

ISOMETRIES ON THE SYMMETRIC PRODUCTS OF THE EUCLIDEAN SPACES WITH USUAL METRICS

知念 直紹 (NAOTSUGU CHINEN)

防衛大学校 (NATIONAL DEFENSE ACADEMY OF JAPAN)

1. INTRODUCTION

As an interesting construction in topology, Borsuk and Ulam [3] introduced the n -th *symmetric product* of a metric space (X, d) , denoted by $F_n(X)$. Namely $F_n(X)$ is the space of non-empty finite subsets of X with at most n elements endowed with the Hausdorff metric d_H , i.e., $F_n(X) = \{A \subset X \mid 1 \leq |A| \leq n\}$ and $d_H(A, B) = \inf\{\epsilon \mid A \subset B_d(B, \epsilon) \text{ and } B \subset B_d(A, \epsilon)\} = \max\{d(a, B), d(b, A) \mid a \in A, b \in B\}$ for any $A, B \in F_n(X)$ (see [10, p.6]).

For the symmetric products of $\mathbb{R}$, it is known that $F_2(\mathbb{R}) \approx \mathbb{R} \times [0, \infty)$ and $F_3(\mathbb{R}) \approx \mathbb{R}^3$ (see Section 3). It was proved in [3] that $F_n(\mathbb{I})$ is homeomorphic to $\mathbb{I}^n$ (written $F_n(\mathbb{I}) \approx \mathbb{I}^n$) if and only if $1 \leq n \leq 3$, and that for $n \geq 4$, $F_n(\mathbb{I})$ can not be embedded into $\mathbb{R}^n$, where $\mathbb{I} = [0, 1]$ has the usual metric. Thus, for $n \geq 4$, $F_n(\mathbb{R}) \not\approx \mathbb{R}^n$. Molski [12] showed that $F_2(\mathbb{I}^2) \approx \mathbb{I}^4$, and that for $n \geq 3$ neither $F_n(\mathbb{I}^2)$ nor $F_2(\mathbb{I}^n)$ can be embedded into $\mathbb{R}^{2n}$. Thus, for $n \geq 3$, $F_n(\mathbb{R}^2) \not\approx \mathbb{R}^{2n}$ and $F_2(\mathbb{R}^n) \not\approx \mathbb{R}^{2n}$.

Turning toward the symmetric product $F_n(\mathbb{S}^1)$ of the circle $\mathbb{S}^1$, Chinen and Koyama [9] prove that for $n \in \mathbb{N}$, both $F_{2n-1}(\mathbb{S}^1)$ and $F_{2n}(\mathbb{S}^1)$ have the same homotopy type of the $(2n-1)$ -sphere $\mathbb{S}^{2n-1}$. In [7] Bott corrected Borsuk's statement [4] and showed that $F_3(\mathbb{S}^1) \approx \mathbb{S}^3$. In [9], another proof of it is given.

For a metric space (X, d) , we denote by $\text{Isom}_d(X)$ ($\text{Isom}(X)$ for short) the group of all isometries from X into itself, i.e., $\phi : X \rightarrow X \in \text{Isom}_d(X)$ if ϕ is a bijection satisfying that $d(x, x') = d(\phi(x), \phi(x'))$ for any $x, x' \in X$. Let $n \in \mathbb{N}$. Every isometry $\phi : X \rightarrow X$ induces an isometry $\chi_{(n)}(\phi) : (F_n(X), d_H) \rightarrow (F_n(X), d_H)$ defined by $\chi_{(n)}(\phi)(A) = \phi(A)$ for each $A \in F_n(X)$. Thus, there exists a natural monomorphism $\chi_{(n)} : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_H}(F_n(X))$. It is clear that $\chi_{(n)} : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_H}(F_n(X))$ is an isomorphism if and only if $\chi_{(n)}$ is an epimorphism, i.e., for every $\Phi \in \text{Isom}_{d_H}(F_n(X))$ there exists $\phi \in \text{Isom}_d(X)$ such that $\Phi = \chi_{(n)}(\phi)$.

