

The Elliptic Quantum Group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

By

Hitoshi KONNO*

Abstract

We survey recent results on a formulation of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ as an H -Hopf algebroid and its representation theory. We put emphasis on a connection of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and a constructive derivation of both finite and infinite-dimensional representations from those of $U_q(\widehat{\mathfrak{sl}}_2)$. Included is an announcement of a new result on a criterion for the finiteness of irreducible pseudo-highest weight representations stated in terms of an elliptic analogue of the Drinfeld polynomials. A derivation of the type I and II vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and its implication in the algebraic analysis of elliptic solvable lattice models are also explained.

§ 1. Introduction

Theory of elliptic quantum groups has been developed in the two different approaches, the one based on H -Hopf algebroids [10] and the other on quasi-Hopf algebras [8].

The H -Hopf algebroid was introduced by Etingof and Varchenko [10], motivated by the work of Felder and Varchenko [12, 13]. There are some structures added by Koelink and Rosengren [19, 31]. See also a survey by van Norden [34]. A similar coalgebra structure was introduced by Lu [30] and Xu [35]. As an H -Hopf algebroid, Felder's elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$ was formulated in terms of the L operator satisfying the RLL relation associated with the elliptic dynamical R matrices [12, 13, 10, 20].

The quasi-Hopf algebra formulation was carried out by Jimbo, Konno, Odake and Shiraishi [17] motivated by the works of Drinfeld [8], Babelon, Bernard and Billey [2] and Frønsdal [15]. There are two types of elliptic quantum groups (quasi-Hopf algebras),

Received February 13, 2008. Accepted October 23, 2008.

2000 Mathematics Subject Classification(s): 16W30, 17B37, 17B67, 17B69, 81R50

Key Words: Elliptic quantum group, Hopf algebroid, Dynamical quantum group

Supported by the Grant-in-Aid for Scientific Research (C)19540033, JSPS Japan.

*Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima 739-8521, Japan.

e-mail: konno@mis.hiroshima-u.ac.jp

© 2009 Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.

the vertex type $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and the face type $\mathcal{B}_{q,\lambda}(\mathfrak{g})$, where $\mathfrak{g}$ is an affine Lie algebra. p is a complex parameter giving the nome of the related elliptic functions. λ denotes a Cartan subalgebra valued parameter which provides the elliptic nome and the dynamical parameters. Both $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ are isomorphic to the corresponding quantum affine algebras $U_q(\mathfrak{g})$ as associative algebras, but their coalgebra structures are deformed from $U_q(\mathfrak{g})$ by the twistors $E(r)$ and $F(\lambda)$, respectively [2, 15, 17]. Here r is related to p by $p = q^{2r}$. Felder's elliptic quantum group also has a formulation as a quasi-Hopf algebra [9].

The classification of the vertex and the face types is based on the fact that the vector representation of the universal dynamical R matrix of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ yields Baxter's elliptic R matrix for the eight-vertex model [3], whereas the one of $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ [17, 21, 22, 25] yields the face type elliptic Boltzmann weight of the SOS face model associated with $\mathfrak{g}$ [1, 16]. Since the latter Boltzmann weights are nothing but the elliptic dynamical R matrices used in Felder's elliptic quantum groups, we classify Felder's ones as the face type. See also [4] for a universal formulation of the vertex-face correspondence in the quasi-Hopf algebra scheme.

Each approach has advantages and disadvantages. An advantage of the quasi-Hopf algebra is that each of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ has an apparent connection to $U_q(\mathfrak{g})$ by the twist procedure. In particular, we can formulate both algebraic and representation theoretical objects of the quasi-Hopf algebra, such as the comultiplication, the universal dynamical R matrices and the vertex operators, from the corresponding objects of $U_q(\mathfrak{g})$ [17]. However a disadvantage is a complication of the coalgebra structure due to the twist procedure mentioned above, so that it is not suitable for a practical calculation.

In contrast, the coalgebra structure of the known H -Hopf algebroid is simple enough for practical use. In fact it was already applied to a study of tensor product representations [13] and of co-representations [19, 20]. In particular, by studying the co-representation, Koelink, van Norden and Rosengren have succeeded to derive the terminating very-well-poised balanced elliptic hypergeometric series ${}_{12}V_{11}$ and their biorthogonality relations, which were introduced by Frenkel-Turaev [14] and developed by Spiridonov and Zhedanov [32, 33]. However a disadvantage is a lack of direct connection to $U_q(\mathfrak{g})$ even as an associative algebra. This defect seems to be an obstacle to extend the known H -Hopf algebroids associated with finite-dimensional simple Lie algebras to those associated with affine Lie algebras and to develop their representation theory in systematic way.

In this paper, we explain a new realization of the face type elliptic quantum group given by the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [26, 27]. It is a realization by the Drinfeld generators of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and has an H -Hopf algebroid structure, so that it provides a complement to the above two approaches.

The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ was introduced in [24] as an elliptic analogue of the algebra of the Drinfeld currents for $U_q(\widehat{\mathfrak{sl}}_2)$. As an associative algebra, $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is isomorphic to the tensor product of $U_q(\widehat{\mathfrak{sl}}_2)$ and the Heisenberg algebra $\{P, e^Q\}$ [18]. A similar algebra was studied in [35]. The generators of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ are treated through the generating functions called the elliptic currents. In terms of the elliptic currents, we can construct the L operator and derive the RLL relation for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [18]. It turns out that the resultant RLL relation is nothing but the one for Felder's elliptic quantum group with a central extension. We hence formulate the H -Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in a way similar to [10, 20], but with a modification due to the existence of the non-zero central element.

Due to a direct connection to $U_q(\widehat{\mathfrak{sl}}_2)$, we can derive all representations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ constructively from those of $U_q(\widehat{\mathfrak{sl}}_2)$. This yields quite a parallel structure to $U_q(\widehat{\mathfrak{sl}}_2)$ for both finite and infinite -dimensional representations. In particular, we can state a criterion for the finiteness of irreducible pseudo-highest weight representations in terms of an elliptic analogue of the Drinfeld polynomials. This provides an elliptic analogue of the works by Drinfeld[7] and by Chari and Pressley[5]. As an example of the application of infinite-dimensional representations, we also report on a formulation and derivation of the type I and II vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ studied in [26].

This paper is organized as follows. In the next section, we review the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. The L operator and the RLL relation are also introduced. In Sect.3, we describe an H -Hopf algebroid structure of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ following [26, 27]. In Sect.4 we summarize some basic facts on the dynamical representations. Theorem 4.2 is fundamental in a constructive derivation of the dynamical representations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Sect.5 is devoted to a study of finite-dimensional irreducible pseudo-highest weight representations and includes an announcement of new results. In Sect.6, we report a result on the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ following [26].

In the forthcoming paper [27], we plan to provide proofs of the statements in Sect. 3, 4, 5 and also report on a structure of the finite-dimensional tensor product representations as well as an alternative derivation of the ${}_{12}V_{11}$.

§ 2. The Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

In this section we review a definition of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and its RLL relation following [18, 26, 27].

§ 2.1. Definition of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

Let us fix a complex number q such that $q \neq 0, |q| < 1$.

Definition 2.1. [8] For a field $\mathbb{K}(\supset \mathbb{C})$, the quantum affine algebra $\mathbb{K}[U_q(\widehat{\mathfrak{sl}}_2)]$ in the Drinfeld realization is an associative algebra over $\mathbb{K}$ generated by the standard

Drinfeld generators a_n ($n \in \mathbb{Z}_{\neq 0}$), $x_n^\pm$ ($n \in \mathbb{Z}$), h, c, d . The defining relations are given as follows.

$$\begin{aligned}
& c : \text{central}, \\
& [h, d] = 0, \quad [d, a_n] = na_n, \quad [d, x_n^\pm] = nx_n^\pm, \\
& [h, a_n] = 0, \quad [h, x^\pm(z)] = \pm 2x^\pm(z), \\
& [a_n, a_m] = \frac{[2n]_q [cn]_q}{n} q^{-c|n|} \delta_{n+m, 0}, \\
& [a_n, x^+(z)] = \frac{[2n]_q}{n} q^{-c|n|} z^n x^+(z), \\
& [a_n, x^-(z)] = -\frac{[2n]_q}{n} z^n x^-(z), \\
& (z - q^{\pm 2} w) x^\pm(z) x^\pm(w) = (q^{\pm 2} z - w) x^\pm(w) x^\pm(z), \\
& [x^+(z), x^-(w)] = \frac{1}{q - q^{-1}} \left(\delta(q^{-c} \frac{z}{w}) \psi(q^{c/2} w) - \delta(q^c \frac{z}{w}) \varphi(q^{-c/2} w) \right).
\end{aligned}$$

where we use $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ and the Drinfeld currents defined by

$$\begin{aligned}
x^\pm(z) &= \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n}, \\
\psi(q^{c/2} z) &= q^h \exp \left((q - q^{-1}) \sum_{n > 0} a_n z^{-n} \right), \quad \varphi(q^{-c/2} z) = q^{-h} \exp \left(-(q - q^{-1}) \sum_{n > 0} a_{-n} z^n \right).
\end{aligned}$$

We also denote by $\mathbb{K}[U'_q(\widehat{\mathfrak{sl}}_2)]$ the subalgebra of $\mathbb{K}[U_q(\widehat{\mathfrak{sl}}_2)]$ generated by the same generators as $\mathbb{K}[U_q(\widehat{\mathfrak{sl}}_2)]$ except d .

