

Geometric Version of the Grothendieck Conjecture for Universal Curves over Hurwitz Stacks: a research announcement

By

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Abstract

Hurwitz stacks are algebraic stacks that parametrize simple coverings. In this paper, we introduce a certain geometric version of the Grothendieck Conjecture for universal curves over Hurwitz stacks. This result generalizes a similar result obtained by Hoshi and Mochizuki in the case of universal curves over moduli stacks of pointed smooth curves. After introducing these results, we give a sketch of the proof of the above version of the Grothendieck Conjecture in the hyperelliptic case.

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§ 1. Introduction

Throughout this paper, for a connected Noetherian algebraic stack X (respectively, fine saturated log algebraic stack $X^{\log}$), we shall write $\pi_1(X)$ (respectively, $\pi_1^{\log}(X^{\log})$)

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for the étale fundamental group of X (respectively, the log étale fundamental group of $X^{\log}$), relative to a suitable choice of a basepoint.

Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$; Σ a nonempty set of prime numbers; k an algebraically closed field of characteristic zero.

Write $\mathcal{M}_{g,r}$ (respectively, $\overline{\mathcal{M}}_{g,r}$) for the moduli stack of r -pointed smooth (respectively, stable) curves of genus g over $\text{Spec } k$ whose r marked points are equipped with an ordering; $\overline{\mathcal{M}}_{g,r}^{\log}$ the log stack obtained by equipping $\overline{\mathcal{M}}_{g,r}$ with the log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$; $\mathcal{C}_{g,r} \rightarrow \mathcal{M}_{g,r}$ for the universal curve over $\mathcal{M}_{g,r}$; $\Pi_{\mathcal{M}_{g,r}} \stackrel{\text{def}}{=} \pi_1(\mathcal{M}_{g,r})$ for the étale fundamental group of the moduli stack $\mathcal{M}_{g,r}$; $\Pi_{g,r}$ for the maximal pro- Σ quotient of the kernel $N_{g,r}$ of the natural surjection $\pi_1(\mathcal{C}_{g,r}) \twoheadrightarrow \pi_1(\mathcal{M}_{g,r}) = \Pi_{\mathcal{M}_{g,r}}$ for suitable choices of base points; $\Pi_{\mathcal{C}_{g,r}}$ for the quotient of the étale fundamental group $\pi_1(\mathcal{C}_{g,r})$ of $\mathcal{C}_{g,r}$ by the kernel of the natural surjection $N_{g,r} \twoheadrightarrow \Pi_{g,r}$. By the various definitions involved, we have an exact sequence of profinite groups

$$1 \longrightarrow \Pi_{g,r} \longrightarrow \Pi_{\mathcal{C}_{g,r}} \longrightarrow \Pi_{\mathcal{M}_{g,r}} \longrightarrow 1.$$

Note that $\Pi_{g,r}$ is a pro- Σ surface group [cf. [MT], Definition 1.2], and that this exact sequence determines an outer representation

$$\rho_{g,r} : \Pi_{\mathcal{M}_{g,r}} \longrightarrow \text{Out}(\Pi_{g,r}).$$

Let G be a topological group and $H \subseteq G$ a closed subgroup of G . Then we shall denote by $Z_G(H)$ the *centralizer* of $H \subseteq G$, i.e.,

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}.$$

Since $\Pi_{g,r}$ is center-free, one may prove that the restriction is an isomorphism

$$\text{Aut}_{\Pi_{\mathcal{M}_{g,r}}}(\Pi_{\mathcal{C}_{g,r}})/\text{Inn}(\Pi_{g,r}) \xrightarrow{\sim} Z_{\text{Out}(\Pi_{g,r})}(\text{Im}(\rho_{g,r})).$$

Write $\text{Out}^C(\Pi_{g,r})$ for the group of outer automorphisms of $\Pi_{g,r}$ which induce bijections on the set of cuspidal inertia subgroups of $\Pi_{g,r}$.

In [CbTpI], Hoshi and Mochizuki proved the following “geometric version of the Grothendieck Conjecture for universal curves over moduli stacks of pointed smooth curves” [cf. Remark 3].

Theorem 1.1 ([CbTpI], Theorem D). *Let $H \subseteq \Pi_{\mathcal{M}_{g,r}}$ be an open subgroup of $\Pi_{\mathcal{M}_{g,r}}$. Suppose that one of the following two conditions is satisfied:*

1. $2g - 2 + r > 1$, i.e., $(g, r) \notin \{(0, 3), (1, 1)\}$.
2. $(g, r) = (1, 1)$, $2 \in \Sigma$, and $H = \Pi_{\mathcal{M}_{g,r}}$.

Then the composite of natural homomorphisms

$$\begin{aligned} \mathrm{Aut}_{\mathcal{M}_{g,r}}(\mathcal{C}_{g,r}) &\longrightarrow \mathrm{Aut}_{\Pi_{\mathcal{M}_{g,r}}}(\Pi_{\mathcal{C}_{g,r}})/\mathrm{Inn}(\Pi_{g,r}) \\ &\xrightarrow{\sim} Z_{\mathrm{Out}(\Pi_{g,r})}(\mathrm{Im}(\rho_{g,r})) \subseteq Z_{\mathrm{Out}(\Pi_{g,r})}(\rho_{g,r}(H)) \end{aligned}$$

determines an **isomorphism**

$$\mathrm{Aut}_{\mathcal{M}_{g,r}}(\mathcal{C}_{g,r}) \xrightarrow{\sim} Z_{\mathrm{Out}^C(\Pi_{g,r})}(\rho_{g,r}(H)).$$

Remark 1. Since $\mathcal{M}_{0,3} = \mathrm{Spec} k$, $\Pi_{\mathcal{M}_{0,3}}$ is a trivial group. On the other hand, one may prove that $Z_{\mathrm{Out}^C(\Pi_{0,3})}(\mathrm{Im}(\rho_{0,3}))$ is an infinite group. Thus, by the fact that $\mathrm{Aut}_{\mathcal{M}_{0,3}}(\mathcal{C}_{0,3})$ is a finite group, we conclude that Theorem 1.1 does not hold in the case where $(g, r) = (0, 3)$.

Remark 2. In the case where $(g, r) = (1, 1)$, it is not clear to the author whether the assumptions $2 \in \Sigma$, and $H = \Pi_{\mathcal{M}_{g,r}}$ can be removed.

Remark 3. In this remark, we compare the original Grothendieck Conjecture proved by Nakamura, Tamagawa, and Mochizuki with the above Theorem 1.1.

