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AN INEQUALITY OF NOETHER TYPE FOR ALGEBRAIC THREEFOLDS

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Abstract. We study the Noether type of inequality for projective three folds of general type.

1. Introduction

Let X be a minimal projective 3-fold of general type. It has been an open problem whether the inequality if true:

$$K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}? \tag{1.1}$$ {ineq}

Here is a brief history of this problem:

(1) M. Kobayashi [10] constructed some examples satisfying $K^3 = \frac{4}{3}p_g - \frac{10}{3}$ in 1992;
(2) M. Chen [5] proved Inequality (1.1) for canonically polarized 3-folds in 2004;
(3) Catanese–Chen–Zhang [1] proved Inequality (1.1) for smooth minimal 3-folds of general type in 2006;
(4) J. Chen and M. Chen [3] proved Inequality (1.1) for Gorenstein minimal 3-folds of general type in 2015.

The aim of this talk is to announce our main statement that Inequality (1.1) is true.

2. The discrepancy of a special resolution for linear systems

First of all, we recall the following result of the first author:

{jk}

Theorem 2.1. *(*[2, Theorem 1.3]*) Let X be an algebraic 3-fold with at worst terminal singularities. For any terminal singularity $P \in X$, there exists a sequence of birational morphisms:*

$$\tau_P : Y = X_m \to X_{m-1} \to \ldots \to X_1 \to X_0 = X,$$

such that Y is smooth on $\tau_P^{-1}(P)$ and, for all i, the morphism $\pi_i : X_{i+1} \to X_i$ is a divisorial contraction to a singular point $P_i \in X_i$ of index $r_i \geq 1$ with discrepancy $1/r_i$.

Definition 2.2. Given a terminal singularity $P \in X$, the birational morphism $\tau_P : Y \to X \ni P$ constructed as above is called a *feasible resolution of $P \in X$*.

Suppose that $|M|$ is a moving linear system (i.e. without fixed part) on the given projective terminal 3-fold X with $\mathrm{Bs}|M| \neq \emptyset$. Similar to usual elimination of indeterminancies, we can have a *feasible elimination of indeterminancies* as follows:

(0) Given a terminal singularity $P \in X$, we may define

$$d(P \in X) := \min\{m | X_m \to \ldots \to X_1 \to X_0 \text{ is a feasible resolution.}\},$$

and set

$$d(X) := \sum_{P \in \mathrm{Sing}(X)} d(P \in X).$$

Note that $d(X) = 0$ if and only if X is non-singular.

(i) If $\mathrm{Bs}|M| \cap \mathrm{Sing}(X) \neq \emptyset$, then there is a singular point $P \in \mathrm{Bs}|M|$. We take the first map of the feasible resolution $X_1 \to X \ni P$ and consider the linear system $|M_1|$, where M_1 is the proper transform of M on X_1. Note that $d(X_1) = d(X) - 1$.

(ii) By induction on $d(X)$, this process must terminates in finite steps. We will end up with a partial resolutions $Y = X_k \to \ldots \to X_1 \to X$ so that $\mathrm{Bs}|M_Y| \cap \mathrm{Sing}(Y) = \emptyset$, where M_Y is the proper transform of M on Y.

(iii) If $\mathrm{Bs}|M_Y| \neq \emptyset$, then $\mathrm{Bs}|M_Y|$ consists of smooth points. We then consider the usual elimination of indeterminancies over $\mathrm{Bs}|M_Y|$, say $Z = X_n \to \ldots \to X_k = Y$, which is composed of a sequence of blow-ups along smooth points or curves by Hironaka's big theorem.

(iv) Thus we end up with a possibly singular 3-fold $Z = X_n$, so that $|M_n|$ is base point free. We call

{Gres} $$\mu \colon Z = X_n \to \ldots \to X_k = Y \to \ldots \to X \tag{2.1}$$

a feasible elimination of indeterminancies of $|M|$. Note that a general member $S \in |M_Z|$ is smooth by Bertini's Theorem.

For any $i > 0$, let E_i be the exceptional divisor of $X_i \to X_{i-1}$. Let K_i be the canonical divisor of X_i. For $i > j$ we write $K_{X_i/X_j} = K_i - \pi_{i,j}^* K_j$, where $\pi_{i,j} : X_i \to X_j$ is the induced map. We also denote $K_{Z/X} := K_Z - \mu^*(K_X)$ and $K_{Y/X}$ similarly.

