

Normal Toeplitz matrices on $\mathbb{C}^n$

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In this lecture, we gave an another proof of the characterization given in [1] and also in [2] of the normal Toeplitz matrices on $\mathbb{C}^n$.

For any positive integer n and for any $a_{j,k} \in \mathbb{C}$ ($j, k = 1, 2, \dots, n$), let

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} = (a_{j,k}) \quad \text{on } \mathbb{C}^n$$

and let

$$W = \overbrace{\begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & & & \vdots \\ 0 & 1 & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}}^{n \text{ times}}$$

Then

$$WAW = \begin{pmatrix} a_{n,n} & \cdots & a_{n,2} & a_{n,1} \\ \vdots & & \vdots & \vdots \\ a_{2,n} & \cdots & a_{2,2} & a_{2,1} \\ a_{1,n} & \cdots & a_{1,2} & a_{1,1} \end{pmatrix} \quad (1)$$

Let ${}_tA = (a_{n+1-k, n+1-j})$ and let ${}^tA = (a_{k,j})$. Then, by (1), we have

$$WAW = {}^t({}_tA) = {}_t({}^tA) \quad (2)$$

$$\begin{aligned} \text{and } {}_t(AB) &\stackrel{(2)}{=} {}^t(WABW) = {}^t\{(WAW)(WBW)\} \\ &= {}^t(WBW) {}^t(WAW) \stackrel{(2)}{=} {}_tB {}_tA. \end{aligned} \quad (3)$$

If T is a Toeplitz matrix on $\mathbb{C}^n$, then

$${}_tT = T \quad (4)$$

$$\text{and } {}_t(T^*T) \underset{(3)}{=} {}_tT {}_t(T^*) \underset{(4)}{=} TT^* \quad (5)$$

and, in particular, we have

$$\begin{aligned} T^*T ; \text{ Toeplitz matrix} &\xrightarrow{(4)} {}_t(T^*T) = T^*T \\ &\xRightarrow{(5)} T ; \text{ normal} \end{aligned}$$

Since ${}_t(T^*T - TT^*) = {}_t(T^*T) - {}_t(TT^*) \underset{(5)}{=} TT^* - T^*T$, we have the following by (4).

Theorem 1. For a Toeplitz matrix T on $\mathbb{C}^n$, T is normal if and only if $T^*T - TT^*$ is a Toeplitz matrix.

Henceforth we assume that the Toeplitz matrix T on $\mathbb{C}^n$ is represented as follows ;

$$T = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-n+1} \\ a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{-1} \\ a_{n-1} & \cdots & a_1 & a_0 \end{pmatrix},$$

where a_j , ($j = 0, \pm 1, \pm 2, \dots, \pm(n-1)$) are some complex numbers. Then $T^*T = (t_{j,k})$, where $t_{j,k} = \sum_{m=1}^n \overline{a_{m-j}} a_{m-k}$, and $T^*T - TT^* = (\alpha_{j,k})$, where

$$\alpha_{j,k} = \sum_{m=1}^n \overline{a_{m-j}} a_{m-k} - \sum_{m=1}^n \overline{a_{k-m}} a_{j-m}$$

and hence, by Theorem 1, we have the following.

Theorem 2. The Toeplitz matrix T on $\mathbb{C}^n$ is normal if and only if, for any $j, k = 1, 2, \dots, n-1$,

$$\overline{a_{-j}} a_{-k} - \overline{a_k} a_j = \overline{a_{n-j}} a_{n-k} - \overline{a_{-(n-k)}} a_{-(n-j)}. \quad (6)$$

Let $n - k$ instead of k in (6). Then, for any $j, k = 1, 2, \dots, n - 1$, we have

$$\overline{a_{-j}}a_{-(n-k)} - \overline{a_{n-k}}a_j = \overline{a_{n-j}}a_k - \overline{a_{-k}}a_{-(n-j)}. \quad (6-1)$$