Remark. We follow the conventions in [18]. In particular, we occasionally treat c as a complex number on the understanding that we make a specialization each time.

Let r be a generic complex number. We set $r^* = r - c$, $p = q^{2r}$ and $p^* = q^{2r^*}$. We define the Jacobi theta functions $[u]$ and $[u]^*$ by

$$\begin{aligned}
[u] &= \frac{q^{\frac{u^2}{r} - u}}{(p; p)_\infty^3} \Theta_p(q^{2u}), \quad [u]^* = \frac{q^{\frac{u^2}{r^*} - u}}{(p^*; p^*)_\infty^3} \Theta_{p^*}(q^{2u}), \\
\Theta_p(z) &= (z; p)_\infty (p/z; p)_\infty (p; p)_\infty,
\end{aligned}$$

where

$$(z; p_1, p_2, \dots, p_m)_\infty = \prod_{n_1, n_2, \dots, n_m=0}^{\infty} (1 - zp_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}).$$

Setting $p = e^{-\frac{2\pi i}{\tau}}$, $[u]$ satisfies the quasi-periodicity $[u+r] = -[u]$, $[u+r\tau] = -e^{-\pi i(2u/r+\tau)}[u]$.

Let $\{P, e^Q\}$ be a Heisenberg algebra commuting with $\mathbb{C}[U_q(\widehat{\mathfrak{sl}}_2)]$ and satisfying

$$(2.1) \quad [P, e^Q] = -e^Q.$$

We take the realization $Q = \frac{\partial}{\partial P}$. We set $H = \mathbb{C}P \oplus \mathbb{C}r^*$ and $H^* = \mathbb{C}Q \oplus \mathbb{C}\frac{\partial}{\partial r^*}$ with the pairing $\langle \cdot, \cdot \rangle$.

$$\langle Q, P \rangle = 1 = \langle \frac{\partial}{\partial r^*}, r^* \rangle,$$

the others are zero. We also consider the abelian group $\bar{H}^* = \mathbb{Z}Q$. We denote by $\mathbb{C}[\bar{H}^*]$ the group algebra over $\mathbb{C}$ of $\bar{H}^*$. We denote by e^α the element of $\mathbb{C}[\bar{H}^*]$ corresponding to $\alpha \in \bar{H}^*$. These e^α satisfy $e^\alpha e^\beta = e^{\alpha+\beta}$ and $(e^\alpha)^{-1} = e^{-\alpha}$. In particular, $e^0 = 1$ is the identity element.

Let M_{H^*} be the field of meromorphic functions on H^* . We regard a meromorphic function $\hat{f} = f(P, r^*)$ of P and r^* as an element of M_{H^*} by $\hat{f}(\mu) = f(\langle \mu, P \rangle, \langle \mu, r^* \rangle)$ $\mu \in H^*$.

Now we take the field $\mathbb{F} = M_{H^*}$ as $\mathbb{K}$ and consider the semi-direct product $\mathbb{C}$ -algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2) = \mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*]$ of $\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]$ and $\mathbb{C}[\bar{H}^*]$, whose multiplication is defined by

$$\begin{aligned} (f(P, r^*)a \otimes e^\alpha) \cdot (g(P, r^*)b \otimes e^\beta) &= f(P, r^*)g(P + \langle \alpha, P \rangle, r^*)ab \otimes e^{\alpha+\beta}, \\ a, b &\in \mathbb{C}[U_q(\widehat{\mathfrak{sl}}_2)], f(P, r^*), g(P, r^*) \in \mathbb{F}, \alpha, \beta \in \bar{H}^*. \end{aligned}$$

The following automorphism ϕ_r of $\mathbb{C}[U_q(\widehat{\mathfrak{sl}}_2)]$ is the key to our “elliptic deformation” [18].

$$\begin{aligned} c &\mapsto c, & h &\mapsto h, & d &\mapsto d, \\ x^+(z) &\mapsto u^+(z, p)x^+(z), & x^-(z) &\mapsto x^-(z)u^-(z, p), \\ \psi(z) &\mapsto u^+(q^{c/2}z, p)\psi(z)u^-(q^{-c/2}z, p), \\ \varphi(z) &\mapsto u^+(q^{-c/2}z, p)\varphi(z)u^-(q^{c/2}z, p). \end{aligned}$$

Here we set

$$u^+(z, p) = \exp \left(\sum_{n>0} \frac{1}{[r^*n]_q} a_{-n}(q^r z)^n \right), \quad u^-(z, p) = \exp \left(- \sum_{n>0} \frac{1}{[rn]_q} a_n(q^{-r} z)^{-n} \right).$$

Definition 2.2. We define the elliptic currents $E(u), F(u), K(u) \in U_{q,p}(\widehat{\mathfrak{sl}}_2)[[u]]$

and $\hat{d}$ by

$$\begin{aligned} E(u) &= \phi_r(x^+(z))e^{2Q}z^{-\frac{P-1}{r^*}}, \\ F(u) &= \phi_r(x^-(z))z^{\frac{P+h-1}{r}}, \\ K(u) &= \exp\left(\sum_{n>0}\frac{[n]_q}{[2n]_q[r^*n]_q}a_{-n}(q^cz)^n\right)\exp\left(-\sum_{n>0}\frac{[n]_q}{[2n]_q[rn]_q}a_nz^{-n}\right) \\ &\quad \times e^Qz^{-\frac{c}{4rr^*}(2P-1)+\frac{1}{2r}h}, \\ \hat{d} &= d - \frac{1}{4r^*}(P-1)(P+1) + \frac{1}{4r}(P+h-1)(P+h+1), \end{aligned}$$

where we set $z = q^{2u}$.

From Definition 2.1 and (2.1), we can derive the following relations.

Proposition 2.3.

c : central,

$$\begin{aligned} [h, a_n] &= 0, \quad [h, E(u)] = 2E(u), \quad [h, F(u)] = -2F(u), \\ [\hat{d}, h] &= 0, \quad [\hat{d}, a_n] = na_n, \\ [\hat{d}, E(u)] &= \left(-z\frac{\partial}{\partial z} - \frac{1}{r^*}\right)E(u), \quad [\hat{d}, F(u)] = \left(-z\frac{\partial}{\partial z} - \frac{1}{r}\right)F(u), \\ [a_n, a_m] &= \frac{[2n]_q[cn]_q}{n}q^{-c|n|}\delta_{n+m,0}, \\ [a_n, E(u)] &= \frac{[2n]_q}{n}q^{-c|n|}z^nE(u), \\ [a_n, F(u)] &= -\frac{[2n]_q}{n}z^nF(u), \end{aligned}$$

$$\begin{aligned} E(u)E(v) &= \frac{[u-v+1]^*}{[u-v-1]^*}E(v)E(u), \\ F(u)F(v) &= \frac{[u-v-1]}{[u-v+1]}F(v)F(u), \\ [E(u), F(v)] &= \frac{1}{q-q^{-1}}\left(\delta\left(q^{-c}\frac{z}{w}\right)H^+(q^{c/2}w) - \delta\left(q^c\frac{z}{w}\right)H^-(q^{-c/2}w)\right), \end{aligned}$$

where $z = q^{2u}$, $w = q^{2v}$,

$$\begin{aligned} H^\pm(z) &= \kappa K\left(u \pm \frac{1}{2}\left(r - \frac{c}{2}\right) + \frac{1}{2}\right)K\left(u \pm \frac{1}{2}\left(r - \frac{c}{2}\right) - \frac{1}{2}\right), \\ \kappa &= \lim_{z \rightarrow q^{-2}} \frac{\xi(z; p^*, q)}{\xi(z; p, q)}, \quad \xi(z; p, q) = \frac{(q^2z; p, q^4)_\infty (pq^2z; p, q^4)_\infty}{(q^4z; p, q^4)_\infty (pz; p, q^4)_\infty}. \end{aligned}$$

Definition 2.4. We call a pair $(\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*], \phi_r)$ the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. We also denote by $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ the subalgebra $\mathbb{F}[U'_q(\widehat{\mathfrak{sl}}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*]$ of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

The following relations are crucial in the formulation of the H -Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. See Sect.3.2.

Proposition 2.5.

$$\begin{aligned} [K(u), P] &= K(u), & [E(u), P] &= 2E(u), & [F(u), P] &= 0, \\ [K(u), P + h] &= K(u), & [E(u), P + h] &= 0, & [F(u), P + h] &= 2F(u). \end{aligned}$$

§ 2.2. The RLL -relation for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

In order to define the L operator, we need the half currents defined by the following formulae.

