

Local cohomology solutions of holonomic D-modules associated with a non-isolated hypersurface singularity

By

Shinichi TAJIMA*

Abstract

A hypersurface with a smooth 2-dimensional singular locus is considered in the context of Computational Algebraic Analysis. The holonomic D-module associated with each root of the reduced b-function is computed. Local cohomology solutions to the holonomic D-module are explicitly computed.

§ 1. Introduction

In this paper, we consider a hypersurface with a smooth 2-dimensional singular locus in the context of Computational Algebraic Analysis. We explicitly compute holonomic D-modules associated with the reduced b-function of the hypersurface by using a computer algebra system and we study holonomic D-modules by computing its local cohomology solutions.

In 1970's, M. Kashiwara studied b-functions by using D-modules. It turned out that holonomic D-modules that he introduced to study b-functions contain a wealth information on singularity. It is important therefore to analyze the structures of the holonomic D-module associated with a root of b-functions.

In a previous paper [15], Y. Umeta and the author of the present paper studied certain kinds of hypersurfaces with a smooth 1-dimensional singular locus. We considered the holonomic D-module associated with a root of the reduced b-function of hypersurfaces. We described in particular a method for computing structures of relevant holonomic D-modules. As a sequel of the previous paper [15], we address the

Received May 7, 2017. Revised October 11, 2017.

2010 Mathematics Subject Classification(s): 32C38, 32S40

Key Words: local cohomology, holonomic D-module, non-isolated singularity.

Supported by JSPS Grant-in-Aid for Scientific Research 15K04891

*Graduate School of Science and Technology, Niigata University, 2-8050 Ikarashi, Nishi-ku, Niigata, Japan. e-mail: tajima@emiritus.niigata-u.ac.jp

case of hypersurface with a 2-dimensional singular locus. We consider one example of hypersurface with non-isolated singularity which is given by A. Zaharia in [17]. In this paper, we present in particular an effective method for studying holonomic D-modules associated with roots of b-functions. The key of our approach is a use of the concept of local cohomology.

In section two, we compute the b-function, local b-functions of the hypersurface. We compute the holonomic D-modules associated with roots of the reduced b-function. In section three, we analyze structures of the holonomic D-modules by computing local cohomology solutions.

§ 2. An example of non-isolated hypersurface singularity

Let $S = \{(x_1, x_2, y_1, y_2) \in X \mid g(x_1, x_2, y_1, y_2) = 0\}$ where $g = y_1^2(y_1 + x_1^3 + x_2^2) + y_2^2$. The singular locus Σ of S is the 2-dimensional plane $\{(x_1, x_2, 0, 0) \mid x_1, x_2 \in \mathbb{C}\}$. The hypersurface above is taken from Table 4 (page 51) in a paper [17]. The table 4 is obtained by A. Zaharia as one of results of classification of simple germs.

The defining function g is a weighted homogeneous polynomial with respect to the weight vector $w = \frac{1}{18}(2, 3, 6, 9)$. Let E denote the Euler operator defined by

$$E = \frac{1}{9}x_1 \frac{\partial}{\partial x_1} + \frac{1}{6}x_2 \frac{\partial}{\partial x_2} + \frac{1}{3}y_1 \frac{\partial}{\partial y_1} + \frac{1}{2}y_2 \frac{\partial}{\partial y_2}.$$

Then, g satisfies, $E(g) = g$.

In this section, we compute the b-function, local b-functions of the defining function g and compute the holonomic D-module associated with each root of the reduced b-function of g . For this purpose, we use a computer algebra system Risa/Asir [11] and four algorithms `bfct`, `ndbf.bf_strat`, `ann` and `cgsd_dx` implemented in Risa/Asir.

By using `bfct`, we compute the b-function of g .

```
[250] G=y1^2*(y1+x1^3+x2^2)+y2^2;
y1^3+(x1^3+x2^2)*y1^2+y2^2
```

```
[251] fctr(bfct(G));
[[1,1],[s+1,2],[2*s+3,1],
[9*s+10,1],[9*s+11,1],[9*s+13,1],[9*s+14,1],[9*s+16,1],[9*s+17,1]]
```

The output means

$$(s+1)^2(s+\frac{3}{2})(s+\frac{10}{9})(s+\frac{11}{9})(s+\frac{13}{9})(s+\frac{14}{9})(s+\frac{16}{9})(s+\frac{17}{9})$$

is the b-function of the polynomial g .