Let $k = j$ in (6) and (6-1) respectively. Then, for any $j = 1, 2, \dots, n - 1$, we have

$$|a_{-j}|^2 - |a_j|^2 = |a_{n-j}|^2 - |a_{-(n-j)}|^2 \quad (6-2)$$

$$\text{and } \overline{a_{-j}}a_{-(n-j)} = \overline{a_{n-j}}a_j \quad (6-3)$$

and hence $|a_{-j}| = |a_{n-j}|$ or $|a_{-j}| = |a_j|$

$$\text{i.e., } \begin{cases} a_{-j} = e^{i\varphi_j} \overline{a_j} \\ \text{or } a_{-j} = e^{i\psi_j} a_{n-j} \end{cases} \text{ for some } \varphi_j, \psi_j \in [0, 2\pi). \quad (6-4)$$

If $a_{-j} = e^{i\varphi_j} \overline{a_j}$, then, by (6-3), we have

$$0 = e^{-i\varphi_j} a_j a_{-(n-j)} - \overline{a_{n-j}} a_j = e^{-i\varphi_j} a_j (a_{-(n-j)} - e^{i\varphi_j} \overline{a_{n-j}})$$

and

$$a_{-j} = e^{i\varphi_j} \overline{a_j} \ (a_j \neq 0) \text{ implies } a_{-(n-j)} = e^{i\varphi_j} \overline{a_{n-j}} \quad (6-5)$$

And if $a_{-j} = e^{i\psi_j} a_{n-j}$, then, by (6-3), we have

$$0 = e^{-i\psi_j} \overline{a_{n-j}} a_{-(n-j)} - \overline{a_{n-j}} a_j = e^{-i\psi_j} \overline{a_{n-j}} (a_{-(n-j)} - e^{i\psi_j} a_j)$$

and

$$a_{-j} = e^{i\psi_j} a_{n-j} \ (a_{n-j} \neq 0) \text{ implies } a_{-(n-j)} = e^{i\psi_j} a_j \quad (6-6)$$

In particular, if $\begin{cases} a_{-j} = e^{i\varphi_j} \overline{a_j} & (a_j \neq 0) \\ a_{-k} = e^{i\psi_k} a_{n-k} & (a_{n-k} \neq 0) \end{cases}$, then, by (6), (6-5) and (6-6), we have

$$\begin{cases} a_{-j} = e^{i\varphi_j} \overline{a_j} = e^{i\varphi_j} e^{-i(\varphi_j - \psi_k)} a_{n-j} = e^{i\psi_k} a_{n-j} \\ \text{or } a_{-k} = e^{i\psi_k} a_{n-k} = e^{i\psi_k} e^{-i(\psi_k - \varphi_j)} \overline{a_k} = e^{i\varphi_j} \overline{a_k} \end{cases}$$

and

$$\begin{cases} a_{-j} = e^{i\varphi_j} \overline{a_j} & (a_j \neq 0) \\ a_{-k} = e^{i\psi_k} \overline{a_{n-k}} & (a_{n-k} \neq 0) \end{cases} \text{ implies } \begin{cases} a_{-j} = e^{i\psi_k} a_{n-j} \\ \text{or } a_{-k} = e^{i\varphi_j} \overline{a_k}. \end{cases} \quad (6-7)$$

Lemma 1. Under the condition (6), if $a_j \neq 0$, $a_k \neq 0$, $a_{n-j} \neq 0$, $a_{n-k} \neq 0$ and if $|\frac{a_{n-j}}{a_j}| \neq |\frac{a_{n-k}}{a_k}|$, then, for some $\varphi, \psi \in [0, 2\pi)$,

$$\begin{cases} a_{-j} = e^{i\varphi} \overline{a_j} \\ a_{-k} = e^{i\varphi} \overline{a_k} \end{cases} \quad \text{or} \quad \begin{cases} a_{-j} = e^{i\psi} a_{n-j} \\ a_{-k} = e^{i\psi} a_{n-k}. \end{cases}$$

