) (\nabla_y t_j(y^0, \eta^0) \cdot \delta y + \nabla_{\eta'} t_j(y^0, \eta^0) \cdot \delta \eta')^2 \\
&\quad + r_j(y^0, \eta^{0'}) (\text{Hess } \alpha_j(y^{0'}, \eta^{0'}))(\delta y', \delta \eta')/2 \\
&\quad + \alpha_j(y^0, \eta^{0'}) (\text{Hess } r_j(y^0, \eta^{0'}))(\delta y, \delta \eta')/2 \}, \\
(\text{Hess } r_j(y^0, \eta^{0'}))(\delta y, \delta \eta') &= \sum_{k, \ell=1}^n \frac{\partial^2 r_j}{\partial y_k \partial y_\ell}(y^0, \eta^{0'}) \delta y_k \delta y_\ell \\
&\quad + 2 \sum_{k=1}^n \sum_{\ell=2}^n \frac{\partial^2 r_j}{\partial y_k \partial \eta_\ell}(y^0, \eta^{0'}) \delta y_k \delta \eta_\ell + \sum_{k, \ell=2}^n \frac{\partial^2 r_j}{\partial \eta_k \partial \eta_\ell}(y^0, \eta^{0'}) \delta \eta_k \delta \eta_\ell.
\end{aligned}$$

Here (3.1) implies that $t_j(y, \eta)$ ($1 \leq j \leq d$) are time functions for $p \circ \chi$ with respect to $(0, \vartheta)$ at (y^0, η^0) .

Example 3.2. Let $n \geq 3$, and put

$$\alpha(y, \eta') = \left(y_1 + \sqrt{y_2^2 + y_n^2} \right)^2 \left(y_1 - \sqrt{y_2^2 + y_n^2} \right)^2 \eta_n^2 \quad (= (y_1^2 - y_2^2 - y_n^2)^2 \eta_n^2).$$

Then $(P-2)_{z^0}$ is satisfied if

$$(3.2) \quad \left| \frac{\partial \lambda}{\partial \eta_2}(y^0, \eta^{0'}) \right|^2 + \left| \frac{\partial \lambda}{\partial \eta_n}(y^0, \eta^{0'}) \right|^2 < 1.$$

Here we have chosen $t_1(y, \eta) = (y_1 + \sqrt{y_2^2 + y_n^2})\eta_n$ and $t_2(y, \eta) = (y_1 - \sqrt{y_2^2 + y_n^2})\eta_n$ which are Lipschitz continuous, and (3.2) implies that $t_j(y, \eta)$ ($j = 1, 2$) are time functions for $p \circ \chi$ with respect to $(0, \vartheta)$ at (y^0, η^0) .

Finally we shall give meaning of time functions. Applying the same arguments as in [9], we have the following

Theorem 3.3. Let $z^0 = (x^0, \xi^0) \in \mathbf{R}^n \times S^{n-1}$, and let $P(x, \xi)$ be a symbol in S^m such that $p(x, \xi)$ is microhyperbolic with respect to $(0, \vartheta) \in \mathbf{R}^{2n}$ at z^0 , where $p(x, \xi)$ denotes the principal symbol of $P(x, \xi)$. Assume that $(P-2)_{z^0}$, $(P-3)_{z^0}$ and at least one of the conditions $(P-4-1)_{z^0}$ and $(P-4-2)_{z^0}$ are satisfied. If $t(x, \xi)$ is a smooth time function for $p(x, \xi)$ with respect to $(0, \vartheta)$ at z^0 , $z^0 \notin WF(P^w(x, D)u)$ and $WF(u) \cap \{t(x, \xi) < 0\} \cap \mathcal{C} = \emptyset$ with some conic neighborhood $\mathcal{C}$ of z^0 , then $z^0 \notin WF(u)$.

Remark. (i) Theorem 3.3 is a microlocal version of Hörmgren's uniqueness theorem. (ii) $(0, \vartheta)$ can be replaced by any non-zero vector in $\mathbf{R}^{2n}$. (iii) We can give the theorem in the form of Theorem 1.3 in [9].