§ 2.1. a stratification and local b-functions

First, we briefly recall the concept of local b-functions. Let X be a complex manifold, $\mathcal{O}_X$ the sheaf on X of holomorphic functions, f a germ of holomorphic function at a point $x \in X$. Let $\mathcal{D}_X$ be the sheaf on X of linear partial differential operators with holomorphic coefficients. There exists a polynomial $b(s)$ in s and partial differential operator $P(s) \in \mathcal{D}_X[s]$ in a neighborhood of x such that

$$P(s)f(x)^{s+1} = b(s)f(x)^s.$$

The monic generator $b_{f,x}(s)$ of the ideal consisting of such polynomials $b(s)$ is the local b-function of f at $x \in X$. This is defined in the context of analytic functions and described in terms of holomorphic linear partial differential operators.

In 1997, T. Oaku showed that local b-functions of a polynomial f in $\mathbb{C}[x_1, x_2, \dots, x_n]$ can be computed by applying computations in Weyl algebra and gave an algorithm for computing local b-functions at all points simultaneously by using a primary decomposition algorithm. The algorithm outputs a stratification of $\mathbb{C}^n$ giving the local b-function $b_{f,x}$ which is constant on each stratum. More recently in 2010, K. Nishiyama and M. Noro proposed alternative method for computing local b-functions([10]). The resulting algorithm `ndbf.bf_strat` is implemented in `Risa/Asir`([11]).

We compute local b-functions of g by executing `ndbf.bf_strat` on `Risa/Asir`. By analyzing the result of computation, we see that the stratification of the space $X = \mathbb{C}^4$ associated with the local b-functions of g consists of 5 strata :

$$\Sigma_0, \Sigma_1, \Sigma_2, S - \Sigma, X - S,$$

where

$$\Sigma_0 = \{(0, 0, 0, 0)\}, \quad \Sigma_1 = \{(x_1, x_2, 0, 0) \mid x_1^3 + x_2^2 = 0\} - \Sigma_0, \quad \Sigma_2 = \Sigma - \Sigma_1 \cup \Sigma_0.$$

We have $b_{g,x}(s) = (s + 1)$ on the non-singular locus $S - \Sigma$, and

$$b_{g,x}(s) = (s + 1)^2 \text{ on } \Sigma_2, \quad b_{g,x}(s) = (s + 1)^2(2s + 3) \text{ on } \Sigma_1,$$

$$b_{g,x}(s) = (s + 1)^2(s + \frac{3}{2})(s + \frac{10}{9})(s + \frac{11}{9})(s + \frac{13}{9})(s + \frac{14}{9})(s + \frac{16}{9})(s + \frac{17}{9}) \text{ on } \Sigma_0.$$

§ 2.2. holonomic D-modules

First, we recall some basics. Let $\text{Ann}_{D_X[s]}(f^s)$ denote the annihilating ideal of f^s in the ring $D_X[s]$:

$$\text{Ann}_{D_X[s]}(f^s) = \{P \in D_X[s] \mid Pf^s = 0\}.$$

Let

$$I = \text{Ann}_{D_X[s]}f^s + D_X[s]f + D_X[s]J_f,$$

where $D_X[s]J_f$ is the ideal generated by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$.

For each root β of the reduced b-function of f , the D_X -module M_β defined by

$$D_X[s]/(I + D_X[s](s - \beta))$$

is holonomic and the support of M_β is a subset of the singular locus of the hypersurface S , in general.

In this subsection we compute holonomic D-modules associated with roots of the reduced b-function of g in question.

We define a list of variables, a matrix for term ordering and a list of generators of the ideal I for preparations. For this purpose, we use the algorithm `ann`.

```
[300] G=y1^2*(y1+x1^3+x2^2)+y2^2$
```

```
[301] W=[[x1,x2,y1,y2],[dx1,dx2,dy1,dy2]]$
```

```
[302] Mat=newmat(11,10,[[0,0,0,0,0,1,1,1,1,0],[0,0,0,0,0,1,0,0,0,0],
[0,0,0,0,0,0,1,0,0,0],[0,0,0,0,0,0,0,1,0,0],[0,0,0,0,0,0,0,0,1,0],
[1,1,1,1,0,0,0,0,0,0],[1,0,0,0,0,0,0,0,0,0],[0,1,0,0,0,0,0,0,0,0],
[0,0,1,0,0,0,0,0,0,0],[0,0,0,1,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,1]])$
```

```
[303] L=append([G,diff(G,x1),diff(G,x2),diff(G,y1),diff(G,y2)], ann(G))$
```