In this paper, it is of interest to know whether $\chi_{(n)} : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_H}(F_n(X))$ is an isomorphism for a metric space (X, d) . Recently, Borovikova and Ibragimov [5] prove that $(F_3(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, d)$ and that $\chi_{(3)} : \text{Isom}_d(\mathbb{R}) \rightarrow \text{Isom}_{d_H}(F_3(\mathbb{R}))$ is an isomorphism, where $\mathbb{R}$ has the usual metric d . The following result is a generalization of the result above and the affirmative answer to [6, p.60, Conjecture 2.1].

Theorem 1.1. *Let $l \in \mathbb{N}$ and let $X = \mathbb{R}^l$ or $X = \mathbb{S}^l$ with the usual metric d . Then $\chi_{(n)} : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_H}(F_n(X))$ is an isomorphism for each $n \in \mathbb{N}$.*

In Section 4, we give the main ideas of proof of Theorem 1.1. In Example 5.2 below, we present a compact metric space (X, d) such that $\chi_{(n)}(\text{Isom}_d(X)) \neq \text{Isom}_{d_H}(F_n(X))$ for all $n \geq 2$, i.e., $\chi_{(n)} : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_H}(F_n(X))$ is not an isomorphism. And, in Section 3, we provide another proof of [5, Theorem 6]. Its proof is based on the proof of [11, Lemma 2.4].

2. PRELIMINARIES

Notation 2.1. Let denote the set of all natural numbers and real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. Let d be the usual metric on $\mathbb{R}^l$, i.e., $d(x, y) = \{\sum_{i=1}^l (x_i - y_i)^2\}^{1/2}$ for any $x = (x_1, \dots, x_l), y = (y_1, \dots, y_l) \in \mathbb{R}^l$. Write $\mathbb{S}^l = \{x = (x_1, \dots, x_{l+1}) \in \mathbb{R}^{l+1} \mid \sum_{i=1}^{l+1} x_i^2 = 1\}$ with the length metric d . Denote the identity map from X into itself by id_X .

Definition 2.2. Let (X, d) be a metric space, let $x \in X$, let Y, Z be subsets of X and let $\epsilon > 0$. Set $d(Y, Z) = \inf\{d(y, x) \mid y \in Y, z \in Z\}$, and $B_d(Y, \epsilon) = \{x \in X \mid d(x, Y) \leq \epsilon\}$. If $Y = \{y\}$, for simplicity of notation, we write $B_d(y, \epsilon) = B_d(Y, \epsilon)$ and $S_d(y, \epsilon) = S_d(Y, \epsilon)$.

For $n \in \mathbb{N}$, the n -th *symmetric product* of X is defined by

$$F_n(X) = \{A \subset X \mid 1 \leq |A| \leq n\},$$

where $|A|$ is the cardinality of A . Write $F_{(m)}(X) = \{A \in 2^X \mid |A| = m\}$ for each $m \in \mathbb{N}$. Let $\text{Isom}(X, Y) = \{\phi \in \text{Isom}(X) \mid \phi(y) = y \text{ for each } y \in Y\}$ for $Y \subset X$. Set $r(A) = \min\{\{1\} \cup \{d(a, a') \mid a, a' \in A, a \neq a'\}\}$ for each $A \in F_n(X)$.

3. A METRIC SPACE IS BI-LIPSCHITZ EQUIVALENT TO THE SYMMETRIC PRODUCT OF $\mathbb{R}$

In this section, we give another proof of [5, Theorem 6] which is based on the proof of [11, Lemma 2.4].

