

Construction of Morse flows to a variational functional of harmonic map type

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In this paper we shall construct solutions of the parabolic differential equations associated to a simple variational functional, the Euler-Lagrange equations of which are linear equations.

Let Ω be a bounded domain in $\mathbf{R}^m$, $m \geq 2$, with C^2 -boundary $\partial\Omega$. In the following, a map u means the one from Ω to $\mathbf{R}^M$, $M \geq 1$. For a map u belonging to Sobolev space $H^1(\Omega)$, we consider the functional

$$F(u) = \int_{\Omega} A_{ij}^{\alpha\beta}(x) D_{\alpha} u^i(x) D_{\beta} u^j(x) dx, \quad (1)$$

where $u = (u^i)$, $D_{\alpha} u^i = \partial u^i / \partial x^{\alpha}$, $1 \leq i \leq M$, $1 \leq \alpha \leq m$. The summation convention is used. The coefficients $A_{ij}^{\alpha\beta}(x)$, $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$, are assumed to be bounded measurable in Ω and to satisfy the elliptic condition: There exists a positive λ such that

$$A_{ij}^{\alpha\beta}(x) \xi_{\alpha}^i \xi_{\beta}^j \geq \lambda |\xi|^2 \quad \text{for } \xi = (\xi_{\alpha}^i) \in \mathbf{R}^{mM} \quad \text{and } x \in \Omega.$$

Hereafter, we use the notation

$$A(x)(Du, Du) = A_{ij}^{\alpha\beta}(x) D_{\alpha} u^i D_{\beta} u^j.$$

'Morse flows' of variational functional F are defined as solutions of parabolic partial differential equations

$$\frac{\partial u^i}{\partial t} = D_{\beta} (A_{ji}^{\alpha\beta}(x) D_{\alpha} u^j) \quad (1 \leq i \leq M). \quad (2)$$

Let u_0 be a given map belonging to $H^1(\Omega)$ and T a positive number. We take a positive integer N and put

$$h = T/N \quad \text{and} \quad t_n = nh \quad (n = 0, 1, \dots, N). \quad (3)$$

In the following, we use a function space

$$H_{u_0}^1(\Omega) = \{u \in H^1(\Omega); u - u_0 \in H_0^1(\Omega)\},$$

$H_0^1(\Omega)$ being the space obtained by taking the closure of $C_0^\infty(\Omega)$ in the space $H^1(\Omega)$. Beginning with u_0 , we inductively construct two sequences of maps u_n and functionals F_n , $1 \leq n \leq N$, as follows: For each n , $1 \leq n \leq N$, we introduce the functional

$$F_n(u) = \int_{\Omega} \left(A(x)(Du, Du) + \frac{1}{h}|u - u_{n-1}|^2 \right) dx \quad (4)$$

and define u_n as a minimizer of F_n in $H_{u_0}^1(\Omega)$, the existence of which is assured by the lower semi-continuity of F_n with respect to weak convergence of $H^1(\Omega)$. We here remark the Euler-Lagrange equations of F_n in $H_{u_0}^1(\Omega)$ are of the form: For n , $1 \leq n \leq N$,

$$\frac{u_n^i - u_{n-1}^i}{h} = D_\beta(A_{ji}^{\alpha\beta} D_\alpha u_n^j) \quad (1 \leq i \leq M), \quad (5)$$

which are Rothe's approximate equations of (2). Upon comparing u_{n-1} with a minimizer u_n of F_n , we infer

$$\int_{\Omega} A(x)(Du_n, Du_n) dx + \int_{\Omega} \frac{1}{h}|u_n - u_{n-1}|^2 dx \leq \int_{\Omega} A(x)(Du_{n-1}, Du_{n-1}) dx$$

and hence have the following result.