Note that in the list L above $\text{ann}(G)$ is a list of generators of the annihilating ideal $\text{Ann}_{D_X[s]}g^s$. The contents are as follows.

```
[304] ann(G);
```

```
[-3*dy2*y1^2+(-2*dy2*x1^3-2*dy2*x2^2)*y1+2*y2*dy1,
-3*dy2*x1^2*y1^2+2*dx1*y2,
-dy2*x2*y1^2+dx2*y2,
9*dy2*y1^3-6*y2*dy1*y1+(4*dx1*x1+6*dx2*x2)*y2,
(-3*x1^2*dy1+3*dx1)*y1+2*dx1*x1^3+2*dx1*x2^2,
(6*x2*dy1-9*dx2)*y1-4*dx1*x2*x1-6*dx2*x2^2,
-3*dx2*x1^2+2*dx1*x2,
18*s-6*dy1*y1-9*dy2*y2-2*dx1*x1-3*dx2*x2]
```

Now we compute holonomic D-modules by using the algorithm `cgs_w_dx` constructed recently in [8] by K. Nabeshima, K. Ohara and the author of the present paper.

[305] cgs_w_dx(L,[s],W,1,Mat);

[[s+1],[1]]

[y₂,y₁,-2*dx₁*x₁-3*dx₂*x₂-3,-3*dx₂*x₁²+2*dx₁*x₂]

[[2*s+3],[1]]

[y₂,y₁²,-6*dy₁*y₁-2*dx₁*x₁-3*dx₂*x₂-18,(-2*x₂*dy₁+dx₂)*y₁-4*x₂,
-3*dx₂*x₁²+2*dx₁*x₂,y₁+x₁³+x₂², (3*x₁²*dy₁-dx₁)*y₁+6*x₁²]

[[9*s+10],[1]]

[y₂,y₁,x₂,x₁]

[[9*s+11],[1]]

[y₂,y₁,x₂, -dx₁*x₁-2,x₁²]

[[9*s+13],[1]]

[y₂,9*dx₂*y₁-16*x₂,y₁²,x₂*y₁,9*y₁+16*x₂²,
-6*dy₁*y₁-2*dx₁*x₁-3*dx₂*x₂-17,x₁*y₁,x₂*x₁, -3*dx₁*y₁+8*x₁²]

[[9*s+14],[1]]

[y₂,9*dx₂*y₁-20*x₂,y₁²,x₂*y₁, -9*y₁-20*x₂²,
-6*dy₁*y₁-2*dx₁*x₁-3*dx₂*x₂-19, -3*dx₁*y₁+10*x₁²]

[[9*s+16],[1]]

[y₂, -6*dy₁*y₁-2*dx₁*x₁-3*dx₂*x₂-23,y₁³,
(-18*x₂*dy₁+9*dx₂)*y₁-46*x₂,x₂*y₁²,9*y₁²+10*x₂²*y₁,
27*x₂*y₁+28*x₂³,x₁*y₁²,x₂*x₁*y₁,9*x₁*y₁+28*x₂²*x₁,
-3*dx₂*x₁²+2*dx₁*x₂, -9*dy₁*y₁²+15*y₁+28*x₁³+28*x₂²,
(9*x₁²*dy₁-3*dx₁)*y₁+23*x₁²]

[[9*s+17],[1]]

[y₂, -6*dy₁*y₁-2*dx₁*x₁-3*dx₂*x₂-25,y₁³,
(-18*x₂*dy₁+9*dx₂)*y₁-50*x₂,x₂*y₁²,
9*y₁²+14*x₂²*y₁,27*x₂*y₁+32*x₂³, -3*dx₂*x₁²+2*dx₁*x₂,
-9*dy₁*y₁²+21*y₁+32*x₁³+32*x₂², (9*x₁²*dy₁-3*dx₁)*y₁+25*x₁²,
x₁²*y₁²,x₂*x₁²*y₁,9*x₁²*y₁+32*x₂²*x₁²]

[[0],[1062882*s⁸+12223143*s⁷+61135398*s⁶+173682792*s⁵

```
+306514368*s^4+344058597*s^3+239861032*s^2+94945468*s+16336320]]
[1]
```

No. of segment is

9

0.0312sec(0.093sec)

The output consists of each factor of the reduced b-function, the Gröbner basis of the holonomic D-module associated with each root of the reduced b-function.