Definition 3.1. Let $n \in \mathbb{N}$. Set $F_n^*(\mathbb{I}) = \{A \in F_n(\mathbb{I}) \mid 0, 1 \in A\}$. It is known that $F_2^*(\mathbb{I}) = \{\{0, 1\}\}$, $F_3^*(\mathbb{I}) = \{\{0, t, 1\} \mid 0 \leq t \leq 1\} \approx \mathbb{S}^1$, and, $F_4^*(\mathbb{I}) = \{\{0, s, t, 1\} \mid 0 \leq s \leq t \leq 1\}$ is homeomorphic to the dance hat (see [16]). In general, $F_{2n}^*(\mathbb{I})$ is contractible but not collapsible, and $F_{2n+1}^*(\mathbb{I})$ has the same homotopy type of $\mathbb{S}^{2n+1}$. In [1], it is called the spaces $F_{2n}^*(\mathbb{I})$, $n \geq 2$, *higher dimensional dunce hats* (see [1]).

Definition 3.2 ([11]). Let (X, d) be a metric space with $\text{diam } X \leq 2$. Set $\text{Cone}^o(X) = X \times [0, \infty) / (X \times \{0\})$, is said to be the *open cone over X* , with the metric $d_C([(x_1, t_1)], [(x_2, t_2)]) = |t_1 - t_2| + \min\{t_1, t_2\} \cdot d(x_1, x_2)$.

Definition 3.3. Let $f : (X, d) \rightarrow (Y, d')$ be a map. The map f is said to be *Lipschitz* (*bi-Lipschitz*, respectively) if there exists $L > 0$ such that

$$d'(f(x_1), f(x_2)) \leq L d(x_1, x_2)$$

$$(L^{-1} d(x_1, x_2) \leq d'(f(x_1), f(x_2)) \leq L d(x_1, x_2), \text{ respectively})$$

for any $x_1, x_2 \in X$. (X, d) is said to be *bi-Lipschitz equivalent* to (Y, d') if there exists a surjective bi-Lipschitz map from (X, d) to (Y, d') .

Theorem 3.4 ([11]). Let $n \in \mathbb{N}$ with $n \geq 2$. Then $(F_n(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R} \times \text{Cone}^o(F_n^*(\mathbb{I})), \rho)$, where $\rho = \sqrt{d^2 + (d_H)_C^2}$.

Sketch of Proof. Let $Z = \{A \in F_n(\mathbb{R}) \mid \min A = 0\}$. For every $A \in Z$ there exists the unique $E \in F_n^*(\mathbb{I})$ such that $A = tE$, where $t = \max A$.

Step1: $(F_n(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R} \times Z, \rho_1)$, where $\rho_1 = \sqrt{d^2 + (d_H)^2}$. In fact, we can show the following.

Step1.1: A map $f : F_n(\mathbb{R}) \rightarrow \mathbb{R} \times Z : A \mapsto (\min A, A - \min A)$ is $\sqrt{5}$ -Lipschitz.

Step1.2: A map $f^{-1} : \mathbb{R} \times Z \rightarrow F_n(\mathbb{R}) : (b, A) \mapsto A + b$ is 2-Lipschitz.

Step2: (Z, d_H) is bi-Lipschitz equivalent to $(\text{Cone}^o(F_n^*(\mathbb{I})), (d_H)_C)$. In fact, we can show the following.

Step2.1: A map $g : Z \rightarrow \text{Cone}^o(F_n^*(\mathbb{I})) : tE \mapsto [(E, t)]$ is 1-Lipschitz.

Step2.2: A map $g^{-1} : \text{Cone}^o(F_n^*(\mathbb{I})) \rightarrow Z : [(E, t)] \mapsto tE$ is 3-Lipschitz.

By the above, $(\text{id}_{\mathbb{R}} \times g) \circ f : F_n(\mathbb{R}) \rightarrow \mathbb{R} \times Z \rightarrow \mathbb{R} \times \text{Cone}^o(F_n^*(\mathbb{I}))$ is a bi-Lipschitz equivalence. $\square$

Corollary 3.5. $(F_2(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R} \times [0, \infty), d)$.