Here, we recall (the automorphism version of) the original Grothendieck Conjecture briefly. Let X be a hyperbolic curve over $\mathbb{Q}$. Write $\Pi_X \stackrel{\mathrm{def}}{=} \pi_1(X)$ for the étale fundamental group of X ; $\Delta_X \stackrel{\mathrm{def}}{=} \pi_1(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})$ for the étale fundamental group of $X \times_{\mathbb{Q}} \overline{\mathbb{Q}}$. Then we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_{\mathbb{Q}} \longrightarrow 1,$$

where $G_{\mathbb{Q}}$ is the absolute Galois group of $\mathbb{Q}$. Note that this exact sequence determines an outer representation

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{Out}(\Delta_X).$$

In this situation, by a special case of [LocAn], Theorem A, the composite of natural homomorphisms

$$\mathrm{Aut}_{\mathbb{Q}}(X) \longrightarrow \mathrm{Aut}_{G_{\mathbb{Q}}}(\Pi_X)/\mathrm{Inn}(\Delta_X) \xrightarrow{\sim} Z_{\mathrm{Out}(\Delta_X)}(\mathrm{Im}(\rho))$$

is an isomorphism.

Thus, we may observe that both (the automorphism version of) the original Grothendieck Conjecture and Theorem 1.1 are the calculation of the centralizer of the monodromy outer representation arising from the respective situations.

One may pose the following question that may be regarded as a certain generalization of Theorem 1.1 [cf. [CbTpI], Remark 6.14.1]:

Question: Let $T \rightarrow \mathcal{M}_{g,r}$ be a (1-)morphism of algebraic stacks. How does the Grothendieck Conjecture for the pullback $T \times_{\mathcal{M}_{g,r}} \mathcal{C}_{g,r} \xrightarrow{\text{pr}_1} T$ hold?

We note that the image of the composite $\rho_T : \pi_1(T) \rightarrow \Pi_{\mathcal{M}_{g,r}} \xrightarrow{\rho_{g,r}} \text{Out}(\Pi_{g,r})$ can be small when the image of $T \rightarrow \mathcal{M}_{g,r}$ is small. Thus, in such a case, it may be difficult to calculate the centralizer $Z_{\text{Out}^c(\Pi_{g,r})}(\text{Im}(\rho_T))$.

In the present paper, in the form of answering this question, we introduce a version of Theorem 1.1 for universal curves over Hurwitz stacks [cf. [CbTpI], Remark 6.14.1] and give a sketch of the proof of the hyperelliptic case. The proof of the general case [i.e., degree ≥ 3 case] is much more complicated than the hyperelliptic case. However, the strategy of the proof is similar. Thus, we do not make a remark about the proof of the general case in this paper.

This paper is organized as follows. In §2, we introduce the notion of “profiled log Hurwitz stacks” (i.e., log algebraic stacks that parametrize Hurwitz coverings for which the marked points are equipped with a certain ordering determined by combinatorial data which we refer to as a “profile”) and state fundamental geometric properties of these stacks without proofs. In §3, we introduce a version of Theorem 1.1 for universal curves over profiled Hurwitz stacks as the main theorem. In §4, we give a sketch of the proof of the main theorem in the hyperelliptic case [i.e., degree 2 case] by applying techniques from combinatorial anabelian geometry [i.e., roughly speaking, a research area which discusses the reconstructibility of combinatorial structure arising from the irreducible components, the cusps, and the nodes of pointed stable curves from the various [for example, discrete, étale, tempered] fundamental groups of the pointed stable curves] to various objects that arise from profiled log Hurwitz stacks.

In the rest of this paper, every algebraic stack and every (1-)morphism between the algebraic stacks are assumed to be defined over $\text{Spec } k$.

§ 2. Profiled Hurwitz stacks

In this section, we introduce the notion of profiled log Hurwitz stacks. First, we recall the notion of admissible coverings.

Definition 2.1 ([GCH], §3.4). Let (g, q, r) be a triple of nonnegative integers.

- (i) Let S_q denote the symmetric group on q letters. Note that we have a natural action of S_q on $\overline{\mathcal{M}}_{g,q+r}^{\log}$ given by permuting the first q marked points. We shall denote by $\overline{\mathcal{M}}_{g,[q]+r}^{\log}$ the (log) stack theoretic quotient of $\overline{\mathcal{M}}_{g,q+r}^{\log}$ by S_q . If $r = 0$, we simply write $\overline{\mathcal{M}}_{g,[q]}^{\log}$. Note that the universal stable log curve $\overline{\mathcal{C}}_{g,q+r}^{\log} \rightarrow \overline{\mathcal{M}}_{g,q+r}^{\log}$ descends to a stable log curve $\overline{\mathcal{C}}_{g,[q]+r}^{\log} \rightarrow \overline{\mathcal{M}}_{g,[q]+r}^{\log}$.

- (ii) Let $S^{\log}$ be a fine log scheme. A (1-)morphism between log stacks $S^{\log} \rightarrow \overline{\mathcal{M}}_{g,[q]+r}^{\log}$ will be referred to as the *data for a $([q] + r)$ -pointed stable log curve of genus g* . Let $C^{\log} \rightarrow S^{\log}$ be the pull-back of the universal stable log curve $\overline{\mathcal{C}}_{g,[q]+r}^{\log} \rightarrow \overline{\mathcal{M}}_{g,[q]+r}^{\log}$ via such a (1-)morphism. By abuse of terminology, we shall refer to such a stable log curve $C^{\log} \rightarrow S^{\log}$ as a *$([q] + r)$ -pointed stable log curve of genus g* . If we forget the log structures of such a stable log curve, the resulting $(f : C \rightarrow S; \mu_f \subseteq C)$ (where $\mu_f \subseteq C$ is the divisor of marked points) will be referred to as a *$([q] + r)$ -pointed stable curve of genus g* , or, when $r = 0$, simply as a *$[q]$ -pointed stable curve of genus g* . When the integers q and g are left *unspecified*, a $[q]$ -pointed stable curve of genus g will be referred to as a *symmetrically pointed stable curve [over S]*.

Definition 2.2 ([GCH], §3.9). Let d be a positive integer; S a scheme; $(f : C \rightarrow S; \mu_f \subseteq C)$ and $(h : D \rightarrow S; \mu_h \subseteq D)$ symmetrically pointed stable curves over S . A finite morphism $\pi : C \rightarrow D$ over S will be called an *admissible covering [of degree d]* if it satisfies the following conditions:

- Each fiber of $h : D \rightarrow S$ admits a dense open subset over which π is finite flat of degree d .
- We have inclusions of effective relative (with respect to the morphism f) divisors $\mu_f \subseteq \pi^{-1}(\mu_h) \subseteq d \cdot \mu_f$ on C .
- The morphism f is smooth at $c \in C$ if and only if the morphism h is smooth at $\pi(c)$.
- The morphism π is étale, except
 - over μ_h , where it is tamely ramified;
 - at nodes of the geometric fibers over S : if $\bar{s}$ is a geometric point of S , λ is a node of $C_{\bar{s}}$, and $\nu = \pi(\lambda)$, then there exist some natural number e satisfying $e \in (\mathcal{O}_{S,\bar{s}}^{\text{sh}})^{\times}$, $a \in \mathfrak{m}_{S,\bar{s}}^{\text{sh}}$, $x, y \in \mathfrak{m}_{C,\lambda}^{\text{sh}}$, and $u, v \in \mathfrak{m}_{D,\nu}^{\text{sh}}$ such that x, y (respectively, u, v) generate $\mathfrak{m}_{f^{-1}(\bar{s}),\lambda}^{\text{sh}}$ (respectively, $\mathfrak{m}_{h^{-1}(\bar{s}),\nu}^{\text{sh}}$), and $xy = a$, $uv = a^e$, $u = x^e$, $v = y^e$.