§ 3. Local systems and local cohomology solutions

In this section we study structures of the holonomic D-modules presented in the previous section. We explicitly compute algebraic local cohomology solutions.

§ 3.1. Algebraic local cohomology solutions supported on Σ_2

Let $s + 1 = 0$. Then a Gröbner basis of the holonomic ideal associated with the root $\beta = -1$ is given by

$$y_1, y_2, 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 3, 2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2}.$$

Let $\mathcal{H}_{[\Sigma_2]}^2(\mathcal{O}_X)$ be the sheaf of algebraic local cohomology supported on Σ_2 , where

$$\Sigma_2 = \{(x_1, x_2, 0, 0) \mid x_1, x_2 \in \mathbb{C}\} - \{(x_1, x_2, 0, 0) \mid x_1^3 + x_2^2 = 0\}.$$

Set

$$H_{\Sigma_2} = \left\{ \psi \in \mathcal{H}_{[\Sigma_2]}^2(\mathcal{O}_X) \mid y_1 \psi = y_2 \psi = 0 \right\}.$$

Then any germ at a point $Q \in \Sigma_2$ of the sheaf H_{Σ_2} can be represented in a form

$$h(x_1, x_2) \begin{bmatrix} 1 \\ y_1 \ y_2 \end{bmatrix},$$

where $[\]$ denotes the Grothendieck symbol and $h(x_1, x_2)$ is a germ at Q of holomorphic functions on Σ_2 .

Algebraic local cohomology solution ψ of the holonomic D-module M_{-1} satisfies the following system of linear partial differential equations

$$(2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 3)h = 0, (2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2})h = 0.$$

Since $\begin{vmatrix} 2x_1 & 3x_2 \\ 2x_2 & -3x_1^2 \end{vmatrix} = -6(x_1^3 + x_2^2)$, the singular locus of the holonomic D-module M_{-1} is

$$\{(x_1, x_2, 0, 0) \mid x_1^3 + x_2^2 = 0\} = \Sigma_1 \cup \Sigma_0.$$

We set $h(x_1, x_2) = (x_1^3 + x_2^2)^\alpha$. Then we have

$$(2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 3)h = (6\alpha + 3)h, \quad (2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2})h = 0.$$

Hence,

$$(x_1^3 + x_2^2)^{-\frac{1}{2}} \begin{bmatrix} 1 \\ y_1 & y_2 \end{bmatrix}$$

is a local cohomology solution.

The monodromy structure on Σ_2 of the holonomic D-module M_{-1} is non trivial. Since the local cohomology solution ψ can not be analytically continued to $\Sigma_1 \cup \Sigma_0$, there exists no non-trivial algebraic local cohomology solution in $\mathcal{H}_{[\Sigma]}^2(\mathcal{O}_X)$ at Σ_0 .

§ 3.2. Algebraic local cohomology solutions supported on Σ_1

Since the local b-function is $(s+1)(s+\frac{3}{2})$ on the stratum Σ_1 , we consider two cases.

(i) Let $2s+3=0$. Then a Gröbner basis of the holonomic ideal associated with the root $\beta = -\frac{3}{2}$ is given by

$$\begin{cases} x_1^3 + x_2^2 + y_1, & y_1^2, & y_2, & 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 18, \\ 2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2}, & y_1 \frac{\partial}{\partial x_1} - 3x_1^2 y_1 \frac{\partial}{\partial y_1} - 6x_1^2, & y_1 \frac{\partial}{\partial x_2} - 2x_2 y_1 \frac{\partial}{\partial y_1} - 4x_2. \end{cases}$$