Lemma 2. Under the condition (6), if $a_j \neq 0$, $a_k \neq 0$, $a_{n-j} \neq 0$, $a_{n-k} \neq 0$ and if $\frac{\overline{a_{n-j}}}{a_j} \neq \frac{\overline{a_{n-k}}}{a_k}$, then

$$\begin{cases} a_{-j} = e^{i\varphi} \overline{a_j} \\ a_{-k} = e^{i\varphi} \overline{a_k} \end{cases} \quad \left(\text{and also } \begin{cases} a_{-j} = e^{i\psi} a_{n-j} \\ a_{-k} = e^{i\psi} a_{n-k} \end{cases} \right)$$

implies that, for any $m = 1, 2, \dots, n-1$,

$$a_{-m} = e^{i\varphi} \overline{a_m} \quad (\text{and } a_{-m} = e^{i\psi} a_{n-m} \text{ respectively}).$$

Lemma 3. Under the condition (6), if $a_j \neq 0$, $a_k \neq 0$, $a_{n-j} \neq 0$, $a_{n-k} \neq 0$ and if $|\frac{a_{n-j}}{a_j}| = |\frac{a_{n-k}}{a_k}| = |\frac{a_k}{a_{n-k}}|$, then

$$\begin{cases} a_{-j} = e^{i\varphi_j} \overline{a_j} = e^{i\psi_j} a_{n-j} \\ a_{-k} = e^{i\varphi_k} \overline{a_k} = e^{i\psi_k} a_{n-k}, \end{cases} \quad \begin{cases} a_{n-j} = e^{i(\varphi_j - \psi_j)} \overline{a_j} \\ a_{n-k} = e^{i(\varphi_k - \psi_k)} \overline{a_k} \end{cases} \quad \text{and} \quad \begin{cases} \varphi_j = \varphi_k \\ \text{or } \psi_j = \psi_k. \end{cases}$$

In particular, if $\frac{a_{n-j}}{a_j} = \frac{a_{n-k}}{a_k}$, then

$$\begin{cases} a_{-j} = e^{i\varphi_j} \overline{a_j} = e^{i\psi_j} a_{n-j} \\ a_{-k} = e^{i\varphi_j} \overline{a_k} = e^{i\psi_j} a_{n-k}. \end{cases}$$

Lemma 4. Under the condition (6), if $a_{-j} = e^{i\varphi} \overline{a_j}$ ($a_j \neq 0$) and if $a_{n-j} = 0$, then

$$a_{-m} = e^{i\varphi} \overline{a_m} \quad (m = 1, 2, \dots, n-1).$$

Lemma 5. Under the condition (6), if $a_{-j} = e^{i\psi} a_{n-j}$ ($a_{n-j} \neq 0$) and if $a_j = 0$, then

$$a_{-m} = e^{i\psi} a_{n-m} \quad (m = 1, 2, \dots, n-1).$$

Theorem 3. ([1], [2]) The Toeplitz matrix T on $\mathbb{C}^n$ is normal if and only if, for some $\theta \in [0, 2\pi)$,

$$\begin{cases} a_{-m} = e^{i\theta} \overline{a_m} & (m = 1, 2, \dots, n-1) \quad (\text{type I}) \\ \text{or} & \\ a_{-m} = e^{i\theta} a_{n-m} & (m = 1, 2, \dots, n-1) \quad (\text{type II}). \end{cases}$$

Proof. By Theorem 2, we have only to prove that the equation (6) implies (type I) or (type II) because (6) is satisfied clearly for each case of a_{-k} 's such as (type I) and (type II).

(case 1) If $a_j = 0$ ($j = 1, 2, \dots, n-1$), then $a_{-j} = 0$ ($j = 1, 2, \dots, n-1$) by (6-4) and hence (type I) and (type II) occur at the same time.

(case 2) If there exists j such that $a_{-j} = e^{i\varphi} \overline{a_j}$ ($a_j \neq 0$) and that $a_{n-j} = 0$, then, by Lemma 4, (type I) occurs.

(case 3) If there exists j such that $a_{-j} = e^{i\psi} a_{n-j}$ ($a_{n-j} \neq 0$) and that $a_j = 0$, then, by Lemma 5, (type II) occurs.