Here, “ $\mathfrak{m}^{\text{sh}}$ ” denotes the maximal ideal of the strict henselization “ $\mathcal{O}^{\text{sh}}$ ” at the specified geometric point of the local ring under consideration.

An admissible covering $\pi : C \rightarrow D$ over S will be called a *simple admissible covering* if the discriminant divisor of π is *étale* over S in some neighborhood of μ_h .

Next, we define the notion of profiled simple admissible coverings.

Definition 2.3 ([T], Definition 1.7). Let (g, d, r) be a triple of nonnegative integers such that $2g - 2 + 2d + r \geq 3$ and $d \geq 2$; $\pi : C \rightarrow D$ a simple admissible covering of degree d from a $([(d-1)(2g-2+2d)] + dr)$ -pointed stable curve $(f : C \rightarrow S; \mu_f \subseteq C)$ of genus g to a $([2g-2+2d] + r)$ -pointed stable curve $(h : D \rightarrow S; \mu_h \subseteq D)$ of genus 0 [cf. Definition 2.1, Definition 2.2]. Then the morphism $\pi : C \rightarrow D$, equipped with these partial orderings on the marked points, will be called an *r -profiled simple admissible covering*, if these partial orderings on the marked points satisfy the following conditions:

- The divisor μ_h consists, étale locally on S , of $2g - 2 + 2d$ unordered sections over which π ramifies and r ordered sections $\sigma_1, \dots, \sigma_r$ over which π is unramified.
- The divisor μ_f consists, étale locally on S , of $(d-1)(2g-2+2d)$ unordered sections over the sections of μ_h over which π ramifies and dr ordered sections over the sections $\sigma_1, \dots, \sigma_r$ such that the sections over σ_m ($1 \leq m \leq r$) are indexed by the natural numbers between $(m-1)d+1$ and md .

When C and D are *smooth*, we shall, on occasion, omit the word “*admissible*” from this terminology “ *r -profiled simple admissible covering*”.

Definition 2.4 ([T], Definition 1.8). Let (g, d, r) be a triple of nonnegative integers such that $2g - 2 + 2d + r \geq 3$ and $d \geq 2$; k an algebraically closed field of characteristic zero. For any scheme S over $\text{Spec } k$, write $\overline{\mathcal{H}}_{g,d,r}(S)$ for the following groupoid [i.e., a category in which every morphism is invertible]:

- Objects: an object is an *r -profiled simple admissible covering* $\pi : C \rightarrow D$ of degree d from a $([(d-1)(2g-2+2d)] + dr)$ -pointed stable curve $(f : C \rightarrow S; \mu_f \subseteq C)$ of genus g to a $([2g-2+2d] + r)$ -pointed stable curve $(h : D \rightarrow S; \mu_h \subseteq D)$ of genus 0.
- Morphisms: a morphism between two objects $\pi : C \rightarrow D$ and $\pi' : C' \rightarrow D'$ is a pair of isomorphisms $\alpha : C \xrightarrow{\sim} C'$ and $\beta : D \xrightarrow{\sim} D'$ that are compatible with the collections of data related to the divisors of marked points such that $\beta \circ \pi = \pi' \circ \alpha$.

We shall denote by $\mathcal{H}_{g,d,r} \subseteq \overline{\mathcal{H}}_{g,d,r}$ the [necessarily open] substack that parametrizes the profiled simple admissible covering $\pi : C \rightarrow D$ such that the curves C and D are smooth. We shall refer to $\mathcal{H}_{g,d,r}$ as the *r -profiled Hurwitz stack of type (g, d)* .

Remark 4. $\mathcal{H}_{g,d,r}$ and $\overline{\mathcal{H}}_{g,d,r}$ are algebraic stacks over $\text{Spec } k$.

Next, we state fundamental geometric properties of these stacks without proofs.

Proposition 2.5 ([T], Proposition 1.10). *Let (g, d, r) be a triple of nonnegative integers such that $2g - 2 + 2d + r \geq 3$, $d \geq 2$; k an algebraically closed field of characteristic zero.*

- (i) *The normalization $\tilde{\mathcal{H}}_{g,d,r}$ of $\overline{\mathcal{H}}_{g,d,r}$ is proper and smooth over $\mathrm{Spec} k$. Moreover, $\mathcal{H}_{g,d,r}$ may be regarded as an open substack of $\tilde{\mathcal{H}}_{g,d,r}$, whose complement [in $\tilde{\mathcal{H}}_{g,d,r}$], equipped with the reduced induced stack structure, is a divisor with normal crossings.*
- (ii) *The divisor with normal crossings of (i) determines a log structure on $\tilde{\mathcal{H}}_{g,d,r}$. Moreover, the resulting log stack $\tilde{\mathcal{H}}_{g,d,r}^{\log}$ is log smooth over $\mathrm{Spec} k$, hence, in particular, log regular.*
- (iii) *There exists a natural (1-)morphism*

$$\overline{\phi}_{g,d,r}^{\log} : \overline{\mathcal{H}}_{g,d,r+1}^{\log} \longrightarrow \overline{\mathcal{H}}_{g,d,r}^{\log}$$

obtained by forgetting the final d sections (respectively, final section) of the domain curve (respectively, codomain curve) of the covering. Now suppose further that $2g - 2 + dr \geq 1$. Then there exists a natural (1-)morphism

$$\overline{\psi}_{g,d,r}^{\log} : \overline{\mathcal{H}}_{g,d,r}^{\log} \longrightarrow \overline{\mathcal{M}}_{g,dr}^{\log},$$

obtained by considering the domain curve of the covering, equipped with its dr ordered marked points. Moreover, we have a (1-)commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{H}}_{g,d,r+1}^{\log} & \xrightarrow{\overline{\psi}_{g,d,r+1}^{\log}} & \overline{\mathcal{M}}_{g,d(r+1)}^{\log} \\ \overline{\phi}_{g,d,r}^{\log} \downarrow & & \downarrow \\ \overline{\mathcal{H}}_{g,d,r}^{\log} & \xrightarrow{\overline{\psi}_{g,d,r}^{\log}} & \overline{\mathcal{M}}_{g,dr}^{\log} \end{array}$$

where the right-hand vertical arrow is the (1-)morphism obtained by forgetting the final d sections.