From $(x_1^3 + x_2^2)^2 = (x_1^3 + x_2^2 - y_1)(x_1^3 + x_2^2 + y_1) + y_1^2$, we see that $(x_1^3 + x_2^2)^2$ is an annihilator. Therefore the holonomic D-module $M_{-\frac{3}{2}}$ is supported on $\Sigma_1 \cup \Sigma_0$, where

$$\Sigma_1 \cup \Sigma_0 = \{(x_1, x_2, 0, 0) \mid x_1^3 + x_2^2 = 0\}.$$

Now consider algebraic local cohomology class ψ in $\mathcal{H}_{\Sigma_1}^3(\mathcal{O}_X)$ supported on Σ_1 of the form

$$\psi = a \begin{bmatrix} 1 \\ (x_1^3 + x_2^2) & y_1^2 & y_2 \end{bmatrix} + b \begin{bmatrix} 1 \\ (x_1^3 + x_2^2)^2 & y_1 y_2 \end{bmatrix}.$$

Note that the weighted degree of ψ is equal to $-\frac{27}{18} = -\frac{3}{2}$.

Then, ψ satisfies, $y_1^2 \psi = y_2 \psi = 0$ and

$$(2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 18)\psi = (2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2})\psi = 0.$$

From

$$(x_1^3 + x_2^2 + y_1)\psi = (a + b) \begin{bmatrix} 1 \\ (x_1^3 + x_2^2) \ y_1 \ y_2 \end{bmatrix}$$

we have $a + b = 0$. Therefore, we set

$$\psi = \begin{bmatrix} 1 \\ (x_1^3 + x_2^2) \ y_1^2 \ y_2 \end{bmatrix} - \begin{bmatrix} 1 \\ (x_1^3 + x_2^2)^2 \ y_1 y_2 \end{bmatrix}.$$

It is easy to verify that the local cohomology class ψ above satisfies

$$(y_1 \frac{\partial}{\partial x_1} - 3x_1^2 y_1 \frac{\partial}{\partial y_1} - 6x_1^2)\psi = (y_1 \frac{\partial}{\partial x_2} - 2x_2 y_1 \frac{\partial}{\partial y_1} - 4x_2)\psi = 0.$$

Since the solution ψ can be analytically continued to Σ_0 , ψ belongs to

$$\mathcal{H}_{\Sigma_1 \cup \Sigma_0}^3(\mathcal{O}_X)$$

and the dimension of the algebraic local cohomology solutions of the holonomic D-module $M_{-\frac{3}{2}}$ is equal to one at the origin Σ_0 .

(ii) Let $s + 1 = 0$. It is easy to see that the holonomic D-module M_{-1} has no non-trivial local cohomology solution in $\mathcal{H}_{\Sigma_1}^3(\mathcal{O}_X)$.

§ 3.3. Local cohomology solutions supported at Σ_0

Since the reduced local b-function at Σ_0 of g is equal to

$$\tilde{b}_{g, \Sigma_0} = (s + 1)(s + \frac{3}{2})(s + \frac{10}{9})(s + \frac{11}{9})(s + \frac{13}{9})(s + \frac{14}{9})(s + \frac{16}{9})(s + \frac{17}{9}),$$

we compute algebraic local cohomology solutions for the holonomic D-module associated with each root of the reduced local b-function.

Let $\mathcal{H}_{[\Sigma_2]}^4(\mathcal{O}_X)$ be the algebraic local cohomology supported on Σ_0 .

(i) $9s + 10 = 0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta = -\frac{10}{9}$ is given by $\{x_1, x_2, y_1, y_2\}$. We see that the local cohomology class

$$\begin{bmatrix} 1 \\ x_1 \ x_2 \ y_1 \ y_2 \end{bmatrix}$$

is a solution of the holonomic D-module $M_{-\frac{10}{9}}$.

(ii) $9s + 11 = 0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta = -\frac{11}{9}$ is given by $\{x_1^2, x_2, y_1, y_2, x_1 \frac{\partial}{\partial x_1} + 2\}$. The local cohomology class

$$\begin{bmatrix} 1 \\ x_1^2 \ x_2 \ y_1 \ y_2 \end{bmatrix}$$

is a solution of the holonomic D-module $M_{-\frac{11}{9}}$.