Theorem 0 ([4]). *For $\{u_n\}$ ($1 \leq n \leq N$) constructed as above, there hold the estimates*

$$\int_{\Omega} A(x)(Du_n, Du_n) dx \leq \int_{\Omega} A(x)(Du_0, Du_0) dx \quad \text{for any } n \ (1 \leq n \leq N) \quad (6)$$

and

$$h \sum_{n=1}^N \int_{\Omega} \left| \frac{u_n - u_{n-1}}{h} \right|^2 dx \leq \int_{\Omega} A(x)(Du_0, Du_0) dx. \quad (7)$$

We define a map $u(t) \in H_{u_0}^1(\Omega)$, $-h \leq t \leq T$, by means of the identities :

$$u(t) = u_n \quad \text{for } t_{n-1} < t \leq t_n \quad (1 \leq n \leq N)$$

and

$$u(t) = u_0 \quad \text{for } -h \leq t \leq 0.$$

(8)

We put

$$\partial_t u(t) = \frac{1}{h}(u_n - u_{n-1}) \quad \text{for } t_{n-1} < t \leq t_n \quad (1 \leq n \leq N)$$

and

$$\tilde{u}(t) = u(t-h) \quad \text{for } 0 \leq t \leq T.$$

(9)

For the gradient of u constructed as above, we have the estimate of higher integrability. To state the result, we shall prepare the notations as follows. We set

$$Q = (0, T) \times \Omega.$$

For $z_0 = (t_{n_0}, x_0) \in Q$, $1 \leq n_0 \leq N$ and positive s , we put

$$Q_s(z_0) = \{t \in (0, T); t_{n_0} - s^2 < t < t_{n_0}\} \times B_s(x_0),$$

where $B_s(x_0) = \{x \in \Omega; |x - x_0| < s\}$.

Theorem 1. *For the map u defined as in (8), there exist positive C and ε not depending on h such that*

$$\begin{aligned} & \left(\int_{Q_{r/2}(z_0)} |Du|^{2+\varepsilon} dz \right)^{1/(2+\varepsilon)} \leq C \left(\int_{Q_{r/2}(z_0)} |Du|^2 dz \right)^{1/2} \\ & + ch^{(\bar{p}-1)(m+2)/2m} \left(\int_{Q_r(z_0)} |\partial_t u|^{(1+\varepsilon/2)\bar{p}} |u - \tilde{u}|^{(1+\varepsilon/2)(2-\bar{p})} dz \right)^{1/(2+\varepsilon)} \end{aligned} \quad (10)$$

holds for any $Q_r(z_0) \subset Q$ and any $\bar{p}$, $1 < \bar{p} < 2$.

Noting the estimates in Theorem 0 and 1 are valid uniformly in h , there holds the existence theorem of a weak solution to (2) with the gradient of higher integrability.

By a weak solution to parabolic system(2), we mean a map $u \in L^\infty((0, \infty), H^1(\Omega)) \cap H^1((0, T), L^2(\Omega))$ such that

$$\int_Q \frac{\partial u^i}{\partial t} \varphi^i dz + \int_Q A_{ij}^{\alpha\beta}(x) D_\alpha u^i D_\beta \varphi^j dz = 0$$

for any $\varphi \in C_0^\infty(Q)$.

Theorem 2. *There exists a weak solution u to (2) satisfying the initial and boundary conditions:*

$$u(t) \in H_{u_0}^1(\Omega) \quad \text{for almost every } t \in (0, T)$$

and

$$\lim_{t \downarrow 0} u(t) = u_0 \quad \text{in } L^2(\Omega).$$

The solution u satisfies the estimate :

$$\left(\int_{Q_{r/2}(z_0)} |Du|^{2+\varepsilon} dz \right)^{1/(2+\varepsilon)} \leq C \left(\int_{Q_r(z_0)} |Du|^2 dz \right)^{1/2}$$

for $Q_r(z_0) \subset Q$, where C and ε are positive numbers as in Theorem 1.

Furthermore, if $A_{ij}^{\alpha\beta}(x)$ are continuous in Ω , u is Hölder continuous in Q with any component α , $0 < \alpha < 1$.