- (iv) *The algebraic stacks $\mathcal{H}_{g,d,r}$, $\overline{\mathcal{H}}_{g,d,r}^{\log}$, and $\tilde{\mathcal{H}}_{g,d,r}^{\log}$ are irreducible.*
- (v) *The (1-)morphism $\tilde{\phi}_{g,d,r}^{\log} : \tilde{\mathcal{H}}_{g,d,r+1}^{\log} \rightarrow \tilde{\mathcal{H}}_{g,d,r}^{\log}$ induced by the (1-)morphism $\overline{\phi}_{g,d,r}^{\log}$ of (iii) is a stable log curve, hence, in particular, has geometrically reduced, geometrically connected fibers.*

Definition 2.6 ([T], Definition 1.13). Let (g, d, r) be a triple of nonnegative integers such that $2g - 2 + dr \geq 1$, $d \geq 2$ [Note that these conditions imply, as is easily verified, that $2g - 2 + 2d + r - 2 \geq 1$.]; k an algebraically closed field of characteristic zero. We shall denote by

$$u_{g,d,r} : \mathcal{C}_{g,d,r} \rightarrow \mathcal{H}_{g,d,r} \quad (\text{respectively, } \tilde{u}_{g,d,r}^{\log} : \tilde{\mathcal{C}}_{g,d,r}^{\log} \rightarrow \tilde{\mathcal{H}}_{g,d,r}^{\log})$$

the pull-back of the universal curve $\mathcal{C}_{g,d,r} \rightarrow \mathcal{M}_{g,d,r}$ (respectively, $\tilde{\mathcal{C}}_{g,d,r}^{\log} \rightarrow \tilde{\mathcal{M}}_{g,d,r}^{\log}$) via $\psi_{g,d,r} : \mathcal{H}_{g,d,r} \rightarrow \mathcal{M}_{g,d,r}$ (respectively, $\tilde{\psi}_{g,d,r}^{\log} : \tilde{\mathcal{H}}_{g,d,r}^{\log} \rightarrow \tilde{\mathcal{M}}_{g,d,r}^{\log}$), where we write $\psi_{g,d,r}$ and $\tilde{\psi}_{g,d,r}^{\log}$ for the (1-)morphisms induced by $\bar{\psi}_{g,d,r}^{\log}$ [cf. Proposition 2.5, (iii)]. We shall refer to $\mathcal{C}_{g,d,r}$ (respectively, $\tilde{\mathcal{C}}_{g,d,r}^{\log}$) as the *universal curve* over $\mathcal{H}_{g,d,r}$ (respectively, $\tilde{\mathcal{H}}_{g,d,r}^{\log}$).

Proposition 2.7 ([T], Proposition 1.14). Let (g, d, r) be a triple of nonnegative integers such that $2g - 2 + 2d + r \geq 3$, $d \geq 2$; $\bar{s}$ (respectively, $\bar{s}^{\log}$) be a geometric point (respectively, a strict log geometric point) of $\mathcal{H}_{g,d,r}$ (respectively, $\tilde{\mathcal{H}}_{g,d,r}^{\log}$). For suitable choices of basepoints, write

$$\begin{aligned} \Pi_{\mathcal{C}_{\bar{s}}} &\stackrel{\text{def}}{=} \pi_1(\mathcal{C}_{g,d,r} \times_{\mathcal{H}_{g,d,r}} \bar{s}) \\ &(\text{respectively, } \Pi_{\mathcal{C}_{\bar{s}^{\log}}}^{\log} \stackrel{\text{def}}{=} \pi_1^{\log}(\tilde{\mathcal{C}}_{g,d,r}^{\log} \times_{\tilde{\mathcal{H}}_{g,d,r}^{\log}} \bar{s}^{\log})); \\ \Pi_{\mathcal{H}_{g,d,r}} &\stackrel{\text{def}}{=} \pi_1(\mathcal{H}_{g,d,r}) \quad (\text{respectively, } \Pi_{\tilde{\mathcal{H}}_{g,d,r}^{\log}}^{\log} \stackrel{\text{def}}{=} \pi_1^{\log}(\tilde{\mathcal{H}}_{g,d,r}^{\log})); \\ \Pi_{\mathcal{C}_{g,d,r}} &\stackrel{\text{def}}{=} \pi_1(\mathcal{C}_{g,d,r}) \quad (\text{respectively, } \Pi_{\tilde{\mathcal{C}}_{g,d,r}^{\log}}^{\log} \stackrel{\text{def}}{=} \pi_1^{\log}(\tilde{\mathcal{C}}_{g,d,r}^{\log})) \\ \Pi_{\mathcal{H}_{\bar{s}}} &\stackrel{\text{def}}{=} \pi_1(\mathcal{H}_{g,d,r+1} \times_{\mathcal{H}_{g,d,r}} \bar{s}) \\ &(\text{respectively, } \Pi_{\mathcal{H}_{\bar{s}^{\log}}}^{\log} \stackrel{\text{def}}{=} \pi_1^{\log}(\tilde{\mathcal{H}}_{g,d,r+1}^{\log} \times_{\tilde{\mathcal{H}}_{g,d,r}^{\log}} \bar{s}^{\log})) \end{aligned}$$

[cf. Definition 2.6].

- (i) Suppose further that $2g - 2 + dr \geq 1$. Then the universal curve $\tilde{u}_{g,d,r}^{\log} : \tilde{\mathcal{C}}_{g,d,r}^{\log} \rightarrow \tilde{\mathcal{H}}_{g,d,r}^{\log}$ is a proper, log smooth (1-)morphism between log regular log stacks [cf. Definition 2.6].
- (ii) Suppose further that $2g - 2 + dr \geq 1$. Then we have a natural commutative diagram of profinite groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Pi_{\mathcal{C}_{\bar{s}}} & \longrightarrow & \Pi_{\mathcal{C}_{g,d,r}} & \longrightarrow & \Pi_{\mathcal{H}_{g,d,r}} \longrightarrow 1 \\ & & \wr \downarrow & & \wr \downarrow & & \wr \downarrow \\ 1 & \longrightarrow & \Pi_{\mathcal{C}_{\bar{s}^{\log}}}^{\log} & \longrightarrow & \Pi_{\tilde{\mathcal{C}}_{g,d,r}^{\log}}^{\log} & \longrightarrow & \Pi_{\tilde{\mathcal{H}}_{g,d,r}^{\log}}^{\log} \longrightarrow 1, \end{array}$$

where the vertical arrows are isomorphisms, and the horizontal sequences are exact.

(iii) We have a natural commutative diagram of profinite groups

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \Pi_{\mathcal{H}_{\bar{s}}} & \longrightarrow & \Pi_{\mathcal{H}_{g,d,r+1}} & \longrightarrow & \Pi_{\mathcal{H}_{g,d,r}} \longrightarrow 1 \\
 & & \wr \downarrow & & \wr \downarrow & & \wr \downarrow \\
 1 & \longrightarrow & \Pi_{\mathcal{H}_{\bar{s}}^{\log}} & \longrightarrow & \Pi_{\tilde{\mathcal{H}}_{g,d,r+1}^{\log}} & \longrightarrow & \Pi_{\tilde{\mathcal{H}}_{g,d,r}^{\log}} \longrightarrow 1,
 \end{array}$$

where the vertical arrows are isomorphisms, and the horizontal sequences are exact.