(iii) $9s + 13 = 0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta = -\frac{13}{9}$ is given by

$$\left\{ \begin{array}{l} x_1x_2, \quad x_1y_1, \quad x_2y_1, \quad y_1^2, \quad y_2, \quad 16x_2^2 + 9y_1, \\ 2x_1\frac{\partial}{\partial x_1} + 3x_2\frac{\partial}{\partial x_2} + 6y_1\frac{\partial}{\partial y_1} + 17, \quad 3y_1\frac{\partial}{\partial x_1} - 8x_1^2, \quad 9y_1\frac{\partial}{\partial x_2} - 16x_2. \end{array} \right.$$

Note that since

$$16x_2^3 = x_2(16x_2^2 + 9y_1) - 9x_2y_1$$

and

$$8x_1^4 = -x_1^2(3y_1\frac{\partial}{\partial x_1} - 8x_1^2) + 3\frac{\partial}{\partial x_1}(x_1^2y_1) - 6x_1y_1,$$

partial differential operators x_1^4, x_2^3 belong to the annihilating ideals. It follows in particular from this fact that the support of the holonomic D-module $M_{-\frac{13}{9}}$ is the origin Σ_0 .

Since the weighted degree of the solution is equal to $-\frac{13}{9}$, we consider algebraic local cohomology class ψ of the form

$$\psi = a \begin{bmatrix} 1 \\ x_1^4 & x_2 & y_1 & y_2 \end{bmatrix} + b \begin{bmatrix} 1 \\ x_1 & x_2^3 & y_1 & y_2 \end{bmatrix} + c \begin{bmatrix} 1 \\ x_1 & x_2 & y_1^2 & y_2 \end{bmatrix}.$$

From

$$(3y_1\frac{\partial}{\partial x_1} - 8x_1^2)\psi = -(8a + 3c) \begin{bmatrix} 1 \\ x_1^2 & x_2 & y_1 & y_2 \end{bmatrix}$$

and

$$(9y_1\frac{\partial}{\partial x_2} - 16x_2)\psi = -(16b + 9c) \begin{bmatrix} 1 \\ x_1 & x_2^2 & y_1 & y_2 \end{bmatrix},$$

we have

$$8a + 3c = 0, \quad 16b + 9c = 0.$$

We thus have

$$\psi = \begin{bmatrix} 1 \\ x_1^4 & x_2 & y_1 & y_2 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 \\ x_1 & x_2^3 & y_1 & y_2 \end{bmatrix} - \frac{8}{3} \begin{bmatrix} 1 \\ x_1 & x_2 & y_1^2 & y_2 \end{bmatrix}$$

as local cohomology solution.

(iv) $9s + 14 = 0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta = -\frac{14}{9}$ is given by

$$\begin{cases} x_2 y_1, y_1^2, y_2, 20x_2^2 + 9y_1, 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 19, \\ 3y_1 \frac{\partial}{\partial x_1} - 10x_1^2, \quad 9y_1 \frac{\partial}{\partial x_2} - 20x_2. \end{cases}$$

A direct computation yields the following local cohomology solution :

$$\psi = \begin{bmatrix} 1 \\ x_1^5 & x_2 & y_1 & y_2 \end{bmatrix} + \frac{3}{4} \begin{bmatrix} 1 \\ x_1^2 & x_2^3 & y_1 & y_2 \end{bmatrix} - \frac{5}{3} \begin{bmatrix} 1 \\ x_1^2 & x_2 & y_1^2 & y_2 \end{bmatrix}.$$

(v) $9s + 16 = 0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta = -\frac{16}{9}$ is given by

$$\begin{cases} 28x_1x_2^2 + 9x_1y_1, x_1x_2y_1, x_1y_1^2, x_2y_1^2, 27x_2y_1 + 28x_2^3, \\ 10x_2^2y_1 + 9y_1^2, y_1^3, y_2, P_1, P_2, Q_1, Q_2, Q_3, \end{cases}$$

where

$$\begin{aligned} P_1 &= 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 23, \quad P_2 = 2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2}, \\ Q_1 &= 3y_1 \frac{\partial}{\partial x_1} - 9x_1^2y_1 \frac{\partial}{\partial y_1} - 23x_1^2, \quad Q_2 = 9y_1 \frac{\partial}{\partial x_2} - 18x_2y_1 \frac{\partial}{\partial y_1} - 46x_2, \\ Q_3 &= 9y_1^2 \frac{\partial}{\partial y_1} - 28x_1^3 - 28x_2^2 - 15y_1. \end{aligned}$$