Proof. By Definition 3.1, $F_2^*(\mathbb{I})$ is one point, thus $(\text{Cone}^o(F_2^*(\mathbb{I})), (d_H)_C)$ is corresponding to $([0, \infty), d)$. By Theorem 3.4, $(F_2(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R} \times [0, \infty), d)$. $\square$

The following result is first proved in [5, Theorem 6]. We give another proof by use of Theorem 3.4.

Corollary 3.6 ([5]). $(F_3(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, d)$.

Sketch of Proof. We note $F_3^*(\mathbb{I}) = \{\{0, t, 1\} \mid 0 \leq t \leq 1\} \approx \mathbb{S}^1$.

Step1: We can show that $(\text{Cone}^o(F_n^*(\mathbb{I})), (d_H)_C)$ is bi-Lipschitz equivalent to $(\text{Cone}^o(\mathbb{S}^1), (d|_{\mathbb{S}^1})_C)$.

Step2: We can show that $(\mathbb{R}^2, d)$ is bi-Lipschitz equivalent to $(\text{Cone}^o(\mathbb{S}^1), (d|_{\mathbb{S}^1})_C)$.

By Theorem 3.4, $(F_3(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, d)$. $\square$

Remark 3.7. We note that $F_2(\mathbb{R}^2) \approx \mathbb{R}^4$. Indeed, we can define a homeomorphism $h : F_2(\mathbb{R}^2) \rightarrow \mathbb{R}^2 \times \text{Cone}^o(\mathbb{S}^1/x \sim -x) (\approx \mathbb{R}^4)$ by

$$h(A) = \begin{cases} \left(m(A), \left[\left(\frac{2(A-m(A))}{\text{diam } A}, \text{diam } A \right) \right] \right) & \text{if } \text{diam } A \neq 0 \\ (m(A), \text{the cone point}) & \text{if } \text{diam } A = 0, \end{cases}$$

where $m(A) = a$ if $A = \{a\}$ and $m(A) = (a + a')/2$ if $A = \{a, a'\}$. In general, we see that $F_2(\mathbb{R}^l) \approx \mathbb{R}^l \times \text{Cone}^o(\mathbb{S}^{l-1}/x \sim -x)$ for each $l \in \mathbb{N}$.

4. ISOMETRIES

Lemma 4.1. *Let $n \in \mathbb{N}$ and let (X, d) be a metric space such that*

- (1) $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$ for each $\Phi \in \text{Isom}(F_n(X))$, and that
- (2) $\text{Isom}(F_n(X), F_1(X)) = \{\text{id}_{F_n(X)}\}$.

Then, $\chi_{(n)} : \text{Isom}(X) \rightarrow \text{Isom}(F_n(X))$ is an isomorphism.

Proof. Let $\Phi \in \text{Isom}(F_n(X))$ and let $A_x = \{x\} \in F_1(X)$ for each $x \in X$. By assumption, $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$. Denote $\Phi(A_x) \in F_1(X)$ by $\{\phi(x)\}$ for each $x \in X$. Then, $\phi : X \rightarrow X : x \mapsto \phi(x)$ is an isometry. Set $\Phi' = \chi_{(n)}(\phi^{-1}) \circ \Phi \in \text{Isom}(F_n(X))$. We claim that $\Phi'|_{F_1(X)} = \text{id}_{F_1(X)}$. Indeed, $\Phi|_{F_1(X)} = \chi_{(n)}(\phi)|_{F_1(X)}$ and $\chi_{(n)}(\phi^{-1}) = (\chi_{(n)}(\phi))^{-1}$. By assumption, we have that $\Phi' = \text{id}_{F_n(X)}$, therefore, $\Phi = \chi_{(n)}(\phi)$, which completes the proof. $\square$

Definition 4.2. Let (X, d) be a metric space, let $n \in \mathbb{N}$, let $\epsilon > 0$ and let $A \in F_n(X)$. Define

$$D_n(A, \epsilon) = \sup\{k \in \mathbb{N} \mid A_1, \dots, A_k \in S_{d_H}(A, \epsilon), d_H(A_i, A_j) = 2\epsilon (i \neq j)\} \in \mathbb{N} \cup \{\infty\}.$$

Lemma 4.3. *Let $l, n \in \mathbb{N}$, let $X = \mathbb{R}^l$ or $X = \mathbb{S}^l$ and let $\Phi \in \text{Isom}(F_n(X))$. Then, $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$.*

Sketch of Proof. Let $n \in \mathbb{N}$ with $n \geq 2$.