§ 3. Main theorem

In this section, we state the following main theorem.

Theorem 3.1 ([T], Theorem A). *Let Σ be a nonempty set of prime numbers; k an algebraically closed field of characteristic zero; (g, d, r) a triple of nonnegative integers such that*

$$\begin{aligned}
 d \geq 2 \wedge (g, r) \notin \{(0, 0), (1, 0)\} \wedge (g, d, r) \notin \{(0, 2, 1), (0, 3, 1)\} \\
 (\Rightarrow 2g - 2 + dr > 1 \wedge 2g + 2d + r - 5 \geq 1).
 \end{aligned}$$

Write $\mathcal{H}_{g,d,r}$ for the r -profiled **Hurwitz stack** of type (g, d) over $\text{Spec } k$ [cf. Definition 2.4], where $\dim \mathcal{H}_{g,d,r} = 2g - 2 + 2d + r - 3 = 2g + 2d + r - 5 \geq 1$; $\mathcal{C}_{g,d,r} \rightarrow \mathcal{H}_{g,d,r}$ for the restriction of the **universal curve** over $\mathcal{M}_{g,dr}$ to $\mathcal{H}_{g,d,r}$ via the natural (1-)morphism $\mathcal{H}_{g,d,r} \rightarrow \mathcal{M}_{g,dr}$ [cf. Proposition 2.5, (iii)]; $\Pi_{\mathcal{H}_{g,d,r}} \stackrel{\text{def}}{=} \pi_1(\mathcal{H}_{g,d,r})$ for the étale fundamental group of the profiled Hurwitz stack $\mathcal{H}_{g,d,r}$; $\Pi_{g,d,r}$ for the maximal pro- Σ quotient of the kernel $N_{g,d,r}$ of the natural surjection $\pi_1(\mathcal{C}_{g,d,r}) \twoheadrightarrow \pi_1(\mathcal{H}_{g,d,r}) = \Pi_{\mathcal{H}_{g,d,r}}$ for suitable choices of base points; $\Pi_{\mathcal{C}_{g,d,r}}$ for the quotient of the étale fundamental group $\pi_1(\mathcal{C}_{g,d,r})$ of $\mathcal{C}_{g,d,r}$ by the kernel of the natural surjection $N_{g,d,r} \twoheadrightarrow \Pi_{g,d,r}$. Thus, we have a natural sequence of profinite groups

$$1 \longrightarrow \Pi_{g,d,r} \longrightarrow \Pi_{\mathcal{C}_{g,d,r}} \longrightarrow \Pi_{\mathcal{H}_{g,d,r}} \longrightarrow 1$$

which determines an outer representation

$$\rho_{g,d,r} : \Pi_{\mathcal{H}_{g,d,r}} \longrightarrow \text{Out}(\Pi_{g,d,r})$$

Then the following hold:

- (i) The profinite group $\Pi_{g,d,r}$ is the maximal pro- Σ quotient of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic zero [i.e., a pro- Σ surface group — cf. [MT], Definition 1.2] and is naturally isomorphic to the profinite group “ $\Pi_{g,r}$ ” of [CbTpI], Theorem 1.1, in the case where one takes the “ (g, r) ” of loc. cit. to be (g, dr) [in the notation of the present discussion].

(ii) Write $\text{Out}^C(\Pi_{g,d,r})$ for the group of outer automorphisms of $\Pi_{g,d,r}$ which induce bijections on the set of cuspidal inertia subgroups of $\Pi_{g,d,r}$. Let $H \subseteq \Pi_{\mathcal{H}_{g,d,r}}$ be an open subgroup of $\Pi_{\mathcal{H}_{g,d,r}}$. Then the composite of natural homomorphisms

$$\begin{aligned} \text{Aut}_{\mathcal{H}_{g,d,r}}(\mathcal{C}_{g,d,r}) &\longrightarrow \text{Aut}_{\Pi_{\mathcal{H}_{g,d,r}}}(\Pi_{\mathcal{C}_{g,d,r}})/\text{Inn}(\Pi_{g,d,r}) \\ &\xrightarrow{\sim} Z_{\text{Out}(\Pi_{g,d,r})}(\text{Im}(\rho_{g,d,r})) \subseteq Z_{\text{Out}(\Pi_{g,d,r})}(\rho_{g,d,r}(H)) \end{aligned}$$

determines an **isomorphism**

$$\text{Aut}_{\mathcal{H}_{g,d,r}}(\mathcal{C}_{g,d,r}) \xrightarrow{\sim} Z_{\text{Out}^C(\Pi_{g,d,r})}(\rho_{g,d,r}(H)).$$

Remark 5. Note that the numerical conditions $(g, r) \notin \{(0, 0), (1, 0)\}$ and $(g, d, r) \neq (0, 2, 1)$ imply the “hyperbolicity condition” for (g, dr) , i.e., $2g - 2 + dr \geq 1$. Note also that, by a similar discussion with Remark 1, one may conclude that Theorem 3.1 does not hold in the case where $(g, d, r) = (0, 3, 1)$.

Remark 6. Assertion (i) of Theorem 3.1 follows immediately from Proposition 2.7, (ii), together with the various definitions involved.

Remark 7. The automorphism group $\text{Aut}_{\mathcal{H}_{g,d,r}}(\mathcal{C}_{g,d,r})$ is isomorphic to

$$\begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } (g, d, r) \in \{(0, 2, 2), (0, 4, 1)\}; \\ \mathbb{Z}/2\mathbb{Z} & \text{if } (g, d, r) \in \{(g, 2, r) \mid (g, r) \neq (0, 2)\} \cup \{(2, d, 0)\}; \\ \{1\} & \text{if } (g, d, r) \notin \{(0, 4, 1), (g, 2, r), (2, d, 0)\}. \end{cases}$$