Let

$$\begin{aligned} \psi_1 &= \begin{bmatrix} 1 \\ x_1^7 & x_2 & y_1 & y_2 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 1 \\ x_1^4 & x_2^3 & y_1 & y_2 \end{bmatrix}, \quad \psi_3 = \begin{bmatrix} 1 \\ x_1^4 & x_2 & y_1^2 & y_2 \end{bmatrix}, \\ \psi_4 &= \begin{bmatrix} 1 \\ x_1 & x_2^5 & y_1 & y_2 \end{bmatrix}, \quad \psi_5 = \begin{bmatrix} 1 \\ x_1 & x_2^3 & y_1^2 & y_2 \end{bmatrix}, \quad \psi_6 = \begin{bmatrix} 1 \\ x_1 & x_2 & y_1^3 & y_2 \end{bmatrix}. \end{aligned}$$

Then,

$$\psi = \psi_1 + \frac{3}{8}\psi_2 - \frac{7}{6}\psi_3 + \frac{27}{16}\psi_4 - \frac{7}{4}\psi_5 + \frac{35}{18}\psi_6$$

is a local cohomology solution.

(vi) $9s + 17 = 0$. Then, a Gröbner basis of the holonomic ideal associated with the root $\beta = -\frac{17}{9}$ is given by

$$\begin{cases} 32x_1^2x_2^2 + 9x_1^2y_1, x_1^2x_2y_1, x_1^2y_1^2, x_2y_1^2, 27x_2y_1 + 32x_2^3, \\ 14x_2^2y_1 + 9y_1^2, y_1^3, y_2, P_1, P_2, Q_1, Q_2, Q_3, \end{cases}$$

where

$$P_1 = 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 25, \quad P_2 = 2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2},$$

$$Q_1 = 3y_1 \frac{\partial}{\partial x_1} - 9x_1^2 y_1 \frac{\partial}{\partial y_1} - 25x_1^2, \quad Q_2 = 9y_1 \frac{\partial}{\partial x_2} - 18x_2 y_1 \frac{\partial}{\partial y_1} - 50x_2, \\ Q_3 = 9y_1^2 \frac{\partial}{\partial y_1} - 32x_1^3 - 32x_2^2 - 21y_1.$$

Let

$$\psi_1 = \begin{bmatrix} 1 \\ x_1^8 & x_2 & y_1 & y_2 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 1 \\ x_1^5 & x_2^3 & y_1 & y_2 \end{bmatrix}, \quad \psi_3 = \begin{bmatrix} 1 \\ x_1^5 & x_2 & y_1^2 & y_2 \end{bmatrix}, \\ \psi_4 = \begin{bmatrix} 1 \\ x_1^2 & x_2^5 & y_1 & y_2 \end{bmatrix}, \quad \psi_5 = \begin{bmatrix} 1 \\ x_1^2 & x_2^3 & y_1^2 & y_2 \end{bmatrix}, \quad \psi_6 = \begin{bmatrix} 1 \\ x_1^2 & x_2 & y_1^3 & y_2 \end{bmatrix}.$$

Then,

$$\psi = \psi_1 + \frac{3}{10}\psi_2 - \frac{16}{15}\psi_3 + \frac{27}{40}\psi_4 - \frac{4}{5}\psi_5 + \frac{56}{45}\psi_6$$

is a local cohomology solution.

(vii) $s + 1 = 0$. Gröbner basis of the holonomic ideal associated with the root $\beta = -1$ is $\{y_1, y_2, 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 3, 2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2}\}$

$$\text{Let } \psi = \begin{bmatrix} 1 \\ x_1^i & x_2^j & y_1 & y_2 \end{bmatrix}. \text{ Then,}$$

$$(2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 3)\psi = (-2i - 3j + 3)\psi.$$

Therefore there is no non-trivial local cohomology solution supported at the origin Σ_0 .

(viii) $2s + 3 = 0$. Recall that

$$\left\{ \begin{array}{l} x_1^3 + x_2^2 + y_1, \quad y_1^2, \quad y_2, \quad 2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 18, \\ 2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2}, \quad y_1 \frac{\partial}{\partial x_1} - 3x_1^2 y_1 \frac{\partial}{\partial y_1} - 6x_1^2, \quad y_1 \frac{\partial}{\partial x_2} - 2x_2 y_1 \frac{\partial}{\partial y_1} - 4x_2 \end{array} \right.$$

is a Gröbner basis of the holonomic ideal associated with the root $\beta = -\frac{3}{2}$

$$\text{Let } \psi = \begin{bmatrix} 1 \\ x_1^i & x_2^j & y_1^k & y_2 \end{bmatrix}. \text{ Then,}$$

$$(2x_1 \frac{\partial}{\partial x_1} + 3x_2 \frac{\partial}{\partial x_2} + 6y_1 \frac{\partial}{\partial y_1} + 18)\psi = (-2i - 3j - 6k + 18)\psi.$$

Therefore, we have $(i, j, k) = (3, 2, 1)$.