Definition 2.6.

$$\begin{aligned} K^+(u) &= K(u + \frac{r+1}{2}), \\ E^+(u) &= a^* \oint_{C^*} E(u') \frac{[u - u' + c/2 - P + 1]^* [1]^*}{[u - u' + c/2]^* [P - 1]^*} \frac{dz'}{2\pi i z'}, \\ F^+(u) &= a \oint_C F(u') \frac{[u - u' + P + h - 1] [1]}{[u - u'] [P + h - 1]} \frac{dz'}{2\pi i z'}. \end{aligned}$$

Here $z' = q^{2u'}$ and $z = q^{2u}$. The contours C^* and C are chosen in such a way that C^* encircles $zq^c p^{*n}$ ($n \geq 1$) but not $zq^c p^{*n}$ ($n \leq 0$), C encircles zp^n ($n \geq 1$) but not zp^n ($n \leq 0$), respectively. The constants a, a^* are chosen to satisfy $\frac{a^* a [1]^* \kappa}{q - q^{-1}} = 1$.

Then we define the operator $\widehat{L}^+(u) \in \text{End}_{\mathbb{C}} V \otimes U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with $V = \mathbb{C}^2$ as follows.

Definition 2.7.

$$\widehat{L}^+(u) = \begin{pmatrix} 1 & F^+(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K^+(u-1) & 0 \\ 0 & K^+(u)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ E^+(u) & 1 \end{pmatrix}.$$

From the relations in Proposition 2.3, we obtain the RLL relation for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Proposition 2.8. *The operator $\widehat{L}^+(u)$ satisfies the following RLL relation.*

$$\begin{aligned} (2.2) \quad & R^{+(12)}(u_1 - u_2, P + h) \widehat{L}^{+(1)}(u_1) \widehat{L}^{+(2)}(u_2) \\ &= \widehat{L}^{+(2)}(u_2) \widehat{L}^{+(1)}(u_1) R^{+*(12)}(u_1 - u_2, P), \end{aligned}$$

where $R^+(u, P + h)$ denotes the elliptic dynamical R matrices given by

$$(2.3) \quad R^+(u, s) = \rho^+(u) \begin{pmatrix} 1 & & & \\ & b(u, s) c(u, s) & & \\ & \bar{c}(u, s) \bar{b}(u, s) & & \\ & & & 1 \end{pmatrix}$$

with

$$\begin{aligned} \rho^+(u) &= z^{\frac{1}{2r}} \frac{\{pq^2z\}^2}{\{pz\}\{pq^4z\}} \frac{\{z^{-1}\}\{q^4z^{-1}\}}{\{q^2z^{-1}\}^2}, \quad \{z\} = (z; p, q^4)_\infty, \\ b(u, s) &= \frac{[s+1][s-1]}{[s]^2} \frac{[u]}{[1+u]}, \quad c(u, s) = \frac{[1]}{[s]} \frac{[s+u]}{[1+u]}, \\ \bar{c}(u, s) &= \frac{[1]}{[s]} \frac{[s-u]}{[1+u]}, \quad \bar{b}(u, s) = \frac{[u]}{[1+u]}, \end{aligned}$$

and $R^{+*}(u, P)$ denotes the R matrix obtained from $R^+(u, P)$ by the replacements $r \rightarrow r^*, p \rightarrow p^*$ and $[\cdot] \rightarrow [\cdot]^*$.

It is also worth while noting the following proposition immediately obtained from Proposition 2.8, which indicates a connection between $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and the quasi-Hopf algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$.

Proposition 2.9. [18] *Let us set $L^+(u, P) = \widehat{L}^+(u) e^{-h \otimes Q}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $L^+(u, P)$ is independent of Q and satisfies the following dynamical RLL relation.*

$$(2.4) \quad \begin{aligned} &R^{+(12)}(u_1 - u_2, P + h) L^{+(1)}(u_1, P) L^{+(2)}(u_2, P + h^{(1)}) \\ &= L^{+(2)}(u_2, P) L^{+(1)}(u_1, P + h^{(2)}) R^{+*(12)}(u_1 - u_2, P). \end{aligned}$$

This is the same dynamical RLL relation that characterizes the quasi-Hopf algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ with the parametrization $\lambda = (r^* + 2)\Lambda_0 + (P + 1)\bar{\Lambda}_1$ [18]. In fact, under this parametrization, the vector representation of the universal dynamical R matrix $\mathcal{R}(\lambda)$ of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ coincides with $R^{+*}(u, P)$ in (2.3). Recalling also that by definition $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ with the usual parameter $\lambda \in \mathfrak{h}$ is isomorphic to $\mathbb{C}[U_q(\widehat{\mathfrak{sl}}_2)]$ as an associative algebra, we have the isomorphism from $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ with $\lambda = (r^* + 2)\Lambda_0 + (P + 1)\bar{\Lambda}_1$ to $\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]$. Combining these facts, we have the isomorphism $U_{q,p}(\widehat{\mathfrak{sl}}_2) \cong \mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2) \otimes \mathbb{C}[\bar{H}^*]$ with $\lambda = (r^* + 2)\Lambda_0 + (P + 1)\bar{\Lambda}_1$ as a semi-direct product algebra.

Note also that the $c = 0$ case of (2.4) is the dynamical RLL relation studied by Felder [12, 9], whereas the $c = 0$ case of (2.2) is the RLL relation studied in [10, 11, 19] for the trigonometric R , and in [20] for the elliptic R .

§ 3. H -Hopf Algebroid Structure

This section is an exposition of the H -Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [26, 27]. Our H -Hopf algebroid is an extension of the one studied in [10, 11] and [19, 20] to the one with the central element c .

§ 3.1. Definition of the H -Hopf Algebroid [10, 11, 19]

Let A be a complex associative algebra, H be a finite dimensional commutative subalgebra of A , and M_{H^*} be the field of meromorphic functions on H^* the dual space of H .

Definition 3.1. An H -algebra is a complex associative algebra A with 1, which is bigraded over H^* , $A = \bigoplus_{\alpha, \beta \in H^*} A_{\alpha\beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r : M_{H^*} \rightarrow A_{00}$ (the left and right moment maps), such that

$$\mu_l(\widehat{f})a = a\mu_l(T_\alpha \widehat{f}), \quad \mu_r(\widehat{f})a = a\mu_r(T_\beta \widehat{f}), \quad a \in A_{\alpha\beta}, \quad \widehat{f} \in M_{H^*},$$

where T_α denotes the automorphism $(T_\alpha \widehat{f})(\lambda) = \widehat{f}(\lambda + \alpha)$ of M_{H^*} .

Definition 3.2. An H -algebra homomorphism is an algebra homomorphism $\pi : A \rightarrow B$ between two H -algebras A and B preserving the bigrading and the moment maps, i.e. $\pi(A_{\alpha\beta}) \subseteq B_{\alpha\beta}$ for all $\alpha, \beta \in H^*$ and $\pi(\mu_l^A(\widehat{f})) = \mu_l^B(\widehat{f})$, $\pi(\mu_r^A(\widehat{f})) = \mu_r^B(\widehat{f})$.

Let A and B be two H -algebras. The tensor product $A \widetilde{\otimes} B$ is the H^* -bigraded vector space with

$$(A \widetilde{\otimes} B)_{\alpha\beta} = \bigoplus_{\gamma \in H^*} (A_{\alpha\gamma} \otimes_{M_{H^*}} B_{\gamma\beta}),$$

where $\otimes_{M_{H^*}}$ denotes the usual tensor product modulo the following relation.

$$(3.1) \quad \mu_r^A(\widehat{f})a \otimes b = a \otimes \mu_l^B(\widehat{f})b, \quad a \in A, b \in B, \widehat{f} \in M_{H^*}.$$

The tensor product $A \widetilde{\otimes} B$ is again an H -algebra with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ and the moment maps

$$\mu_l^{A \widetilde{\otimes} B} = \mu_l^A \otimes 1, \quad \mu_r^{A \widetilde{\otimes} B} = 1 \otimes \mu_r^B.$$

Let $\mathcal{D}$ be the algebra of automorphisms $M_{H^*} \rightarrow M_{H^*}$

$$\mathcal{D} = \left\{ \sum_i \widehat{f}_i T_{\beta_i} \mid \widehat{f}_i \in M_{H^*}, \beta_i \in H^* \right\}.$$

Equipped with the bigrading $\mathcal{D}_{\alpha\alpha} = \{ \widehat{f}T_{-\alpha} \mid \widehat{f} \in M_{H^*}, \alpha \in H^* \}$, $\mathcal{D}_{\alpha\beta} = 0$ ($\alpha \neq \beta$) and the moment maps $\mu_l^{\mathcal{D}}, \mu_r^{\mathcal{D}} : M_{H^*} \rightarrow \mathcal{D}_{00}$ defined by $\mu_l^{\mathcal{D}}(\widehat{f}) = \mu_r^{\mathcal{D}}(\widehat{f}) = \widehat{f}T_0$, $\mathcal{D}$ is an H -algebra. For any H -algebra A , we have the canonical isomorphism as an H -algebra

$$(3.2) \quad A \cong A \widetilde{\otimes} \mathcal{D} \cong \mathcal{D} \widetilde{\otimes} A$$

by $a \cong a \widetilde{\otimes} T_{-\beta} \cong T_{-\alpha} \widetilde{\otimes} a$ for all $a \in A_{\alpha\beta}$. Hence $\mathcal{D}$ plays the role of unit object in the category of H -algebras.