Step1: Let $A = \{a_1\} \in F_1(X)$ and let $\epsilon > 0$ with $\epsilon < r(A)$. We can show that $D_n(A, \epsilon) = 3$.

Step2: Let $m \in \mathbb{N}$ with $m \geq 2$, let $A = \{a_1, \dots, a_m\} \in F_{(m)}(X)$ and let $\epsilon > 0$ with $\epsilon < r(A)/5$. We can show that $D_n(A, \epsilon) > 3$.

Let $\Phi \in \text{Isom}(F_n(X))$ and let $A \in F_n(X)$. From the definition of $D_n(A, \epsilon)$, we obtain $D_n(A, \epsilon) = D_n(\Phi(A), \epsilon)$ for each $0 < \epsilon < \min\{r(A), r(\Phi(A))\}$. By the above, we see that $A \in F_1(X)$ if and only if $\Phi(A) \in F_1(X)$. Therefore, $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$. $\square$

Lemma 4.4. *Let $l, n \in \mathbb{N}$. Then, $\text{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l)) = \{\text{id}_{F_n(\mathbb{R}^l)}\}$.*

Sketch of Proof.

Step1: Let $l, n \in \mathbb{N}$ and let $\Phi \in \text{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l))$. Then, $\Phi|_{F_2(\mathbb{R}^l)} = \text{id}_{F_2(\mathbb{R}^l)}$.

Step2: Let $n \in \mathbb{N}$ with $n \geq 2$ and let $\Phi \in \text{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l))$ and let $A \in F_{(m)}(\mathbb{R}^l)$. We can show that $\Phi(A) \subset A$. If similar arguments apply to $\Phi(A)$ and Φ^{-1} , we obtain $A = \Phi^{-1}(\Phi(A)) \subset \Phi(A)$, therefore, $A = \Phi(A)$. $\square$

Lemma 4.5. *Let $l, n \in \mathbb{N}$. Then $\text{Isom}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l)) = \{\text{id}_{F_n(\mathbb{S}^l)}\}$.*

Proof. Let $\Phi \in \text{Isom}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l))$, $m \in \mathbb{N}$ with $2 \leq m \leq n$ and let $A \in F_{(m)}(\mathbb{S}^l)$. We show that $A = \Phi(A)$. Let $a \in A$ and let $a' \in \mathbb{S}^l$ be the anti-point of a . Since $d_H(\{a'\}, \Phi(A)) = d_H(\Phi(\{a'\}), \Phi(A)) = d_H(\{a'\}, A) = \pi$, we have $a \in \Phi(A)$, therefore, $A \subset \Phi(A)$. If similar arguments apply to $\Phi(A)$ and Φ^{-1} , we obtain $\Phi(A) \subset \Phi^{-1}(\Phi(A)) = A$, therefore, $A = \Phi(A)$, which completes the proof. $\square$

The proof of Theorem 1.1. By Lemmas 4.3, 4.4 and 4.5, the conditions in Lemma 4.1 hold for (X, d) , which completes the proof. $\square$

5. QUESTIONS

Question 5.1. *Let $l, n \in \mathbb{N}$ with $n \geq 2$. When (X, d) is a following space, is $\chi_n : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_H}(F_n(X))$ an isomorphism?*

- (1) $X = \mathbb{R}^l$ has a metric d_∞ , where $d_\infty(x, y) = \max\{|x_i - y_i| \mid i = 1, \dots, l\}$ for any $x = (x_1, \dots, x_l), y = (y_1, \dots, y_l) \in X$.
- (2) X is a convex subset of $\mathbb{R}^l$.
- (3) X is an $\mathbb{R}$ -tree (see [2] for $\mathbb{R}$ -trees).
- (4) X is the hyperbolic l -space (see [8] for the hyperbolic l -space).