Since

$$(2x_2 \frac{\partial}{\partial x_1} - 3x_1^2 \frac{\partial}{\partial x_2}) \begin{bmatrix} 1 \\ x_1^3 & x_2^2 & y_1 & y_2 \end{bmatrix} \neq 0.$$

Therefore there is no non-trivial local cohomology solution supported at the origin Σ_0 .

We have verified in this section that

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{D_X}(M_{\beta}, \mathcal{H}_{[\Sigma_2]}^4(\mathcal{O}_X)) = 1, \quad \text{for } \beta = -\frac{10}{9}, -\frac{11}{9}, -\frac{13}{9}, -\frac{14}{9}, -\frac{16}{9}, -\frac{17}{9}$$

and

$$\mathrm{Hom}_{D_X}(M_{\beta}, \mathcal{H}_{[\Sigma_2]}^4(\mathcal{O}_X)) = \{0\}, \quad \text{for } \beta = -1, -\frac{3}{2}.$$

References

- [1] R. Bahloul and T. Oaku, *Local Bernstein-Sato ideals ; algorithms and examples*, Journal of Symbolic Computation **45** (2010), 46–59.
- [2] J. Briançon and P. Maisonobe, *Remarques sur l'idéal de Bernstein associé à des polynômes*, prépublication Univ. Nice-Sophia Antipolis, n° 650, Mai, 2002.
- [3] P. Deligne, *Le formalisme des cycles évanescents*, SGA 7, II, exposés 13, Lecture Notes in Math. **340** (1973), 82–115.
- [4] M. Kashiwara, *b-functions and singularity of a hypersurface*, (in Japanese, Noted by T. Miwa), RIMS Kôkyûroku **225** (1975), 16–53.
- [5] M. Kashiwara, *On the maximally overdetermined system of linear differential equations, I*, Publ. Res. Inst. Math. Sci. **10** (1974-1975) 563–579.
- [6] M. Kashiwara, *B-functions and holonomic systems*, Invent. Math. **38** (1976) 33–53.
- [7] M. Kashiwara, *On the holonomic systems of linear differential equations, II*, Invent. Math. **49** (1978) 121–135.
- [8] K. Nabeshima, K. Ohara and S. Tajima, *Comprehensive Gröbner systems in rings of differential operators, holonomic D-modules and b-functions*, ISSAC2016, Proceedings of the International Symposium on Symbolic and Algebraic Computation, (2016), 349–356.
- [9] L. Narváez-Macarro, *Cycles évanescents et faisceaux pervers : cas des courbes planes irréductibles*, Compositio Math. **65** (1988), 321–347.
- [10] K. Nishiyama and M. Noro, *Stratification associated with local b-functions*, J. Symbolic Computation **45** (2010) 462–480.
- [11] M. Noro and T. Takeshima, *Risa/Asir- A computer algebra system*, ISSAC1992, Proceedings of the International Symposium on Symbolic and Algebraic Computation (1992), 387–396.
- [12] T. Oaku, *Algorithms for the b-function and D-modules associated with a polynomial*, J. Pure and Applied Algebra **117** & **118** (1997) 495–518.
- [13] T. Oaku, *Algorithms for b-functions, restrictions, and algebraic local cohomology groups of D-modules*, Adv. Appl. Math. **19** (1997), 61–105.
- [14] S. Tajima, *On b-functions and algebraic local cohomology classes attached to hypersurfaces with line singularities*, RIMS Kôkyûroku Bessatsu **52** (2014), 175–191.
- [15] S. Tajima and Y. Umeta, *Computing structure of holonomic D-modules associated with a simple line singularity*, RIMS Kôkyûroku Bessatsu. **57** (2016), 125–140.
- [16] T. Yano, *On the theory of b-functions*, Publ. Res. Inst. Math. Sci. **14** (1978), 111–202.
- [17] A. Zaharia, *On the bifurcation variety of some non-isolated singularities*, Math. Scand. **77** (1995), 45–59.