

On Laplace transforms of certain special functions

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1 Introduction

In this note we consider a Laplace transform of the following multiple integral

$$\begin{aligned} g(d; n; y) &= c\pi^{d/2} \frac{1}{y^{\{n(d+1)-d\}/2+1}} \exp\left\{-\frac{a_n^2}{y}\right\} \\ &= \int \cdots \int_{0 \leq v_1 + \cdots + v_{n-1} \leq 1; 0 \leq v_j} \exp\left\{\sum_{j=1}^{n-1} (a_n^2 - a_j^2) v_j / y\right\} \\ &\quad \left[v_1 \cdots v_{n-1} (1 - \sum_{j=1}^{n-1} v_j)\right]^{(d+1)/2-1} dv_1 \cdots dv_{n-1} \end{aligned} \quad (1)$$

where c is a constant. This multiple integral is coming from the normed product of multidimensional Cauchy densities, namely

$$f(d; a_1, \dots, a_n; x) = \frac{c_1}{\pi_{j=1}^n (a_j^2 + |x|^2)^{(d+1)/2}},$$

where c_1 is the normalised constant and $0 < a_j$; $a_j \neq a_k$ if $j \neq k$, and $x = (x_1, \dots, x_d) \in \mathbf{R}^d$. We note that the density function $g(d; 1; u)$ can be expressed by a confluent hypergeometric function,

$$\begin{aligned} g(d; 1; u) &= \frac{c\pi^{d/2} \Gamma((d+1)/2)^2}{\Gamma(d+1)} \\ &\quad \cdot \frac{1}{u^{d/2+2}} \exp\left\{-\frac{b^2}{u}\right\} {}_1F_1\left(\frac{d+1}{2}; d+1; \frac{(b^2 - a^2)}{u}\right) \end{aligned} \quad (2)$$

for all dimension d . For the dimension $d = 3$ we obtain

$$\begin{aligned} g(3; 1; u) &= \frac{c\pi^{3/2}}{(b^2 - a^2)^2} \left[\frac{1}{u^{3/2}} (\exp\{-\frac{a^2}{u}\} + \exp\{-\frac{b^2}{u}\}) \right. \\ &\quad \left. - \frac{2}{(b^2 - a^2)u^{1/2}} (\exp\{-\frac{a^2}{u}\} - \exp\{-\frac{b^2}{u}\}) \right] \end{aligned} \quad (3)$$

and for the dimension $d = 5$ we obtain

$$\begin{aligned}
 g(5; 1; u) = & \frac{c\pi^{5/2}}{(b^2 - a^2)^3} \left[\frac{2!}{u^{3/2}} (\exp\{-\frac{a^2}{u}\} - \exp\{-\frac{b^2}{u}\}) \right. \\
 & - \frac{2 \cdot 3!}{(b^2 - a^2)u^{1/2}} (\exp\{-\frac{a^2}{u}\} + \exp\{-\frac{b^2}{u}\}) \\
 & \left. + \frac{4!u^{1/2}}{(b^2 - a^2)^2} (\exp\{-\frac{a^2}{u}\} - \exp\{-\frac{b^2}{u}\}) \right]. \quad (4)
 \end{aligned}$$

2 The Laplace transforms of the case $n = 1$

Let us denote the Laplace transform of the functions $g(d; 1; u)$ by $\eta(d; 1; s)$ and let us take a branch of the value $\eta(d; +0) = 1$. We obtain

$$\begin{aligned}
 \eta(3; 1; s) = & \frac{c\pi^2}{(b^2 - a^2)^2} \left\{ \left(\frac{1}{a} \exp\{-2a\sqrt{s}\} + \frac{1}{b} \exp\{-2b\sqrt{s}\} \right) \right. \\
 & \left. - \frac{2}{(b^2 - a^2)\sqrt{s}} (\exp\{-2a\sqrt{s}\} - \exp\{-2b\sqrt{s}\}) \right\}, \quad (5)
 \end{aligned}$$

