

ON THE ORDER OF STRONGLY MEROMORPHIC STARLIKENESS OF STRONGLY MEROMORPHIC CONVEX FUNCTIONS

MAMORU NUNOKAWA [布川 護 群馬大学教育学部]

AKIRA IKEDA [池田 彰 福岡大学理学部]

SHIGEYOSHI OWA [尾和 重義 近畿大学理工学部]

IN HWA KIM AND NAK EUN CHO

ABSTRACT. In [1], Nunokawa proved that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $|z| < 1$ and

$$\left| \arg \left(1 + \frac{z f''(z)}{f'(z)} \right) \right| < \frac{\pi}{2} \alpha(\beta) \quad \text{in} \quad |z| < 1.$$

where $\alpha(\beta)$ satisfies the condition of Theorem A, then

$$\left| \arg \left(\frac{z f'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \beta \quad \text{in} \quad |z| < 1.$$

It is the purpose of the present paper that if $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ is analytic in $0 < |z| < 1$ and

$$\left| \arg \left(1 + \frac{z f''(z)}{f'(z)} \right) \right| > \frac{\pi}{2} (1 + \delta(\beta)) \quad \text{in} \quad |z| < 1.$$

where $\delta(\beta)$ satisfies the condition of the Main Theorem, then we have

$$\left| \arg \left(\frac{z f'(z)}{f(z)} \right) \right| > \frac{\pi}{2} (1 + 1 - \beta) \quad \text{in} \quad |z| < 1.$$

1. Introduction

Let Σ denote the class of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

which are analytic and univalent in the punctured disk $\mathcal{E} = \{z : 0 < |z| < 1\}$.

A function $f(z) \in \Sigma$ is called to be strongly meromorphic starlike of order β ($-1 < \beta < 1$) if

$$\left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| > \frac{\pi}{2}(1 + \beta) \quad \text{in} \quad \mathcal{E}.$$

We denote by $SMS(\beta)$ the class of all strongly meromorphic starlike functions of order β . Similarly, a function $f(z) \in \Sigma$ is called to be strongly meromorphic convex of order β ($-1 < \beta < 1$) if

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| > \frac{\pi}{2}(1 + \beta) \quad \text{in} \quad \mathcal{E}.$$

We denote by $SMC(\beta)$ the class of all strongly meromorphic convex function of order β .

In [1], Nunokawa obtained the following theorem.

Theorem A. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $|z| < 1$ and

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2}\alpha(\beta) \quad \text{in} \quad |z| < 1$$

where $0 < \beta \leq 1$,

$$\alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \frac{\beta q(\beta) \sin \frac{\pi}{2}(1 - \beta)}{p(\beta) + \beta q(\beta) \cos \frac{\pi}{2}(1 - \beta)},$$

$$p(\beta) = (1 + \beta)^{\frac{1+\beta}{2}} \quad \text{and} \quad q(\beta) = (1 - \beta)^{\frac{\beta-1}{2}},$$

then we have

$$\left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2}\beta \quad \text{in} \quad |z| < 1.$$

It is the purpose of the present paper to obtain an analogous result for meromorphic starlike and convex functions.

2. Preliminary

Lemma. Let $p(z)$ be analytic in $\mathcal{U} = \{z : |z| < 1\}$, $p(0) = 1$, $p(z) \neq 0$ in $\mathcal{U}$ and suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$|\arg p(z)| < \frac{\pi\alpha}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\alpha}{2}$$

where $0 < \alpha$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i k \alpha$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi\alpha}{2}$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi\alpha}{2}$$

where

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia, \quad \text{and} \quad a > 0.$$

We owe this lemma to [1].

3. Main result

Theorem. If $f(z) \in SMC(\delta(\beta))$, then $f(z) \in SMS(1 - \beta)$, where

$$\delta(\beta) = \beta - 1 + \frac{2}{\pi} \tan^{-1} \frac{\beta q(\beta) \sin \frac{\pi}{2}(1 - \beta)}{\beta q(\beta) \cos \frac{\pi}{2}(1 - \beta) - p(\beta)},$$

$$p(\beta) = (1 + \beta)^{\frac{1+\beta}{2}}, \quad q(\beta) = (1 - \beta)^{\frac{\beta-1}{2}},$$

and $0 < \beta < 1$.