The existence proof of solutions follows from the estimates in Theorem 0. Noting the higher integrability (10) of Du and the estimate (7) and paralleling the method developed in [2], it follows from Campanato's fundamental result [1] that Hölder continuity of u is derived. The estimate (10) in Theorem 1 is derived from the following estimate of Caccioppoli type, for the verification of which we have only to follow the method due to Giaquinta-Stuwe([3]).

For positive s satisfying $B_s(x_0) \subset \Omega$ and $u \in L^1(Q)$, we put ([6])

$$u_s = u_s(t) = \int_{B_s(x_0)} \eta(x) u(t, x) dx \quad \text{for } 0 < t < T, \quad (11)$$

where $\eta(x) = 1$ on $B_{s/2}(x_0)$ and $|D\eta| \leq 4/s$.

Lemma (Caccioppoli type estimate). *For the map u defined as in (8), there exists a positive C not depending on h such that*

$$\begin{aligned} \int_{Q_r(z_0)} |Du|^2 dz &\leq Cr^{-2} \int_{Q_{2r}(z_0)} |u - u_{2r}|^2 dz \\ &+ Ch^{\bar{p}-1} \int_{Q_{2r}(z_0)} |\partial_t u|^{\bar{p}} |u - \tilde{u}|^{2-\bar{p}} dz \end{aligned}$$

holds for any $Q_{2r}(z_0) \subset Q$, $z_0 = (t_{n_0}, x_0)$, $1 \leq n_0 \leq N$ and for any $\bar{p}$, $1 < \bar{p} < 2$, where $|\partial_t u|^{\bar{p}} |u - \tilde{u}|^{2-\bar{p}}$, $1 < \bar{p} < 2$, belongs to $L^p(Q)$ with some $p, p > 1$, satisfying $p \leq m/(m - 2 + \bar{p})$.

We shall only sketch our proof. Let k and l be positive numbers satisfying $r < k < l < 2r$. As a comparison map in functional F_n , we adopt v_n , $1 \leq n \leq N$, defined by

$$v_n = u_n - h\eta(u_n - u_{n,l}),$$

where u_n is a minimizer of F_n in $H_{u_0}^1(\Omega)$ and $u_{n,l}$ is defined as in (11).

We make the classification between $(l - k)^2$ and h ([5]):

$$(l - k)^2 \leq 4h, \tag{12}$$

$$(l - k)^2 > 4h. \tag{13}$$

We treat each case of (12) and (13) and follow the iteration procedure ([2]) to obtain each estimate. By adding both the estimates, we arrive at the estimate in Lemma, which is available under no restriction of (12) and (13).

The term $|\partial_t u|^{\bar{p}} |u - \tilde{u}|^{2-\bar{p}}$ is assured to belong to $L^p(Q)$ with some $p, p > 1$, satisfying $p \leq m/(m - 2 + \bar{p})$, which is verified to hold from the global estimates (6) and (7).

References

- [1] Campanato, S., *Equazioni paraboliche del secondo ordine e spazi $\mathcal{L}^{2,\theta}(\Omega, \delta)$* , Ann. Mat. Pura Appl. **73**. (1966) 55–102.
- [2] Giaquinta, M. and E. Giusti, *On the regularity of the minima of variational integrals*, Acta Math. **148**. (1982) 31–46.

- [3] Giaquinta, M. and M. Struwe, *On the partial regularity of weak solutions of non-linear parabolic systems*, Math. Zeit. **142**. (1982) 473–451.
- [4] Kikuchi, N., *An approach to the construction of Morse flows for variational functionals*, in Nematics-Mathematical and Physical Aspects, ed. J-M.Coron, J-M.Ghidaglia and F.Hélein (NATO Adv. Sci. Inst. Ser.C: Math.Phys. Sci. **332**. Kluwer Acad. Publ., Dordrecht-Boston-London) (1991) 195–198,
- [5] Kikuchi, N., *Hölder estimates of solutions for difference-differential equations of elliptic-parabolic type*, Research Report (1992) 1–23.
- [6] Struwe, M., *On the Hölder continuity of bounded weak solutions of quasilinear parabolic systems*, manus. math. **35**. (1981) 125–145.