Definition 3.3. An H -bialgebroid is an H -algebra A equipped with two H -algebra homomorphisms $\Delta : A \rightarrow A \widetilde{\otimes} A$ (the comultiplication) and $\varepsilon : A \rightarrow \mathcal{D}$ (the counit) such that

$$\begin{aligned} (\Delta \widetilde{\otimes} \text{id}) \circ \Delta &= (\text{id} \widetilde{\otimes} \Delta) \circ \Delta, \\ (\varepsilon \widetilde{\otimes} \text{id}) \circ \Delta &= \text{id} = (\text{id} \widetilde{\otimes} \varepsilon) \circ \Delta, \end{aligned}$$

under the identification (3.2).

Definition 3.4. An H -Hopf algebroid is an H -bialgebroid A equipped with a $\mathbb{C}$ -linear map $S : A \rightarrow A$ (the antipode), such that

$$\begin{aligned} S(\mu_r(\widehat{f})a) &= S(a)\mu_l(\widehat{f}), \quad S(a\mu_l(\widehat{f})) = \mu_r(\widehat{f})S(a), \quad \forall a \in A, \widehat{f} \in M_{H^*}, \\ m \circ (\text{id} \widetilde{\otimes} S) \circ \Delta(a) &= \mu_l(\varepsilon(a)1), \quad \forall a \in A, \\ m \circ (S \widetilde{\otimes} \text{id}) \circ \Delta(a) &= \mu_r(T_\alpha(\varepsilon(a)1)), \quad \forall a \in A_{\alpha\beta}, \end{aligned}$$

where $m : A \widetilde{\otimes} A \rightarrow A$ denotes the multiplication and $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1 \in M_{H^*}$.

§ 3.2. H -Hopf Algebroid Structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

Now let us consider the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. The commutative subalgebra H is given in §2.1. Let $\bar{\mathfrak{h}} = \mathbb{C}h$ be the Cartan subalgebra, α_1 the simple root and $\bar{\Lambda}_1$ the fundamental weight of $\mathfrak{sl}(2, \mathbb{C})$. We set $\mathcal{Q} = \mathbb{Z}\alpha_1$ and $\bar{\mathfrak{h}}^* = \mathbb{C}\bar{\Lambda}_1$. Let $\langle, \rangle$ be the standard pairing of $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}^*$. Using the isomorphism $\phi : \mathcal{Q} \rightarrow \bar{H}^*$ by $n\alpha_1 \mapsto nQ$, we define the $\bar{H}^*$ -bigrading of $U_{q,p} = U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by

$$(3.3) \quad \begin{aligned} U_{q,p} &= \bigoplus_{\alpha, \beta \in \bar{H}^*} (U_{q,p})_{\alpha\beta}, \\ (U_{q,p})_{\alpha\beta} &= \left\{ x \in U_{q,p} \left| \begin{array}{l} q^h x q^{-h} = q^{\langle \phi^{-1}(\alpha-\beta), h \rangle} x, \\ q^P x q^{-P} = q^{\langle \beta, P \rangle} x \end{array} \right. \right\}. \end{aligned}$$

Noting $\langle \phi^{-1}(\alpha), h \rangle = \langle \alpha, P \rangle$, we have $q^{P+h} x q^{-(P+h)} = q^{\langle \alpha, P \rangle} x$ for $x \in (U_{q,p})_{\alpha\beta}$.

Let M_{H^*} be the field of meromorphic functions given in §2.1. We define two moment maps $\mu_l, \mu_r : M_{H^*} \rightarrow (U_{q,p})_{00}$ as follows

$$(3.4) \quad \mu_l(\widehat{f}) = f(P + h, r^* + c), \quad \mu_r(\widehat{f}) = f(P, r^*).$$

From (3.3), one finds for $x \in (U_{q,p})_{\alpha\beta}$

$$\begin{aligned} \mu_l(\widehat{f})x &= f(P + h, r^* + c)x = xf(P + h + \langle \alpha, P \rangle, r^* + c) = x\mu_l(T_\alpha \widehat{f}), \\ \mu_r(\widehat{f})x &= f(P, r^*)x = xf(P + \langle \beta, P \rangle, r^*) = x\mu_r(T_\beta \widehat{f}), \end{aligned}$$

where $T_\alpha = e^\alpha$ denotes a shift operator $M_{H^*} \rightarrow M_{H^*}$ defined by

$$(T_\alpha \widehat{f}) = e^\alpha f(P, r^*) e^{-\alpha} = f(P + \langle \alpha, P \rangle, r^*).$$

Equipped with the bigrading structure (3.3) and the two moment maps (3.4), the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is an H -algebra.

We also have the H -algebra $\mathcal{D}$ of the shift operators on M_{H^*}

$$\mathcal{D} = \left\{ \sum_i \widehat{f}_i T_{\alpha_i} \mid \widehat{f}_i \in M_{H^*}, \alpha_i \in \bar{H}^* \right\}$$

whose bigrading structure and moment maps are given as in §3.1.

The tensor product among the H -algebras $U_{q,p}$ and $\mathcal{D}$ is defined as in §3.1. In particular, we have the H -algebra isomorphism $U_{q,p} \widetilde{\otimes} \mathcal{D} \cong U_{q,p} \cong \mathcal{D} \widetilde{\otimes} U_{q,p}$ by $x \widetilde{\otimes} T_{-\beta} = x = T_{-\alpha} \widetilde{\otimes} x$ for $x \in (U_{q,p})_{\alpha\beta}$.

Now let us consider the coalgebra structure of $U_{q,p}$. Let $\widehat{L}^+(u)$ be the L operator introduced in Sect.2.2. We write the entries of $\widehat{L}^+(u)$ as

$$\widehat{L}^+(u) = \begin{pmatrix} \widehat{L}_{++}^+(u) & \widehat{L}_{+-}^+(u) \\ \widehat{L}_{-+}^+(u) & \widehat{L}_{--}^+(u) \end{pmatrix}.$$

From Proposition 2.5 and Definition 2.7, one finds

$$\widehat{L}_{\varepsilon_1 \varepsilon_2}^+(u) \in (U_{q,p})_{-\varepsilon_1 Q, -\varepsilon_2 Q}.$$

It is also easy to check the relations

$$\begin{aligned} f(P + h, r^* + c) \widehat{L}_{\varepsilon_1 \varepsilon_2}^+(u) &= \widehat{L}_{\varepsilon_1 \varepsilon_2}^+(u) f(P + h - \varepsilon_1, r^* + c), \\ f(P, r^*) \widehat{L}_{\varepsilon_1 \varepsilon_2}^+(u) &= \widehat{L}_{\varepsilon_1 \varepsilon_2}^+(u) f(P - \varepsilon_2, r^*). \end{aligned}$$

Definition 3.5. We define H -algebra homomorphisms, $\varepsilon : U_{q,p} \rightarrow \mathcal{D}$ and $\Delta :$

$U_{q,p} \rightarrow U_{q,p} \widetilde{\otimes} U_{q,p}$ by

$$\begin{aligned}\varepsilon(\widehat{L}_{\varepsilon_1 \varepsilon_2}^+(u)) &= \delta_{\varepsilon_1, \varepsilon_2} T_{-\varepsilon_2 Q}, \\ \varepsilon(e^Q) &= e^Q, \quad \varepsilon(\mu_l(\widehat{f})) = \varepsilon(\mu_r(\widehat{f})) = f(P, r^*)T_0, \\ \Delta(\widehat{L}_{\varepsilon_1 \varepsilon_2}^+(u)) &= \sum_{\varepsilon'} \widehat{L}_{\varepsilon_1 \varepsilon'}^+(u) \widetilde{\otimes} \widehat{L}_{\varepsilon' \varepsilon_2}^+(u), \\ \Delta(e^Q) &= e^Q \widetilde{\otimes} e^Q, \\ \Delta(\mu_l(\widehat{f})) &= \mu_l(\widehat{f}) \widetilde{\otimes} 1, \quad \Delta(\mu_r(\widehat{f})) = 1 \widetilde{\otimes} \mu_r(\widehat{f}).\end{aligned}$$

We also define an H -algebra anti-homomorphism $S : U_{q,p} \rightarrow U_{q,p}$ by

$$\begin{aligned}S(\widehat{L}_{++}^+) &= \widehat{L}_{--}^+(u-1), \quad S(\widehat{L}_{+-}^+(u)) = -\frac{[P+h+1]}{[P+h]} \widehat{L}_{+-}^+(u-1), \\ S(\widehat{L}_{-+}^+(u)) &= -\frac{[P]^*}{[P+1]^*} \widehat{L}_{-+}^+(u-1), \quad S(\widehat{L}_{--}^+(u)) = \frac{[P+h+1][P]^*}{[P+h][P+1]^*} \widehat{L}_{++}^+(u-1), \\ S(e^Q) &= e^{-Q}, \quad S(\mu_r(\widehat{f})) = \mu_l(\widehat{f}), \quad S(\mu_l(\widehat{f})) = \mu_r(\widehat{f}).\end{aligned}$$

One can check that Δ and S preserve the RLL relation (2.2). Furthermore one finds that ε, Δ and S are the counit, the comultiplication and the antipode satisfying the following relations.