Example 5.2. Let $n, m \in \mathbb{N}$ with $2 \leq n \leq m$ and let (X, d) be an m -points discrete metric space satisfying that $d(x, x') = 1$ whenever $x \neq x'$. Then, $F_n(X)$ is a discrete metric space such that $d_H(A, A') = 1$ for any $A, A' \in F_n(X)$ with $A \neq A'$. Thus, $|\text{Isom}(X)| = |X|! < |F_n(X)|! = |\text{Isom}(F_n(X))|$, therefore, $\chi_{(n)} : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_H}(F_n(X))$ is not an isomorphism.

This drives us to the following question as the generalization of Theorem 1.1.

Question 5.3. *Let (X, d) be a connected metric space. Then, is $\chi_{(n)} : \text{Isom}(X) \rightarrow \text{Isom}(F_n(X))$ an isomorphism?*

Question 5.4. *It is known that $F_3(\mathbb{S}^1) \approx \mathbb{S}^3$. Is $F_3(\mathbb{S}^1)$ bi-Lipschitz equivalent to $\mathbb{S}^3$?*

REFERENCES

- [1] R. N. Andersen, M. M. Marjanović, and R. M. Schori, *Symmetric products and higher-dimensional dunce hats*, Topology Proc. 18 (1993), 7–17.
- [2] M. Bestvina, *$\mathbb{R}$ -trees in topology, geometry, and group theory*, Handbook of geometric topology, 55–91, North-Holland, Amsterdam, 2002.
- [3] K. Borsuk and S. Ulam, *On symmetric products of topological spaces*, Bull. A. M. S. 37(1931), 875–882.
- [4] K. Borsuk, *On the third symmetric potency of the circumference*, Fund. Math. 36 (1949), 236–244.
- [5] M. Borovikova and Z. Ibragimov, *The third symmetric product of $\mathbb{R}$* , Comput. Methods Funct. Theory 9 (2009), 255–268.
- [6] M. Borovikova, Z. Ibragimov and H. Yousefi, *Symmetric products of the real line*, J. Anal. 18 (2010), 53–67.
- [7] R. Bott, *On the third symmetric potency of S_1* , Fund. Math. 39 (1952), 264–268.
- [8] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999.
- [9] N. Chinen and A. Koyama, *On the symmetric hyperspace of the circle*, Topology Appl. 157 (2010), 2613–2621.
- [10] A. Illanes and S. B. Nadler, Jr., *Hyperspaces*, Marcel Dekker, 1999.
- [11] L. V. Kovalev, *Symmetric products of the line: embeddings and retractions*, arXiv:1212.1698v1 [math.MG] 7 Dec 2012.
- [12] R. Molski, *On symmetric product*, Fund. Math. 44 (1957), 165–170.
- [13] H. R. Morton, *Symmetric product of the circle*, Proc. Camb. Phil. Soc. 63 (1967), 349–352.
- [14] R. M. Schori, *Hyperspaces and symmetric products of topological spaces*, Fund. Math. 39 (1968), 77–87.
- [15] W. Wu, *Note sur les produits essentiels symétriques des espaces topologiques*, C. R. Acad. Sci. Paris 224 (1947), 1139–1141.
- [16] E. C. Zeeman, *On the dunce hat*, Topology 39 (1964), 341–358.

DEPARTMENT OF MATHEMATICS, NATIONAL DEFENSE ACADEMY OF JAPAN, YOKOSUKA
239-8686, JAPAN

E-mail address: naochin@nda.ac.jp