Proof. Let us put

$$p(z) = -\frac{zf'(z)}{f(z)}, \quad (p(0) = 1)$$

then we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} - p(z).$$

From the assumption of the theorem, we have

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| > \frac{\pi}{2} (1 + \delta(\beta)) \quad \text{in} \quad \mathcal{U}.$$

Therefore, we have $f'(z) \neq 0$ in $\mathcal{U}$, because if $f'(z)$ has a zero at $z = z_0$ in $\mathcal{U}$, then $(1 + \frac{zf''(z)}{f'(z)})$ can be infinite and $\arg(1 + \frac{zf''(z)}{f'(z)})$ can take any value of θ , for $0 \leq \theta \leq 2\pi$ when z approaches to z_0 from certain direction. This shows that

$$p(z) \neq 0 \quad \text{in} \quad \mathcal{U}.$$

If there exists a point $z_0 \in \mathcal{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2} \beta \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2} \beta$$

then from the lemma, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta$$

where k is a real

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi\beta}{2}$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi\beta}{2}$$

where

$$p(z_0)^{\frac{1}{\beta}} = \pm ia, \quad \text{and} \quad a > 0.$$

When $\arg p(z_0) = \frac{\pi\beta}{2}$, then from Lemma and applying the same method as the proof of [1, p.236], we have

$$\begin{aligned} \arg \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) &= \arg p(z_0) + \arg \left(\frac{z_0 p'(z_0)}{p(z_0)^2} - 1 \right) \\ &= \frac{\pi\beta}{2} + \arg \left(e^{i(1-\beta)\frac{\pi}{2}} \beta k \frac{1}{a^\beta} - 1 \right) \\ &\leq \frac{\pi\beta}{2} + \arg \left(e^{i(1-\beta)\frac{\pi}{2}} \frac{\beta}{2} (a^{1-\beta} + a^{-1-\beta}) - 1 \right) \\ &\leq \frac{\pi\beta}{2} + \arg \left\{ e^{i(1-\beta)\frac{\pi}{2}} \frac{\beta}{2} \left(\left(\frac{1+\beta}{1-\beta} \right)^{\frac{1-\beta}{2}} + \left(\frac{1+\beta}{1-\beta} \right)^{-\frac{1+\beta}{2}} \right) - 1 \right\} \\ &= \frac{\pi}{2} \beta + \tan^{-1} \frac{(\frac{\beta}{1-\beta})(\frac{1-\beta}{1+\beta})^{\frac{1+\beta}{2}} \sin \frac{\pi}{2} (1-\beta)}{(\frac{\beta}{1-\beta})(\frac{1-\beta}{1+\beta})^{\frac{1+\beta}{2}} \cos \frac{\pi}{2} (1-\beta) - 1} \\ &= \frac{\pi}{2} \beta + \tan^{-1} \frac{\beta q(\beta) \sin \frac{\pi}{2} (1-\beta)}{\beta q(\beta) \cos \frac{\pi}{2} (1-\beta) - p(\beta)} \\ &= \frac{\pi}{2} (1 + \delta(\beta)) \end{aligned}$$

and if $\arg p(z_0) = -\frac{\pi\beta}{2}$, applying the same method as the above, we have

$$\arg \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) \geq -\frac{\pi}{2} (1 + \delta(\beta)).$$

These contradict the assumption of the theorem, therefore we have

$$\left| \arg \left(-\frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi\beta}{2} \quad \text{in} \quad \mathcal{U}$$

or

$$\left| \arg \frac{zf'(z)}{f(z)} \right| > \frac{\pi}{2} (2 - \beta) \quad \text{in} \quad \mathcal{U}.$$

This shows that

$$f(z) \in SMS(1 - \beta).$$

This completes the proof.

Remark. *It is trivial that $-1 < \delta(\beta) < 0$ for $0 < \beta < 1$.*

REFERENCES

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MAMORU NUNOKAWA:

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GUNMA
4-2 ARAMAKI MAEBASHI GUNMA, 371, JAPAN

E-mail address: nunokawa@storm.edu.gunma-u.ac.jp

AKIRA IKEDA:

DEPARTMENT OF APPLIED MATHEMATICS, FUKUOKA UNIVERSITY
8-19-1 NANAKUMA JONAN-KU FUKUOKA, 814-80, JAPAN

E-mail address: aikeda@sf.sm.fukuoka-u.ac.jp

SHIGEYOSHI OWA:

DEPARTMENT OF MATHEMATICS, KINKI UNIVERSITY
HIGASHI-OSAKA, OSAKA 577, JAPAN

E-mail address: owa@math.kindai.ac.jp

IN HWA KIM:

NAK EUN CHO:

DEPARTMENT OF MATHEMATICS, PUKYONG NATIONAL UNIVERSITY
PUSAN, 608, KOREA