Proposition 3.6. *[27] The maps ε, Δ and S satisfy*

$$\begin{aligned}(\Delta \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \Delta) \circ \Delta, \\ (\varepsilon \otimes \text{id}) \circ \Delta &= \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta, \\ m \circ (\text{id} \otimes S) \circ \Delta(x) &= \mu_l(\varepsilon(x)1), \quad \forall x \in U_{q,p}, \\ m \circ (S \otimes \text{id}) \circ \Delta(x) &= \mu_r(T_\alpha(\varepsilon(x)1)), \quad \forall x \in (U_{q,p})_{\alpha\beta}.\end{aligned}$$

Equipped with (Δ, ε, S) , the H -algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is an H -Hopf algebroid.

Definition 3.7. We call the H -Hopf algebroid $(U_{q,p}(\widehat{\mathfrak{sl}}_2), H, M_{H^*}, \mu_l, \mu_r, \Delta, \varepsilon, S)$ the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

§ 4. Representations

In this section, we summarize some basic facts on the dynamical representations of H -algebras [10, 19] and their application to $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Let us consider a vector space $\widehat{V}$ over $\mathbb{F}$, which is $\bar{\mathfrak{h}}$ -diagonalizable,

$$\widehat{V} = \bigoplus_{\mu \in \bar{\mathfrak{h}}^*} \widehat{V}_\mu, \quad \widehat{V}_\mu = \{v \in \widehat{V} \mid q^h v = q^\mu v \quad (h \in \bar{\mathfrak{h}})\}.$$

Let us define the H -algebra $\mathcal{D}_{H,\widehat{V}}$ of the $\mathbb{C}$ -linear operators on $\widehat{V}$ by

$$\begin{aligned} \mathcal{D}_{H,\widehat{V}} &= \bigoplus_{\alpha,\beta \in \bar{H}^*} (\mathcal{D}_{H,\widehat{V}})_{\alpha\beta}, \\ (\mathcal{D}_{H,\widehat{V}})_{\alpha\beta} &= \left\{ X \in \text{End}_{\mathbb{C}} \widehat{V} \mid \begin{aligned} &X(f(P, r^*)v) = f(P - \langle \beta, P \rangle, r^*)X(v), \quad v \in \widehat{V}, \\ &f(P, r^*) \in \mathbb{F}, \quad X(\widehat{V}_\mu) \subseteq \widehat{V}_{\mu + \phi^{-1}(\alpha - \beta)} \end{aligned} \right\}, \\ \mu_l^{\mathcal{D}_{H,\widehat{V}}}(\widehat{f})v &= f(P + \mu, r^* + c)v, \quad \mu_r^{\mathcal{D}_{H,\widehat{V}}}(\widehat{f})v = f(P, r^*)v, \quad \widehat{f} \in M_{H^*}, \quad v \in \widehat{V}_\mu. \end{aligned}$$

Note that the subspace $(\mathcal{D}_{H,\widehat{V}})_{\alpha\beta}$ consists of the $\mathbb{C}$ -linear operators on $\widehat{V}$ of the form $x\widetilde{e}^{-\beta}$, where x is a $\mathbb{C}$ -linear operator carrying the weight $\phi^{-1}(\alpha - \beta)$.

Definition 4.1. [10, 19] A dynamical representation of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ on $\widehat{V}$ is an H -algebra homomorphism $\widehat{\pi} : U'_{q,p}(\widehat{\mathfrak{sl}}_2) \rightarrow \mathcal{D}_{H,\widehat{V}}$.

Let $(\widehat{\pi}_V, \widehat{V}), (\widehat{\pi}_W, \widehat{W})$ be two dynamical representations of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$. We define the tensor product $\widehat{V} \widetilde{\otimes} \widehat{W}$ by

$$\widehat{V} \widetilde{\otimes} \widehat{W} = \bigoplus_{\mu \in \bar{\mathfrak{h}}^*} (\widehat{V} \widetilde{\otimes} \widehat{W})_\mu, \quad (\widehat{V} \widetilde{\otimes} \widehat{W})_\mu = \bigoplus_{\nu \in \bar{\mathfrak{h}}^*} \widehat{V}_\nu \otimes_{M_{H^*}} \widehat{W}_{\mu - \nu},$$

where $\otimes_{M_{H^*}}$ denotes the usual tensor product modulo the relation

$$(4.1) \quad f(P, r^*)v \otimes w = v \otimes f(P + \nu, r^* + c)w$$

for $w \in \widehat{W}_\nu$. The action of scalars $f(P, r^*) \in \mathbb{F}$ on the tensor space $\widehat{V} \widetilde{\otimes} \widehat{W}$ is defined by

$$f(P, r^*). (v \widetilde{\otimes} w) = \Delta(\mu_r(\widehat{f}))(v \widetilde{\otimes} w) = v \widetilde{\otimes} f(P, r^*)w.$$

Then one finds a natural H -algebra embedding $\theta_{\widehat{V}\widetilde{\otimes}\widehat{W}} : \mathcal{D}_{H,\widehat{V}} \widetilde{\otimes} \mathcal{D}_{H,\widehat{W}} \rightarrow \mathcal{D}_{H,\widehat{V}\widetilde{\otimes}\widehat{W}}$ by $X_{\widehat{V}} \widetilde{\otimes} X_{\widehat{W}} \in (\mathcal{D}_{H,\widehat{V}})_{\alpha\gamma} \widetilde{\otimes} (\mathcal{D}_{H,\widehat{W}})_{\gamma\beta} \mapsto X_{\widehat{V}} \widetilde{\otimes} X_{\widehat{W}} \in (\mathcal{D}_{H,\widehat{V}\widetilde{\otimes}\widehat{W}})_{\alpha\beta}$. Hence the map $\theta_{\widehat{V}\widetilde{\otimes}\widehat{W}} \circ (\widehat{\pi}_V \otimes \widehat{\pi}_W) \circ \Delta : U'_{q,p}(\widehat{\mathfrak{sl}}_2) \rightarrow \mathcal{D}_{H,\widehat{V}\widetilde{\otimes}\widehat{W}}$ gives a dynamical representation of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ on $\widehat{V} \widetilde{\otimes} \widehat{W}$.

Now let us consider a construction of dynamical representations of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$. Let V be an $\bar{\mathfrak{h}}^*$ -diagonalizable vector space over $\mathbb{F}$. Let V_Q be a vector space over $\mathbb{C}$, and assume that an action of e^Q on V_Q is defined appropriately. Two important examples of V_Q are $V_Q = \mathbb{C}1$ and $V_Q = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}e^{nQ}$, where 1 denotes the vacuum state satisfying

$e^Q.1 = 1$. We then consider the vector space $\widehat{V} = V \otimes_{\mathbb{C}} V_Q$ over $\mathbb{F}$, where the actions of $f(P, r^*) \in \mathbb{F}$ and e^Q on $\widehat{V}$ are defined by

$$\begin{aligned} f(P, r^*)(v \otimes \xi) &= f(P, r^*)v \otimes \xi, \\ e^Q(f(P, r^*)v \otimes \xi) &= f(P + 1, r^*)v \otimes e^Q\xi \end{aligned}$$

for $v \otimes \xi \in V \otimes V_Q$.

Then the following theorem is fundamental in our construction of dynamical representations.