§ 4. Sketch of the proof of the hyperelliptic case

In this section, we give a sketch of the proof of Theorem 3.1, (ii), in the hyperelliptic case by applying the language of semi-graph of anabelioids of PSC-type [i.e., roughly speaking, the dual semi-graph of a pointed stable curve with the Galois categories attached to each of the vertices and the edges of the dual semi-graph] [cf. [SemiAn], §2; [CmbGC], Definition 1.1, (i)] and the technique of combinatorial anabelian geometry. Especially, we apply combinatorial Grothendieck Conjecture [cf. [CmbGC], [NodNon]] and the theory of profinite Dehn multi-twists [cf. [CbTpI], §4]. In the present paper, we shall refer to profinite Dehn multi-twists as profinite Dehn twists. We recall the definition of profinite Dehn twists briefly. Let $\mathcal{G}$ be a semi-graph of anabelioids of pro- Σ PSC-type. Write $\Pi_{\mathcal{G}}$ for the fundamental group [defined naturally in [SemiAn], §2, preceding discussion of Definition 2.2] of $\mathcal{G}$. Note that $\Pi_{\mathcal{G}}$ is naturally isomorphic to the maximal pro- Σ quotient of $\text{Ker}(\pi_1^{\log}(X^{\log}) \rightarrow \pi_1^{\log}((\text{Spec } k)^{\log}))$ induced by the structure morphism of a stable log curve $X^{\log} \rightarrow (\text{Spec } k)^{\log}$. Let α be an element of $\text{Out}(\Pi_{\mathcal{G}})$. When α satisfies the following conditions, we shall refer to α as a profinite Dehn twist:

- α preserves and fixes the conjugacy classes of the vertical subgroups [the subgroups associated to the Galois categories attached to the vertices of the underlying semi-graph of $\mathcal{G}$; [CmbGC], Definition 1.1, (ii)], the nodal subgroups [the subgroups associated to the Galois categories attached to the closed edges of the underlying semi-graph of $\mathcal{G}$; [CmbGC], Definition 1.1, (ii)], and the cuspidal subgroups [the subgroups associated to the Galois categories attached to the open edges of the underlying semi-graph of $\mathcal{G}$; [CmbGC], Definition 1.1, (ii)] of $\Pi_{\mathcal{G}}$;
- α induces inner automorphisms on each of the vertical subgroups of $\Pi_{\mathcal{G}}$. Note that this is well-defined by applying [CmbGC], Proposition 1.2, (ii).

Since any non-trivial automorphism of a hyperbolic curve over an algebraically closed field of characteristic $\notin \Sigma$ induces a non-trivial outer automorphism of the maximal pro- Σ quotient of the étale fundamental group of the hyperbolic curve [cf., e.g., [LocAn], the proof of Theorem 14.1], the injectivity follows immediately. Thus, it suffices to show that the homomorphism

$$\mathrm{Aut}_{\mathcal{H}_{g,d,r}}(\mathcal{C}_{g,d,r}) \hookrightarrow Z_{\mathrm{Out}^C(\Pi_{g,dr})}(\rho_{g,d,r}(H))$$

induced by the composite homomorphism

$$\mathrm{Aut}_{\mathcal{H}_{g,d,r}}(\mathcal{C}_{g,d,r}) \rightarrow Z_{\mathrm{Out}(\Pi_{g,dr})}(\rho_{g,d,r}(H))$$

in Theorem 3.1 is surjective. We give a sketch of the proof of this surjectivity in the case where $(d, r) = (2, 0)$.

Roughly speaking, the proof proceeds by the following two steps:

- (1) We first prove the surjectivity in the case where $g = 0, d = 2, r \geq 3$. This case is reduced to a version of the Grothendieck Conjecture for configuration spaces that applies to certain more combinatorially complicated spaces that arise from (log) Hurwitz stacks and then proved by applying the combinatorial Grothendieck Conjecture [i.e., the graphicity of outer automorphisms of pro- Σ surface groups satisfying certain combinatorial conditions [cf. [NodNon], Theorem A]] and elementary topological and graph-theoretic considerations.
- (2) Then we prove the surjectivity in the case where $d = 2, r = 0$. This case is reduced to the $g = 0, d = 2, r \geq 3$ case by using [the profiled Hurwitz stack version of] the theory of clutching morphisms [cf. [Knud], Definition 3.6], and the theory of profinite Dehn twists [cf. [CbTpI], §4].

Proof. (Sketch of the proof in the case where $g = 0, d = 2, r \geq 3$)

Write

- Π_B for the kernel of the outer homomorphism $\Pi_{\mathcal{H}_{0,2,r}} \rightarrow \Pi_{\mathcal{H}_{0,2,r-1}}$ induced by the restriction $\phi_{0,2,r-1}$ of $\tilde{\phi}_{0,2,r-1}^{\log}$ [cf. Proposition 2.5, (iii)] to $\mathcal{H}_{0,2,r}$;
- Π_T for the kernel of the outer homomorphism $\Pi_{\mathcal{C}_{0,2,r}} \rightarrow \Pi_{\mathcal{H}_{0,2,r-1}}$ induced by $\phi_{0,2,r-1} \circ u_{0,2,r}$ [cf. Definition 2.6].

Then we have a natural commutative diagram of profinite groups [cf. Proposition 2.7, (ii)]

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & \Pi_{0,2,r} & \longrightarrow & \Pi_T & \longrightarrow & \Pi_B & \longrightarrow & 1 \\
 & & \parallel & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \Pi_{0,2,r} & \longrightarrow & \Pi_{\mathcal{C}_{0,2,r}} & \longrightarrow & \Pi_{\mathcal{H}_{0,2,r}} & \longrightarrow & 1 \\
 & & & & & & \downarrow & & \\
 & & & & & & \Pi_{\mathcal{H}_{0,2,r-1}} & &
 \end{array}$$

The upper exact sequence determines an outer representation

$$\rho_B : \Pi_B \rightarrow \text{Out}(\Pi_{0,2,r})$$

which factors through the outer representation

$$\rho_{0,2,r} : \Pi_{\mathcal{H}_{0,2,r}} \longrightarrow \text{Out}(\Pi_{0,2,r}).$$

Since the hyperelliptic involution determines a nontrivial element of $\text{Aut}_{\mathcal{H}_{0,2,r}}(\mathcal{C}_{0,2,r})$, it suffices to show that the cardinality of the centralizer $Z_{\text{Out}^C(\Pi_{0,2,r})}(\rho_B(H_B))$ is equal to 2 for each open subgroup H_B of Π_B . Let us fix a geometric point of $\mathcal{H}_{0,2,r-1}$, and write

- $\pi : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1, (x_{1,1}, x_{1,2}), \dots, (x_{r-1,1}, x_{r-1,2})$ for the $(r-1)$ -profiled simple covering corresponding to the geometric point of $\mathcal{H}_{0,2,r-1}$, where $x_{i,j}$ is the ordered marked point for each $i = 1, \dots, r-1, j = 1, 2$;
- $M_{\text{unr}} \stackrel{\text{def}}{=} \{x_{i,j} \in \mathbb{P}_k^1(k) \mid (1 \leq i \leq r-1, 1 \leq j \leq 2)\}$;
- M_{ram} for the set of the ramification points of π ;
- $M \stackrel{\text{def}}{=} M_{\text{ram}} \cup M_{\text{unr}}$;
- $\Delta \subseteq (\mathbb{P}_k^1 \setminus M) \times (\mathbb{P}_k^1 \setminus M_{\text{unr}})$ (respectively, $\Delta' \subseteq (\mathbb{P}_k^1 \setminus M) \times (\mathbb{P}_k^1 \setminus M_{\text{unr}})$) for the diagonal divisor (respectively, the graph divisor determined by the hyperelliptic involution);
- K for the kernel of the outer surjection $\pi_1((\mathbb{P}_k^1 \setminus M) \times (\mathbb{P}_k^1 \setminus M_{\text{unr}}) \setminus \Delta \cup \Delta') \rightarrow \pi_1(\mathbb{P}_k^1 \setminus M)$ induced by the first projection;
- K^Σ for the maximal pro- Σ quotient of K ;

- K_Σ for the kernel of the natural quotient $K \twoheadrightarrow K^\Sigma$.