Given a $\mathbb{Q}$-Cartier divisor D on X, let D_i be the proper transform of D in X_i. Similarly, we define $D_{X_i/X_j} := \pi_{i,j}^* D_j - D_i$ write $D_{Z/X} := \mu^*(D) - D_Z$. $D_{Y/X}$ is defined similarly.

{key}

Theorem 2.3. *Let $|M|$ be a moving linear system on a projective terminal 3-fold X and $D \in |M|$ be a general member. Let $\mu : Z = X_n \to X$ be the feasible elimination of indeterminancies as in (2.1). Then $D_{Y/X} \geq K_{Y/X}$ and $2D_{Z/Y} \geq K_{Z/Y}$.*

Lemma 2.4. *Keep the notation as above. Suppose that $\alpha_i + \beta_i = a(D_{Z/X}, E_i) + a(K_{Z/X}, E_i) \leq 2$, then $i \in J$.*

3. **The case of canonical family of curves**

Let X be a projective minimal 3-fold of general type. We may assume that $p_g \geq 3$ and always consider the non-trivial canonical map φ_1. Set $d := \dim \overline{\varphi_1(X)}$.

The following inequalities are already known:

I. if $p_g(X) \geq 3$, then $K_X^3 \geq 1$ and if $p_g(X) \geq 4$, then $K_X^3 \geq 2$ (cf. [7, Theorem 1.5]).

II. If $d = 2$ and X is canonically fibred by curves C of genus $g(C) \geq 3$, then $K_X^3 \geq 2p_g(X) - 4$ by [6, Theorem 4.1(ii)].

From now on, we consider $d = 2$ and X is canonically fibred by curves C of genus $g(C) = 2$.

{g2}

Theorem 3.1. *Let X be a projective minimal smooth 3-fold of general type. Suppose that $d = 2$ and X is canonically fibred by curves of genus 2. Then*

$$K_X^3 \geq \frac{1}{3}(4p_g(X) - 10).$$

The inequality is sharp.

4. **The case of canonical family of surfaces**

Assume $d = 1$. We have an induced fibration $f : X' \longrightarrow \Gamma$. Take a general fiber F of f. By assumption, we know that $H^0(X', f_*\omega_{X'})$ naturally generate an invertible sheaf $\mathcal{L} \subset f_*\omega_{X'}$. We may always assume $p_g(X) \geq 5$. Thus, by [4, Theorem 1], we have

$$\text{either} \quad 0 \leq b := g(\Gamma) \leq 1 \qquad (4.1)$$ {b2}

$$\text{or} \quad b > 1 \text{ and } p_g(F) = 1. \qquad (4.2)$$ {b1}

Denote by $\sigma : F \to F_0$ the birational contraction onto the minimal model. Recall that we have $\pi^*(K_X) \sim M + E' \equiv \theta F + E'$ where

$$\theta := \deg \mathcal{L} \geq p_g(X) - 1,$$

$|M| := \mathrm{Mov}|K_{X'}|$ and E' is an effective $\mathbb{Q}$-divisor. When $K_{F_0}^2 \geq 2$, by Chen–Zhang [8, Lemma 3.7, Lemma 4.7], we have

$$K_X^3 \geq (1 - \frac{1}{p_g(X)})^2 \cdot K_{F_0}^2 \cdot (p_g(X) - 1) > \frac{4}{3}p_g(X) - \frac{10}{3}.$$

Since $p_g(X) > 0$, we have $p_g(F) > 0$. Thus, when $K_{F_0}^2 = 1$, the Noether inequality for surfaces implies $1 \leq p_g(F) \leq 2$.

{1,1}

Theorem 4.1. *Let X be a minimal projective 3-fold of general type. Assume $p_g(X) \geq 5$ and $d = 1$. If F is a $(1,1)$ surface, then $K_X^3 \geq \frac{27}{20}p_g(X) - \frac{9}{5}$.*

Remark 4.2. Since we are only concerned with the inequality $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$, Theorem 4.1 may be improved to some extent.

Theorem 4.3. *(=Claim) Let X be a minimal projective 3-fold of general type. Assume $p_g(X) \geq 5$ and $d = 1$. If F is a $(1, 2)$ surface, then $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$.* {1,2}

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