Theorem 4.2. [27] *Let $\widehat{V}$ be as above and $\pi_V : \mathbb{F}[U'_q(\widehat{\mathfrak{sl}}_2)] \rightarrow \text{End}_{\mathbb{F}} V$ be an algebra homomorphism. Define a map $\widehat{\pi}_V = \pi_V \otimes \text{id} : U'_{q,p}(\widehat{\mathfrak{sl}}_2) = \mathbb{F}[U'_q(\widehat{\mathfrak{sl}}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*] \rightarrow \text{End}_{\mathbb{C}} \widehat{V}$ by*

$$\begin{aligned} \widehat{\pi}_V(E(u)) &= \pi_V(\phi_r(x^+(z))) e^{2Q} z^{-\frac{P-1}{r^*}}, \\ \widehat{\pi}_V(F(u)) &= \pi_V(\phi_r(x^-(z))) z^{\frac{P+\pi_V(h)-1}{r}}, \\ \widehat{\pi}_V(K(u)) &= \exp \left(\sum_{n>0} \frac{[n]_q}{[2n]_q[r^*n]_q} \pi_V(a_{-n})(q^c z)^n \right) \exp \left(- \sum_{n>0} \frac{[n]_q}{[2n]_q[rn]_q} \pi_V(a_n) z^{-n} \right) \\ &\quad \times e^Q z^{-\frac{c}{4rr^*}(2P-1) + \frac{1}{2r}\pi_V(h)}. \end{aligned}$$

Then $(\widehat{\pi}_V, \widehat{V})$ is a dynamical representation of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ on $\widehat{V}$.

Through this paper we consider dynamical representations obtained in this way.

Let us also state the Poincaré-Birkhoff-Witt theorem for $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Definition 4.3. Let $\mathcal{H}$ (resp. $\mathcal{N}_{\pm}$) be the subalgebras of $\mathbb{F}[U'_{q,p}(\widehat{\mathfrak{sl}}_2)]$ generated by c, h and a_k ($k \in \mathbb{Z}_{\neq 0}$) (resp. by $x_n^{\pm}$ ($n \in \mathbb{Z}$)).

Theorem 4.4. [27]

$$U'_{q,p}(\widehat{\mathfrak{sl}}_2) = (\mathcal{N}_- \otimes \mathcal{H} \otimes \mathcal{N}_+) \otimes \mathbb{C}[\bar{H}^*].$$

Here the last $\otimes$ should be understood as a semi-direct product.

§ 5. Finite-Dimensional Representations

This section is an announcement of some new results on the finite-dimensional dynamical representations of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$. Detailed discussion will be published elsewhere [27].

§ 5.1. Pseudo-highest Weight Representations

We begin by stating a characteristic feature of the finite-dimensional irreducible dynamical representations.

Theorem 5.1. *Every finite-dimensional irreducible dynamical representation $(\widehat{\pi}_V, \widehat{V} = V \otimes_{\mathbb{C}} V_Q)$ of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ contains a non-zero vector of the form $\widehat{\Omega} = \Omega \otimes 1, \Omega \in V$ such*

that

- 1) $x_n^+.\widehat{\Omega} = 0 \quad \forall n \in \mathbb{Z},$
- 2) $\psi_n.\widehat{\Omega} = d_n^+.\widehat{\Omega}, \quad \phi_{-n}.\widehat{\Omega} = d_{-n}^-.\widehat{\Omega} \quad \forall n \in \mathbb{Z}_{\geq 0},$
- 3) $e^Q.\widehat{\Omega} = \widehat{\Omega},$
- 4) $\widehat{V} = U'_{q,p}.\widehat{\Omega}.$

with some complex numbers $d_{\pm n}^{\pm}$, $d_0^+ d_0^- = 1$. Furthermore q^c acts as 1 or -1 on $\widehat{V}$.

Definition 5.2. We define a pseudo-highest weight representation, a pseudo-highest weight vector and a pseudo-highest weight to be a dynamical representation (not necessarily irreducible) $(\pi_V, \widehat{V})$, a vector $\widehat{\Omega} \in \widehat{V}$ and a set of complex numbers $\mathbf{d} = \{d_{\pm n}^{\pm}\}_{n \in \mathbb{Z}_{\geq 0}}$ satisfying the conditions 1) – 4) in Theorem 5.1, respectively.

The following theorem is useful.

Theorem 5.3. For a vector $\widehat{\Omega} \in \widehat{V}$ satisfying $e^Q.\widehat{\Omega} = \widehat{\Omega}$, the conditions 1) and 2) in Definition 5.2 are equivalent to the following.

- i) $\widehat{L}_{-+}^+(u).\widehat{\Omega} = 0 \quad \forall u,$
- ii) $q^h.\widehat{\Omega} = q^\lambda \widehat{\Omega} \quad \exists \lambda \in \mathbb{C},$
- $\widehat{L}_{++}^+(u).\widehat{\Omega} = A(u)\widehat{\Omega}, \quad \widehat{L}_{--}^+(u).\widehat{\Omega} = D(u)\widehat{\Omega}$

with some meromorphic functions $A(u)$ and $D(u)$ satisfying $D(u-1)^{-1} = A(u)$ and

$$(5.1) \quad A(u) = z^{\frac{\lambda}{2r}} \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}} A_{m,n} z^m p^n \quad A_{m,n} \in \mathbb{C}, z = q^{2u}, p = q^{2r}.$$

$U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ admits the universal pseudo-highest weight representation defined as follows.

Definition 5.4. Let $\mathbf{d} = \{d_{\pm n}^{\pm}\}_{n \in \mathbb{Z}_{\geq 0}}$ be any sequence of complex numbers. The Verma module $M(\mathbf{d})$ is the quotient of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ by the two sided ideal generated by $q^c - 1$ and the left ideal generated by $\{x_k^+ \ (k \in \mathbb{Z}), \psi_n - d_n^+ \cdot 1, \phi_{-n} - d_{-n}^- \cdot 1 \ (n \in \mathbb{Z}_{\geq 0}), e^Q - 1\}$.

Proposition 5.5. The Verma module $M(\mathbf{d})$ is pseudo-highest weight with pseudo-highest weight $\mathbf{d}$. Every pseudo-highest weight representation with pseudo-highest weight $\mathbf{d}$ is isomorphic to a quotient of $M(\mathbf{d})$. Moreover $M(\mathbf{d})$ has a unique irreducible pseudo-highest weight module.

§ 5.2. Elliptic Analogue of the Drinfeld Polynomials

We state a necessary and sufficient condition for an irreducible dynamical representations of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ to be finite-dimensional. We introduce a natural elliptic analogue of the Drinfeld polynomials.

Theorem 5.6. *The irreducible pseudo-highest weight representation $(\widehat{\pi}_V, \widehat{V})$ of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is finite-dimensional if and only if there exists an entire and quasi-periodic function $P_V(u)$ such that*

$$\begin{aligned} H^\pm(u) \cdot \widehat{\Omega} &= c_V \frac{P_V(u+1)}{P_V(u)} \widehat{\Omega} \\ P_V(u+r) &= (-)^{\deg P} P_V(u), \\ P_V(u+r\tau) &= (-)^{\deg P} e^{-\pi i \sum_{j=1}^{\deg P} (\frac{2(u-\alpha_j)}{r} + \tau)} P_V(u). \end{aligned}$$

Here $\widehat{\Omega}$ denotes the pseudo-highest weight vector in $\widehat{V}$, and $\tau = -\frac{2\pi i}{\log p}$. The symbol c_V denotes a constant given by

$$c_V = q^{\frac{r-1}{r} \deg P} \prod_{j=1}^{\deg P} a_j^{\frac{1}{r}},$$

where $\deg P$ is the number of zeros of $P_V(u)$ in the period parallelogram $(1, \tau)$ (= the degree of the Drinfeld polynomial $P(z) = \lim_{r \rightarrow \infty} P_V(u)$, $z = q^{2u}$), and $a_j = q^{2\alpha_j}$ with α_j being a zero of $P_V(u)$ in the period parallelogram. The function $P_V(u)$ is unique up to a scalar multiple.

Remark. We can take $c_V = 1$ by the gauge transformation given by (2.11) in [18]. An example is given in Theorem 5.8.

The following Proposition is a direct consequence of the comultiplication formula for $\widehat{L}^+(u)$ and Definition 2.7.

Proposition 5.7. *Let $\widehat{V}$ and $\widehat{W}$ be finite dimensional dynamical representations of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ and assume that the tensor product $\widehat{V} \widetilde{\otimes} \widehat{W}$ is irreducible. Let $P_V(u), P_W(u)$ and $P_{V \widetilde{\otimes} W}(u)$ be the entire quasi-periodic function associated to $\widehat{V}, \widehat{W}$ and $\widehat{V} \widetilde{\otimes} \widehat{W}$ in Theorem 5.6. Then*

$$P_{V \widetilde{\otimes} W}(u) = P_V(u) P_W(u).$$

§ 5.3. Evaluation Representations

An important example of finite-dimensional irreducible dynamical representations of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ is the evaluation representation. We here give a summary on the $l+1$ -dimensional evaluation representation obtained from the one of $\mathbb{F}[U'_q(\widehat{\mathfrak{sl}}_2)]$.