If $a_j = 0$, then, by (6-3), $\overline{a_{-j}} a_{-(n-j)} = 0$ and we have $a_{-j} = 0$ or $a_{-(n-j)} = 0$. In the case where $a_{-j} \neq 0$, by (6-4), $a_{-j} = e^{i\varphi_j} \overline{a_j} = 0$ or $a_{-j} = e^{i\psi_j} a_{n-j}$ and we have $a_{(n-j)} \neq 0$ and hence (case 3) occurs. In the case where $a_{-(n-j)} \neq 0$, by (6-4), $a_{-(n-j)} = e^{i\varphi_{n-j}} \overline{a_{n-j}}$ or $a_{-(n-j)} = e^{i\psi_{n-j}} a_j = 0$ and we have $a_{(n-j)} \neq 0$ and hence (case 2) occurs. Therefore we may consider the case that $a_{-j} = a_{-(n-j)} = 0$. And then, by (6), we have $\overline{a_{n-j}} a_{n-k} = 0$ ($k = 1, 2, \dots, n-1$) and we may assume $a_{n-j} = 0$ because, in the other case, (case 1) occurs. And henceforth we may assume that

$$a_j = 0 \quad \text{implies} \quad a_{n-j} = 0 \quad \text{and then} \quad a_{-j} = a_{-(n-j)} = 0 \quad (\#)$$

by (6-4). And hence

$$\begin{aligned} & \text{the equations of (type I) and (type II)} \\ & \text{are satisfied for each } j \text{ such as } a_j = 0. \end{aligned} \quad (b)$$

And henceforth we may assume that there exists some j_0 such that $a_{j_0} \neq 0$ and $a_{n-j_0} \neq 0$. Then, by (6-4), (6-5) and (6-6), we have

$$\begin{cases} a_{-j_0} = e^{i\varphi_{j_0}} \overline{a_{j_0}} \\ a_{-(n-j_0)} = e^{i\varphi_{j_0}} \overline{a_{n-j_0}} \end{cases} \quad \text{or} \quad \begin{cases} a_{-j_0} = e^{i\psi_{j_0}} a_{n-j_0} \\ a_{-(n-j_0)} = e^{i\psi_{j_0}} a_{j_0}. \end{cases}$$

(case 4) If $a_j = 0$ for all $j \neq j_0$, then $a_{n-j} = a_{-j} = a_{-(n-j)} = 0$ for all $j \neq j_0$ by (#) and (type I) or (type II) occurs by (b).

(case 5) If there exists some j_1 such that $a_{j_1} \neq 0$ and that $|\frac{a_{n-j_0}}{a_{j_0}}| \neq |\frac{a_{n-j_1}}{a_{j_1}}|$, then, by Lemmas 1 and 2, (type I) or (type II) occurs because $a_{j_1} \neq 0$ implies $a_{n-j_1} \neq 0$ by (#).

(case 6) If there exists some j_2 such that $a_{j_2} \neq 0$, $|\frac{a_{n-j_0}}{a_{j_0}}| = |\frac{a_{n-j_2}}{a_{j_2}}| = |\frac{a_{j_2}}{a_{n-j_2}}|$ and that $\frac{a_{n-j_0}}{a_{j_0}} \neq \frac{a_{n-j_2}}{a_{j_2}}$, then, by Lemmas 3 and 2, (type I) or (type II) occurs because $a_{j_2} \neq 0$ implies $a_{n-j_2} \neq 0$ by (#).

(case 7) If $|\frac{a_{n-j_0}}{a_{j_0}}| = |\frac{a_{n-j}}{a_j}| = |\frac{a_j}{a_{n-j}}|$ and $\frac{a_{n-j_0}}{a_{j_0}} = \frac{a_{n-j}}{a_j}$ for all $j \neq j_0$ such as $a_j \neq 0$, then, by Lemma 3 and by (b), (type I) and (type II) occur at the same time because $a_j \neq 0$ implies $a_{n-j} \neq 0$ by (#).

References

- [1] Farenik, D. R., Krupnik, M., Krupnik, N. and Lee, W. Y., *Normal Toeplitz matrices*, SIAM J. Matrix Anal. Appl., 17(1996), pp. 1037–1043.
- [2] Ito, T., *Every normal Toeplitz matrix is either of type I or of type II*, SIAM J. Matrix Anal. Appl., 17(1996), pp. 998–1006.