It follows from Proposition 2.7, (iii), together with the various definitions involved that

$$\begin{aligned}\Pi_B &\xleftarrow{\sim} \pi_1(\mathbb{P}_k^1 \setminus M), \\ \Pi_T &\xleftarrow{\sim} \pi_1((\mathbb{P}_k^1 \setminus M) \times (\mathbb{P}_k^1 \setminus M_{\text{unr}}) \setminus \Delta \cup \Delta')/K_\Sigma.\end{aligned}$$

Let α be an element of $Z_{\text{Out}^c(\Pi_{0,2r})}(\rho_B(H_B))$.

It suffices to verify the following assertions:

Claim 1: α preserves each conjugacy classes of the cuspidal inertia subgroups of $\Pi_{0,2r}$ up to the composite with the nontrivial element determined by the hyperelliptic involution.

Claim 2: If α preserves each conjugacy classes of the cuspidal inertia subgroups of $\Pi_{0,2r}$, then α is the identity.

First, we give a sketch of the proof of Claim 1. For each cusp x of $\mathbb{P}_k^1 \setminus M$, write Π_x for a cuspidal inertia subgroup associated to x . Note that we have the following commutative diagram of profinite groups:

$$\begin{array}{ccccc}\widehat{\mathbb{Z}} & \xrightarrow{\sim} & H_B \cap \Pi_x & \longrightarrow & \text{Out}(\Pi_{\mathcal{G}_x}) \\ \parallel & & \parallel & & \wr \downarrow \\ \widehat{\mathbb{Z}} & \xrightarrow{\sim} & H_B \cap \Pi_x & \longrightarrow & \text{Out}(\Pi_{\mathcal{G}_x}),\end{array}$$

where $\mathcal{G}_x$ is the semi-graph of anabelioids of pro- Σ PSC-type [cf. [CmbGC], Definition 1.1, (i)] obtained by considering the stable log curve

$$C_x^{\log} \stackrel{\text{def}}{=} x^{\log} \times_{\widetilde{\mathcal{H}}_{0,2,r}^{\log}} \widetilde{\mathcal{C}}_{0,2,r}^{\log}$$

over $x^{\log} \stackrel{\text{def}}{=} x \times_{\widetilde{\mathcal{H}}_{0,2,r}} \widetilde{\mathcal{H}}_{0,2,r}^{\log}$ determined by the (1-)morphism $\widetilde{u}_{0,2,r}^{\log} : \widetilde{\mathcal{C}}_{0,2,r}^{\log} \rightarrow \widetilde{\mathcal{H}}_{0,2,r}^{\log}$; $\Pi_{\mathcal{G}_x}$ is the [pro- Σ] fundamental group of $\mathcal{G}_x$; the right vertical arrow is the isomorphism induced by a specialization isomorphism $\Pi_{\mathcal{G}_x} \xleftarrow{\sim} \Pi_{0,2r}$ and α ; the horizontal arrows are the homomorphisms induced by the outer representation $\rho_B : \Pi_B \rightarrow \text{Out}(\Pi_{0,2r})$.

The commutativity of the above diagram of profinite groups follows from the fact that α is an element of the centralizer $Z_{\text{Out}^c(\Pi_{0,2r})}(\rho_B(H_B))$. Thus, it follows from [NodNon], Theorem A, that the element of $\text{Out}(\Pi_{\mathcal{G}_x})$ determined by the specialization isomorphism and α is graphic. By varying x , we conclude that α preserves each conjugacy classes of the cuspidal inertia subgroups of $\Pi_{0,2r}$ up to the composite with the nontrivial element determined by the hyperelliptic involution by the graph-theoretical consideration. This completes the proof of Claim 1.

Next, we give a sketch of the proof of Claim 2. Write

- Π_Δ (respectively, $\Pi_{\Delta'}$) for a cuspidal inertia subgroup of $\Pi_{0,2r}$ associated to Δ (respectively, Δ');
- $N_{\Delta'}$ (respectively, $N_{\Delta,\Delta'}$) for the normal subgroup of $\Pi_{0,2r}$ topologically normally generated by $\Pi_{\Delta'}$ (respectively, Π_Δ and $\Pi_{\Delta'}$);
- $\alpha_{\Delta'}$ (respectively, $\alpha_{\Delta,\Delta'}$) for the outer automorphism of $\Pi_{0,2r}/N_{\Delta'}$ (respectively, $\Pi_{0,2r}/N_{\Delta,\Delta'}$) induced by α .

The proof proceeds as follows:

Step 1: $\alpha_{\Delta,\Delta'}$ is the identity.

Step 2: $\alpha_{\Delta'}$ is the identity.

Step 3: α is the identity.

We explain the idea of the proof of each step briefly.

The point of the proof of Step 1 is to relate H_B to $\Pi_{0,2r}/N_{\Delta,\Delta'}$ through a decomposition group associated to Δ . Let us fix a decomposition group associated to Δ . Since $\Pi_{0,2r}$ is center-free, as a lift of α , we have an automorphism of $\Pi_T \times_{\Pi_B} H_B$ over H_B that preserves the fixed decomposition group. It follows from the various definitions involved that one may take a lift of $\alpha_{\Delta,\Delta'}$ in $\text{Aut}(\Pi_{0,2r}/N_{\Delta,\Delta'})$ that restricts to the identity automorphism on some open normal subgroup of $\Pi_{0,2r}/N_{\Delta,\Delta'}$. Since $r \geq 3$, $\Pi_{0,2r}/N_{\Delta,\Delta'}$ is a pro- Σ surface group, hence slim [i.e., for each open subgroup $O \subseteq \Pi_{0,2r}/N_{\Delta,\Delta'}$, $Z_{\Pi_{0,2r}/N_{\Delta,\Delta'}}(O) = \{1\}$]. Thus, by an elementary group theoretic consideration, we conclude that $\alpha_{\Delta,\Delta'}$ is the identity.