Let $V^{(l)} = \oplus_{m=0}^l \mathbb{F} v_m^l$, $V_w^{(l)} = V^{(l)} \otimes \mathbb{C}[w, w^{-1}]$, and consider the operators $h, S^\pm$ on $V^{(l)}$ defined by

$$h v_m^l = (l - 2m) v_m^l, \quad S^\pm v_m^l = v_{m \mp 1}^l, \quad v_m^l = 0 \quad \text{for } m < 0, \quad m > l.$$

In terms of the Drinfeld generators, the $l + 1$ -dimensional evaluation representation $(\pi_{l,w}, V_w^{(l)})$ of $\mathbb{F}[U'_q(\widehat{\mathfrak{sl}}_2)]$ is given by

$$(5.2) \quad \begin{aligned} \pi_{l,w}(q^c) &= 1, \\ \pi_{l,w}(a_n) &= \frac{w^n}{n} \frac{1}{q - q^{-1}} ((q^n + q^{-n})q^{nh} - (q^{(l+1)n} + q^{-(l+1)n})), \\ \pi_{l,w}(x^\pm(z)) &= S^\pm \left[\frac{\pm h + l + 2}{2} \right]_q \delta \left(q^{h \pm 1} \frac{w}{z} \right). \end{aligned}$$

Now let us consider the vector space $\widehat{V}^{(l)}(w) = V^{(l)}(w) \otimes \mathbb{C}1$ and the map $\widehat{\pi}_{l,w} = \pi_{l,w} \otimes \text{id}$ on $U'_{q,p}(\widehat{\mathfrak{sl}}_2) \cong \mathbb{F}[U'_q(\widehat{\mathfrak{sl}}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*]$. Applying (5.2) to Theorem 4.2 and noting Definitions 2.2 and 2.6, we can derive the evaluation representation of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ as follows.

Theorem 5.8. *$(\widehat{\pi}_{l,w}, \widehat{V}^{(l)}(w))$ is the $l+1$ -dimensional irreducible dynamical representation of $U'_{q,p}(\widehat{\mathfrak{sl}}_2)$ with the pseudo-highest weight vector $v_0^l \otimes 1$. In particular, the images of the matrix elements of $\widehat{L}^+(u)$ by the map $\widehat{\pi}_{l,w}$ are given, up to fractional powers of z, w and q , by*

$$\begin{aligned} \widehat{\pi}_{l,w}(\widehat{L}_{++}^+(u)) &= -\frac{[u - v + \frac{h+1}{2}][P - \frac{l-h}{2}][P + \frac{l+h+2}{2}]}{\varphi_l(u-v)[P][P+h+1]} e^Q, \\ \widehat{\pi}_{l,w}(\widehat{L}_{+-}^+(u)) &= -S^- \frac{[u - v + \frac{h-1}{2} + P][\frac{l-h+2}{2}]}{\varphi_l(u-v)[P+h-1]} e^{-Q}, \\ \widehat{\pi}_{l,w}(\widehat{L}_{-+}^+(u)) &= S^+ \frac{[u - v - \frac{h+1}{2} - P][\frac{l+h+2}{2}]}{\varphi_l(u-v)[P]} e^Q, \\ \widehat{\pi}_{l,w}(\widehat{L}_{--}^+(u)) &= -\frac{[u - v - \frac{h-1}{2}]}{\varphi_l(u-v)} e^{-Q}, \end{aligned}$$

where $z = q^{2u}$, $w = q^{2v}$, and

$$\begin{aligned} \varphi_l(u) &= -z^{-\frac{l}{2r}} \rho_{1l}^+(z, p)^{-1} [u + \frac{l+1}{2}], \\ \rho_{kl}^+(z, p) &= q^{\frac{kl}{2}} \frac{\{pq^{k-l+2}z\}\{pq^{-k+l+2}z\}\{q^{k+l+2}/z\}\{q^{-k-l+2}/z\}}{\{pq^{k+l+2}z\}\{pq^{-k-l+2}z\}\{q^{k-l+2}/z\}\{q^{-k+l+2}/z\}}. \end{aligned}$$

Corollary 5.9. *The elliptic analogue of the Drinfeld polynomial associated to $\widehat{V}^{(l)}(q^{2v})$ is given by*

$$P_{l,v}(u) = [u - v - \frac{l-1}{2}][u - v - \frac{l-1}{2} + 1] \cdots [u - v + \frac{l-1}{2}].$$

Obviously the zeros of $P_{l,v}(u)$ modulo $\mathbb{Z}r + \mathbb{Z}r\tau$ coincide with those of the Drinfeld polynomial corresponding to the evaluation representation $V^{(l)}(q^{2v})$ of $U'_q(\widehat{\mathfrak{sl}}_2)$.

The following Proposition indicates a consistency of our construction of $\widehat{\pi}_{l,w}$ and the standard fusion construction of the dynamical R matrices (=face type Boltzmann weights).

Proposition 5.10. *Let us define the matrix elements of $\widehat{\pi}_{l,w}(\widehat{L}_{\varepsilon_1\varepsilon_2}^+(u))$ by*

$$\widehat{\pi}_{l,w}(\widehat{L}_{\varepsilon_1\varepsilon_2}^+(u))v_m^l = \sum_{m'=0}^l (\widehat{L}_{\varepsilon_1\varepsilon_2}^+(u))_{\mu_{m'}\mu_m} v_{m'}^l,$$

where $\mu_m = l - 2m$. Then we have

$$(\widehat{L}_{\varepsilon_1\varepsilon_2}^+(u))_{\mu_{m'}\mu_m} = R_{1l}^+(u-v, P)_{\varepsilon_1\mu_{m'}}^{\varepsilon_2\mu_m}.$$

Here $R_{1l}^+(u-v, P)$ is the R matrix from (C.17) in [18]. In the case $l = 1$, $R_{11}^+(u-v, P)$ coincides with the image $(\pi_{1,z} \otimes \pi_{1,w})$ of the universal R matrix $\mathcal{R}^+(\lambda)$ [17] given in (2.3). In the case $l > 1$, $R_{1l}^+(u-v, P)$ coincides with the R matrix obtained by the standard fusion procedure from $R_{11}^+(u-v, P)$. In particular the matrix element $R_{1l}^+(u-v, P)_{\varepsilon\mu}^{\varepsilon'\mu'}$ is gauge equivalent to the fusion face weight $W_{l1}(P+\varepsilon', P+\varepsilon'+\mu', P+\mu, P|u-v)$ from (4) in [6].

§ 6. Infinite-dimensional Representations and Vertex Operators

Theorem 4.2 is valid also for infinite-dimensional representations. Let $(\pi, V(\lambda_l))$ be the level- k ($c = k$) irreducible highest weight representation of $\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]$ with the highest weight $\lambda_l = (k-l)\Lambda_0 + l\Lambda_1$ ($0 \leq l \leq k$). Here Λ_i ($i = 0, 1$) denote the fundamental weights of $\widehat{\mathfrak{sl}}(2, \mathbb{C})$. Then $(\widehat{\pi} = \pi \otimes \text{id}, \widehat{V}(\lambda_l) = \bigoplus_{m \in \mathbb{Z}} V(\lambda_l) \otimes \mathbb{C}e^{-mQ})$ is the level- k highest weight irreducible dynamical representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

A realization of $\widehat{V}(\lambda_l)$ in terms of the Drinfeld generators a_n ($n \in \mathbb{Z}_{\neq 0}$) and the q -deformed $\mathbb{Z}_k$ -parafermion algebra was given in [23, 26]. In [24, 18], we also studied the Wakimoto representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ labeled by an integer J ($0 \leq J \leq k$), which is nothing but the dynamical representation $\widehat{V}(\lambda_J)$.

The H -Hopf algebroid structure allows us to define the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ as follows.

Definition 6.1. The type I and II vertex operators of spin $n/2$ are the intertwiners of the $U_{q,p}$ -modules of the form

$$\widehat{\Phi}(u) : \widehat{V}(\lambda) \rightarrow \widehat{V}_z^{(n)} \widetilde{\otimes} \widehat{V}(\nu), \quad \widehat{\Psi}^*(u) : \widehat{V}(\lambda) \widetilde{\otimes} \widehat{V}_z^{(n)} \rightarrow \widehat{V}(\nu),$$

where $z = q^{2u}$, and $\widehat{V}(\lambda)$ and $\widehat{V}(\nu)$ denote the level- k highest weight $U_{q,p}$ -modules of highest weights λ and ν , respectively. They satisfy the intertwining relations with respect to the comultiplication Δ in Definition 3.5.

$$(6.1) \quad \Delta(x)\widehat{\Phi}(u) = \widehat{\Phi}(u)x, \quad x\widehat{\Psi}^*(u) = \widehat{\Psi}^*(u)\Delta(x) \quad \forall x \in U_{q,p}.$$

The physically interesting cases are $n = k, \lambda = \lambda_l, \nu = \lambda_{k-l}$ for the type I and $n = 1, \lambda = \lambda_l, \nu = \lambda_{l\pm 1}$ for the type II vertex operators. See for example [23].

Let us define the components of the vertex operators as follows.

$$(6.2) \quad \widehat{\Phi}(v - \frac{1}{2}) = \sum_{m=0}^n v_m^n \widetilde{\otimes} \Phi_m(v), \quad \widehat{\Psi}^*(v - \frac{c+1}{2})(\cdot \widetilde{\otimes} v_m^n) = \Psi_m^*(v).$$

By using the comultiplication formula for $\widehat{L}^+(u)$ in Definition 3.5 and Proposition 5.10, we obtain the following theorem.