Assume that the hypotheses of Theorem 3.3 are satisfied for z^0 replaced by $z = (x, \xi) \in \Omega \cap \{|\xi| = 1\}$, where Ω is an open conic set in $T^*\mathbf{R}^n \setminus 0$ and contains z^0 . Let $t(x, \xi)$ be a smooth time function for $p(x, \xi)$ with respect to $\tilde{\vartheta} \in \mathbf{R}^{2n}$ in Ω , i.e., $t(x, \xi)$ is a real-valued smooth function in $T^*\mathbf{R}^n \setminus 0$ and positively homogeneous of degree 0 and $-H_t(z) \in \Gamma(p_z, \tilde{\vartheta})$ for $z \in \Omega$. If $WF(P^w(x, D)u) \cap \Omega = \emptyset$, and if u is not smooth at the present time (i.e., $WF(u) \cap \{t(x, \xi) = 0\} \cap \Omega \neq \emptyset$), then u was not smooth in the past (i.e., $WF(u) \cap \{t(x, \xi) < 0\} \cap \Omega \neq \emptyset$). So time functions give measure of time concerning propagation of singularities.

REFERENCES

1. C. Fefferman and D. H. Phong, *On positivity of pseudodifferential operators*, Proc. Nat. Acad. Sc. **75** (1978), 4673–4674.
2. L. Hörmander, *The Cauchy problem for differential equations with double characteristics*, J. Analyse Math. **32** (1977), 118–196.
3. L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer, Berlin-Heidelberg-New York-Tokyo, 1983.
4. L. Hörmander, *The Analysis of Linear Partial Differential Operators III*, Springer, Berlin-Heidelberg-New York-Tokyo, 1985.
5. N. Iwasaki, *The Cauchy problem for effectively hyperbolic equations (general case)*, J. Math. Kyoto Univ. **25** (1985), 727–743.
6. V. Ja. Ivrii, *Sufficient conditions for regular and completely regular hyperbolicity*, Trudy Moskov. Mat. Obšč. **33** (1976), 3–66; Moscow Math. Soc. **33** (1978), 1–65.
7. V. Ja. Ivrii, *The well-posedness of the Cauchy problem for nonstrictly hyperbolic operators. III. The energy integral*, Trudy Moskov. Mat. Obšč. **34** (1977), 151–170; Moscow Math. Soc. **34** (1978), 149–168.
8. K. Kajitani and S. Wakabayashi, *Microhyperbolic operators in Gevrey classes*, Publ. RIMS, Kyoto Univ. **25** (1989), 169–221.
9. K. Kajitani and S. Wakabayashi, *Propagation of singularities for several classes of pseudodifferential operators*, Bull. Sc. math., 2^e série **115** (1991), 397–449.
10. K. Kajitani and S. Wakabayashi, *Microlocal a priori estimates and the Cauchy problem I*, to appear.
11. K. Kajitani and S. Wakabayashi, *Microlocal a priori estimates and the Cauchy*

problem II, to appear.

12. K. Kajitani and S. Wakabayashi, *The Cauchy problem for a class of hyperbolic operators with double characteristics*, in preparation.
13. K. Kajitani, S. Wakabayashi and T. Nishitani, *The Cauchy problem for hyperbolic operators of strong type*, to appear.
14. T. Nishitani, *Local energy integrals for effectively hyperbolic operators, I*, J. Math. Kyoto Univ. **24** (1984), 623–658.
15. T. Nishitani, *Local energy integrals for effectively hyperbolic operators, II*, J. Math. Kyoto Univ. **24** (1984), 659–666.
16. O. A. Oleinik, *On the Cauchy problem for weakly hyperbolic equations*, Comm. Pure Appl. Math. **23** (1970), 569–586.
17. S. Wakabayashi, *Generalized Hamilton flows and singularities of solutions of hyperbolic Cauchy problem*, Proc. Hyperbolic Equations and Related Topics, Taniguchi Symposium, Kinokuniya, Tokyo, 1984, pp. 415–423.