Here, we recall that α induces the graphic outer automorphism of $\Pi_{\mathcal{G}_x}$ for each cusp x . Since the proof of Step 2 and the proof of Step 3 are similar, we only explain the idea of the proof of Step 3. The point of the proof of Step 3 is to find appropriate cusps x, x' and verticial subgroups $\Pi_v \subseteq \Pi_{\mathcal{G}_x}$, $\Pi_{v'} \subseteq \Pi_{\mathcal{G}_{x'}}$ satisfying the following two conditions:

- The image of Π_v by the specialization isomorphism $\Pi_{\mathcal{G}_x} \xleftarrow{\sim} \Pi_{0,2r}$ and the image of $\Pi_{v'}$ by the specialization isomorphism $\Pi_{\mathcal{G}_{x'}} \xleftarrow{\sim} \Pi_{0,2r}$ generate $\Pi_{0,2r}$.
- The natural composites

$$\Pi_v \hookrightarrow \Pi_{\mathcal{G}_x} \xleftarrow{\sim} \Pi_{0,2r} \twoheadrightarrow \Pi_{0,2r}/N_{\Delta'}$$

and

$$\Pi_{v'} \hookrightarrow \Pi_{\mathcal{G}_{x'}} \xleftarrow{\sim} \Pi_{0,2r} \twoheadrightarrow \Pi_{0,2r}/N_{\Delta'}$$

are injective.

If we obtain these cusps and verticial subgroups, by discussing various indeterminacy concerning inner automorphisms appropriately, we conclude that α is the identity. $\square$

Proof. (Sketch of the proof in the case where $d = 2, r = 0$)

Since every projective curve of genus 2 is a hyperelliptic curve, Theorem 3.1 in the case where $g = 2, d = 2, r = 0$ follows immediately from Theorem 1.1. Thus, we may assume that $g \geq 3$. Since the hyperelliptic involution determines a nontrivial element of $\text{Aut}_{\mathcal{H}_{g,2,0}}(\mathcal{C}_{g,2,r})$, it suffices to show that the cardinality of the centralizer $Z_{\text{Out}^c(\Pi_{g,0})}(\rho_{g,2,0}(H))$ is equal to 2.

For each $x \in \tilde{\mathcal{H}}_{g,2,0}(k)$ [cf. Proposition 2.5, (i)], write

$$\mathcal{G}_x$$

for the semi-graph of anabelioids of pro- Σ PSC-type associated to the geometric fiber of $\tilde{\mathcal{C}}_{g,2,0}^{\log} \rightarrow \tilde{\mathcal{H}}_{g,2,0}^{\log}$ [cf. Definition 2.6] over $x^{\log} \stackrel{\text{def}}{=} x \times_{\tilde{\mathcal{H}}_{g,2,0}} \tilde{\mathcal{H}}_{g,2,0}^{\log}$.

The points of the proof are the following:

- (1) There exists $x \in \tilde{\mathcal{H}}_{g,2,0}(k)$ satisfying the following conditions [cf. [NodNon], Definition 1.1]:
 - Write $\text{Vert}(\mathcal{G}_x)$ for the set of vertices of the underlying semi-graph of $\mathcal{G}_x$. Then $\text{Vert}(\mathcal{G}_x) = \{v_1, v_2\}$, and the hyperelliptic involution permutes v_1, v_2 ;
 - Write $\text{Node}(\mathcal{G}_x)$ for the set of closed edges of the underlying semi-graph of $\mathcal{G}_x$. Then $\text{Node}(\mathcal{G}_x) = \{e_1, e_2, \dots, e_{g+1}\}$, and the hyperelliptic involution permutes the branches of each e_j , for $j = 1, \dots, g+1$;
 - Write $\mathcal{N}(v_1)$ (respectively, $\mathcal{N}(v_2)$) for the set of closed edges of the underlying semi-graph of $\mathcal{G}_x$ that abut to v_1 (respectively, v_2). Then $\mathcal{N}(v_1) = \mathcal{N}(v_2) = \text{Node}(\mathcal{G}_x)$;
 - for $i = 1, 2$, v_i is of type $(0, g+1)$.
- (2) For each $j = 1, \dots, g+1$, there exists $y_j \in \tilde{\mathcal{H}}_{g,2,0}(k)$ corresponds to $(\mathcal{G}_x)_{\rightsquigarrow \{e_j\}}$ [cf. [CbTpI], Definition 2.8].
- (3) $y_j \in \tilde{\mathcal{H}}_{g,2,0}(k)$ is contained in the image of the “clutching morphism”

$$\mathcal{H}_{0,2,g} \times \mathcal{H}_{0,2,1} \times \dots \times \mathcal{H}_{0,2,1} \longrightarrow \tilde{\mathcal{H}}_{g,2,0} \xrightarrow{\sim} \overline{\mathcal{H}}_{g,2,0},$$

where the number of factors in the above product is $g+1$; the first factor in the product corresponds to the vertex of $(\mathcal{G}_x)_{\rightsquigarrow \{e_j\}}$ that arises from v_1, v_2 , and e_j ; the factors other than the first factor correspond to the $e_{j'}$, for $j' \in \{1, 2, \dots, g+1\} \setminus \{j\}$.

By using these k -valued points, we verify the following assertions:

Claim 3: $\alpha \in Z_{\text{Out}^C(\Pi_{g,2,0})}(\rho_{g,2,0}(H))$ determines an element of $\text{Aut}^{|\text{Node}|}(\mathcal{G}_x)$ [cf. [CbTpI], Definition 2.6, (i)]. In particular, for each $j = 1, \dots, g+1$, α determines an element of $\text{Aut}^{|\text{Node}|}((\mathcal{G}_x)_{\rightsquigarrow \{e_j\}})$.

Claim 4: If $\alpha \in \text{Aut}^{|\text{grph}|}(\mathcal{G}_x)$ [cf. [CbTpI], Definition 2.6, (i)], then α is the identity.

Claim 3 is highly nontrivial, however, formally follows from the logarithmic structure of $\tilde{\mathcal{H}}_{g,2,0}^{\log}$; [CbTpI], Proposition 5.6, (ii); [CbTpI], Theorem 5.14, (ii).

The proof of Claim 4 proceeds as follows. By the assumption $\alpha \in \text{Aut}^{|\text{grph}|}(\mathcal{G}_x)$, $\alpha_j \in \text{Aut}^{|\text{grph}|}((\mathcal{G}_x)_{\rightsquigarrow \{e_j\}}) = \text{Aut}^{|\text{grph}|}(\mathcal{G}_{y_j})$. Since $y_j \in \tilde{\mathcal{H}}_{g,2,0}(k)$ is contained in the image of the above clutching morphism, and Theorem 3.1 holds for $\mathcal{H}_{0,2,g}$, we obtain that $\alpha_j \in \text{Dehn}((\mathcal{G}_x)_{\rightsquigarrow \{e_j\}})$ [cf. [CbTpI], Definition 4.4]. Thus, by varying j , we conclude that α is the identity by [CbTpI], Theorem 4.8, (iv) [i.e., the structure theorem of profinite Dehn twists]. This completes the sketch of the proof of Claim 4, hence the sketch of the proof of Theorem 3.1, (ii), in the case where $d = 2, r = 0$. $\square$

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