Theorem 6.2. [26] *The vertex operators satisfy the following linear equations.*

$$(6.3) \quad \widehat{\Phi}(u)\widehat{L}^+(v) = R_{1n}^{+(12)}(v - u, P + h)\widehat{L}^+(v)\widehat{\Phi}(u),$$

$$(6.4) \quad \widehat{L}^+(v)\widehat{\Psi}^*(u) = \widehat{\Psi}^*(u)\widehat{L}^+(v)R_{1n}^{+*(13)}(v - u, P - h^{(1)} - h^{(3)}).$$

The relation (6.3) should be understood on $\widehat{V}_w^{(1)} \widetilde{\otimes} \widehat{V}(\lambda)$, whereas (6.4) on $\widehat{V}_w^{(1)} \widetilde{\otimes} \widehat{V}(\lambda) \widetilde{\otimes} \widehat{V}_z^{(n)}$.

Equations (6.3) and (6.4) coincide with (5.3) and (5.4) in [18], respectively. In [18], those equations were derived by using the quasi-Hopf algebra structure on $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ and the isomorphism $U_{q,p}(\widehat{\mathfrak{sl}}_2) \cong \mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2) \otimes \mathbb{C}[\bar{H}^*]$ as a semi-direct product algebra. Under certain analyticity conditions, these equations determine the vertex operators uniquely up to normalization. Combining the results obtained here and those in [29, 24, 18, 28, 23], we have established the algebraic analysis scheme for the fusion RSOS models as well as for the fusion eight-vertex models on the basis of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Acknowledgments

The author would like to thank the organizers Susumu Ariki and Masato Okado for a kind invitation to the workshop. This work is supported by the Grant-in-Aid for Scientific Research (C)19540033, JSPS Japan.

References

- [1] G. E. Andrews, R. J. Baxter and P. J. Forrester, Eight-vertex SOS Model and Generalized Rogers-Ramanujan-type Identities, *J. Stat. Phys.*, **35**, 1984, 193–266.
- [2] O. Babelon, D. Bernard, and E. Billey. A Quasi-Hopf Algebra Interpretation of Quantum $3j$ - and $6j$ -symbols and Difference Equations. *Phys. Lett. B*, **375**, 1996, 89–97.
- [3] R. J. Baxter. Partition Function of the Eight-vertex Lattice Model. *Ann. of Phys.*, **70**, 1972, 193–228.
- [4] E. Buffenoir, Ph. Roche and V. Terras, Quantum Dynamical coBoundary Equation for Finite Dimensional Simple Lie Algebras. *Adv.in Math.*, **214**, 2007, 181–229; Universal Vertex-IRF Transformation for Quantum Affine Algebras. arXiv:0707.0955.

- [5] V. Chari and A. Pressley, Quantum Affine Algebras, *Comm. Math. Phys.*, **142**, 1991, 261–283
- [6] E.Date, M.Jimbo, T.Miwa and M.Okado, Fusion of the Eight-Vertex SOS Model , *Lett.Math.Phys.*, **12**, 1986, 209–215.
- [7] V.G.Drinfeld, A New Realization of Yangians and Quantized Affine Algebras, *Soviet Math. Dokl.*, **36**, 1988, 212–216.
- [8] V.G.Drinfeld, Quasi-Hopf Algebras, *Leningrad Math. J.*, **1**, 1990, 1419–1457.
- [9] B. Enriquez and G. Felder, Elliptic Quantum Groups $E_{\tau,\eta}(\mathfrak{sl}_2)$ and Quasi-Hopf Algebra , *Comm.Math.Phys.*, **195**, 1998, 651–689.
- [10] P.Etingof and A.Varchenko, Solutions of the Quantum Dynamical Yang-Baxter Equation and Dynamical Quantum Groups, *Comm.Math.Phys.*, **196**, 1998, 591–640.
- [11] P.Etingof and A.Varchenko, Exchange Dynamical Quantum Groups, *Comm.Math.Phys.*, **205**, 1999, 19–52.
- [12] G. Felder, Elliptic Quantum Groups, *Proc. ICMP Paris-1994*, 1995, 211–218.
- [13] G. Felder and A. Varchenko, On Representations of the Elliptic Quantum Groups $E_{\tau,\eta}(\mathfrak{sl}_2)$, *Comm.Math.Phys.*, **181**, 1996, 741–761.
- [14] I.B.Frenkel and V.G.Turaev, Elliptic Solutions of the Yang-Baxter Equation and Modular Hypergeometric Functions, in V.I.Arnold et.al.(eds.), The Arnold-Gelfand Mathematical Seminars, 171–204, Birkhauser, Boston, 1997.
- [15] C. Frønsdal. Quasi-Hopf deformation of quantum groups. *Lett. Math. Phys.*, **40**, 1997, 117–134.
- [16] M. Jimbo, T. Miwa and M. Okado, Solvable Lattice Models Related to the Vector Representation of Classical Simple Lie Algebras, *Comm. Math. Phys.*, **116**, 1988, 507–525.
- [17] M. Jimbo, H. Konno, S. Odake and J. Shiraishi, Quasi-Hopf Twistors for Elliptic Quantum Groups, *Transformation Groups*, **4**, 1999, 303–327.
- [18] M. Jimbo, H. Konno, S. Odake and J. Shiraishi, Elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$: Drinfeld currents and vertex operators, *Comm. Math. Phys.* , **199**, 1999, 605–647
- [19] E.Koelink and H.Rosengren, Harmonic Analysis on the $SU(2)$ Dynamical Quantum Group, *Acta.Appl.Math.*, **69**, 2001, 163–220.
- [20] E.Koelink, Y.van Norden and H.Rosengren, Elliptic $U(2)$ Quantum Group and Elliptic Hypergeometric Series, *Comm.Math.Phys.*, **245**, 2004, 519–537.
- [21] T. Kojima and H. Konno, The Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and the Drinfeld Realization of the Elliptic Quantum Group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$, *Comm. Math. Phys.*, **239**, 2003, 405–447.
- [22] T. Kojima and H. Konno, The Drinfeld Realization of the Elliptic Quantum Group $\mathcal{B}_{q,\lambda}(A_2^{(2)})$, *J. Math. Phys.*, **45**, 2004, 3146–3179.
- [23] T. Kojima, H. Konno and R. Weston, The Vertex-Face Correspondence and Correlation Functions of the Fusion Eight-Vertex Models I: The General Formalism, *Nucl. Phys.*, **B720**, 2005, 348–398.
- [24] H. Konno, An Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and the Fusion RSOS Models, *Comm. Math. Phys.*, **195**, 1998, 373–403.
- [25] H. Konno, Dynamical R Matrices of Elliptic Quantum Groups and Connection Matrices for the q -KZ Equations, *SIGMA*, **2**, 2006, Paper 091, 25 pages.
- [26] H. Konno, Elliptic Quantum Group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and Vertex Operators, *J.Phys.A:Math.and Theor.*, **41**, 194012 (12 pages), 2008. Special issue on Recent Developments in Infinite Dimensional Algebras and Their Applications to Quantum Integrable Systems 2007.
- [27] H. Konno, Elliptic Quantum Group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, Hopf Algebroid Structure and Elliptic Hypergeometric Series, arXiv:0803.2292.

- [28] M. Lashkevich and Y. Pugai. Free Field Construction for Correlation Functions of the Eight-Vertex Model. *Nucl. Phys.*, **B516**, 1998, 623–651.
- [29] S. Lukyanov and Y. Pugai, Multi-point Local Height Probabilities in the Integrable RSOS Models. *Nucl. Phys.*, **B473**, 1996, 631–658.
- [30] J.-H. Lu, Hopf Algebroids and Quantum Groupoids, *Int. J. Math.*, **7**, 1996, 47-70.
- [31] H. Rosengren, Duality and Self-duality for Dynamical Quantum Groups, *Algebr. Represent. Theory*, **7**, 2004, 363-393.
- [32] V.P. Spiridonov, and A. Zhedanov, Spectral Transformation Chains and Some New Biorthogonal Rational Functions, *Comm. Math. Phys.* **210**, 2000, 49-83; Generalized Eigenvalue Problem and a New Family of Rational Functions Biorthogonal on Elliptic Grids, in J. Bustoz et al. (eds.), *Special Functions 2000: Current Perspective and Future Directions*, 365-388, Kluwer Acad. Publ., Dordrecht 2001.
- [33] V.P. Spiridonov, An Elliptic Incarnation of the Bailey Chain, *Int. Math. Res. Notices* **37**, 2002, 1945-1977.
- [34] Y. van Norden, Dynamical Quantum Groups, Duality and Special Functions, PhD thesis, 2005.
- [35] P. Xu, Quantum Groupoids, *Comm. Math. Phys.*, **216**, 2001, 539-581.