for the dimension $d = 3$ and for the dimension $d = 5$ we obtain

$$\begin{aligned}
 \eta(5; 1; s) = & \frac{c\pi^3}{(b^2 - a^2)^3} \left[2! \left(\frac{1}{a} \exp\{-2a\sqrt{s}\} - \frac{1}{b} \exp\{-2b\sqrt{s}\} \right) \right. \\
 & - \frac{2 \cdot 3!}{(b^2 - a^2)\sqrt{s}} (\exp\{-2a\sqrt{s}\} + \exp\{-2b\sqrt{s}\}) \\
 & + \frac{4!}{(b^2 - a^2)^2 s} \left\{ a \exp\{-2a\sqrt{s}\} \left(1 + \frac{1}{2a\sqrt{s}} \right) \right. \\
 & \left. \left. - b \exp\{-2b\sqrt{s}\} \left(1 + \frac{1}{2b\sqrt{s}} \right) \right\} \right]. \quad (6)
 \end{aligned}$$

Making use of the formula

$$K_\nu(z) = \frac{1}{2} \left(\frac{1}{2} z \right)^\nu \int_0^\infty \exp\left\{-t - \frac{z^2}{4t}\right\} \frac{dt}{t^{\nu+1}}, \quad (7)$$

we obtain Laplace transforms for the odd dimensional case,

$$\begin{aligned}
 \eta(d; 1; s) = & \frac{c\pi^{d/2}}{\{(b^2 - a^2)^{l+1}\} \sum_{j=0}^l \frac{(-1)^{2l-j}(l+j)!}{(b^2 - a^2)^j} \binom{l}{j} 2^{j+1/2}} \cdot \\
 & \left\{ a^{2j-1} \frac{K_{(j-1)+1/2}(2a\sqrt{s})}{(2a\sqrt{s})^{j-1+1/2}} + (-1)^{l+1+j} b^{2j-1} \frac{K_{(j-1)+1/2}(2b\sqrt{s})}{(2b\sqrt{s})^{j-1+1/2}} \right\}, \quad (8)
 \end{aligned}$$

where

$$K_{n+1/2}(z) = \left(\frac{\pi}{2z} \right)^{1/2} \exp\{-z\} \sum_{r=0}^n \frac{(n+r)!}{r!(n-r)!(2z)^r} \quad (9)$$

(cf. G. Watson [7]).

3 Nonzero region of the Laplace transform

In general, we obtain

$$\begin{aligned} \eta(d; 1; s) &= c\pi^{l+1} \int_0^1 \frac{v^l(1-v)^l}{(b^2 - (b^2 - a^2)v)^{(d+2)/2}} \\ &\exp\{-2(b^2 - (b^2 - a^2)v)^{1/2}\sqrt{s}\} \\ &\sum_{r=0}^{l+1} \frac{(l+1+r)!}{r!(l+1-r)!2^r} \{(b^2 - (b^2 - a^2)v)^{1/2}\sqrt{s}\}^{l+1-r} dv \end{aligned} \quad (10)$$

for the odd dimension $d = 2l + 1$ and we see that $\eta(d; 1; s) \neq 0$ in a neighborhood of the origin except at the origin and $\eta(d; 1; s)$ is bounded in a neighborhood of the origin. From the above Laplace transforms (8) we can see that the Laplace transforms $\eta(d; 1; s)$ do not vanish in a neighborhood of the infinity point.

4 The Laplace transform for the general case

Let us obtain the Laplace transform of $g(d; n; y)$ for the odd dimension. The following equality holds:

$$\begin{aligned} &\exp\{-a_n^2 t\} \cdot t^{nl+n-1} \\ &\cdot \int \cdots \int_{0 \leq v_1 + \cdots + v_{n-1} \leq 1; 0 \leq v_j} \exp\left\{\sum_{j=1}^{n-1} (a_n^2 - a_j^2)tv_j\right\} \\ &\cdot [v_1 \cdots v_{n-1} (1 - \sum_{j=1}^{n-1} v_j)]^l dv_1 \cdots dv_{n-1} \\ &= (-1)^{(l+1)(n-1)} \sum_{r=1}^n \exp\{-a_r^2 t\} \\ &\cdot \left[\sum_{j_{n-1}=0}^l \frac{l!(l+j_{n-1})!}{j_{n-1}!(l-j_{n-1}+1)!} \frac{1}{B(r; n, n-1)^{l+j_{n-1}+1}} \right. \\ &\cdot \sum_{j_{n-2}=0}^{l-j_{n-1}} \frac{(l-j_{n-1})!(l+j_{n-2})!}{j_{n-2}!(l-j_{n-1}-j_{n-2})!} \frac{1}{B(r; n, n-2)^{l+j_{n-2}+1}} \\ &\cdot \sum_{j_{n-3}=0}^{l-j_{n-1}-j_{n-2}} \frac{(l-j_{n-1}-j_{n-2})!(l+j_{n-3})!}{j_{n-3}!(l-j_{n-1}-j_{n-2}-j_{n-3})!} \\ &\cdot \frac{1}{B(r; n, n-3)^{l+j_{n-3}+1}} \cdots \\ &\cdot \sum_{j_1=0}^{l-j_{n-1}-j_{n-2}-\cdots-j_2} \frac{(l-j_{n-1}-j_{n-2}-\cdots-j_2)!(l+j_1)!}{j_1!(l-j_{n-1}-j_{n-2}-\cdots-j_2-j_1)!} \\ &\cdot \left. \frac{1}{B(r; n, 1)^{l+j_1+1}} t^{l-j_1-\cdots-j_{n-2}-j_{n-1}} \right] \end{aligned} \quad (11)$$

where $B(r; n, 1) = a_r^2 - a_{r+1}^2, \dots, B(r; n, n-r) = a_r^2 - a_n^2, B(r; n, n-r+1) = a_r^2 - a_1^2, \dots, B(r; n, n-1) = a_r^2 - a_{r-1}^2$.

Theorem 1 *The Laplace transform is given in the following form.*

$$\begin{aligned}
 \eta(d; n; s) &= \int_0^\infty \exp\{-sy\} g(d; n; y) dy \\
 &= c_1 \pi^{l+1/2} (-1)^{(l+1)(n-1)} \\
 &\quad \sum_{r=1}^n \left[\sum_{j_{n-1}=0}^l \frac{l!(l+j_{n-1})!}{j_{n-1}!(l-j_{n-1}+1)!} \frac{1}{B(r; n, n-1)^{l+j_{n-1}+1}} \right. \\
 &\quad \sum_{j_{n-2}=0}^{l-j_{n-1}} \frac{(l-j_{n-1})!(l+j_{n-2})!}{j_{n-2}!(l-j_{n-1}-j_{n-2})!} \frac{1}{B(r; n, n-2)^{l+j_{n-2}+1}} \\
 &\quad \sum_{j_{n-3}=0}^{l-j_{n-1}-j_{n-2}} \frac{(l-j_{n-1}-j_{n-2})!(l+j_{n-3})!}{j_{n-3}!(l-j_{n-1}-j_{n-2}-j_{n-3})!} \\
 &\quad \frac{1}{B(r; n, n-3)^{l+j_{n-3}+1}} \cdots \\
 &\quad \sum_{j_1=0}^{l-j_{n-1}-j_{n-2}-\cdots-j_2} \frac{(l-j_{n-1}-j_{n-2}-\cdots-j_2)!(l+j_1)!}{j_1!(l-j_{n-1}-j_{n-2}-\cdots-j_2-j_1)!} \\
 &\quad \left. \frac{1}{B(r; n, 1)^{l+j_1+1}} 2 \left(\frac{2}{2a_r \sqrt{s}} \right)^{\sum_{p=1}^{n-1} j_p - 1/2} K_{\sum_{p=1}^{n-1} j_p - 1/2} (2a_r \sqrt{s}) \right] \quad (12)
 \end{aligned}$$

Here $K_{\sum_{p=1}^{n-1} j_p - 1/2} (2a_r \sqrt{s})$ is given by (9).

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