

# Recollement of Homotopy Categories and Cohen-Macaulay Modules \*

Kiriko Kato  
Osaka Prefecture University

## Abstract

We introduce the homotopy category of unbounded complexes with bounded homologies. We study a recollement of its a quotient by the homotopy category of bounded complexes. This leads to the existence of quotient categories which are equivalent to a homotopy category of acyclic complexes, that is a stable derived category. In the case of a coherent ring  $R$  of self-injective dimension both sides, we show that the above recollement are triangulated equivalent to a recollement of the stable module category of Cohen-Macaulay  $R$ -modules.

## 1 Introduction

We study two types of triangulated categories in this paper. One is the categories of homotopy classes of chain complexes, equipped with triangles induced by chain maps and mapping cones. The other is stable module categories that are module categories mod projective modules. A stable module category is not triangulated in general. Suppose that the module category is Frobenius; it has enough projectives and injectives, and the class of projectives coincide with that of injectives. Then it's projective stabilization is triangulated. This type of triangulated categories are called algebraic triangulated categories. A well-known example is the stable module category of Cohen-Macaulay modules over Gorenstein rings.

Let  $R$  be a two-sided noetherian ring. The categories of right  $R$ -modules, of finitely generated right  $R$ -modules and of finitely generated projective right  $R$ -modules are denoted by  $\text{Mod}R$  and  $\text{mod}R$ , and  $\text{proj} R$  respectively. Let  $K = K(\text{proj}R)$  be the category of homotopy classes of complexes of finitely generated  $R$ -projective complexes. The following triangulated subcategories of  $K$  are of our concern.

$$K^{\infty,b} = \{C \in K \mid H^i(C) = 0 \text{ (except for finite } i\text{'s)}\}$$

---

\*This is joint work with Osamu Iyama and Jun-ichi Miyachi.

$$K^{-,b} = \{C \in K^{\infty,b} \mid C^i = 0 \text{ (for sufficiently large } i)\}$$

$$K^{\infty,\emptyset} = \{C \in K^{\infty,b} \mid H^i(C) = 0 \text{ (} i \in \mathbb{Z})\}$$

$$K^b = \{C \in K \mid C^i = 0 \text{ (except for finite } i\text{'s)}\}$$

Those triangulated categories are all thick, so we have quotient categories which are triangulated.

**Definition 1.1 ([Iw])** *A two-sided noetherian ring is called Iwanaga-Gorenstein if  $\text{id}_R R < \infty$  and  $\text{id}_{R^{\text{op}}} R < \infty$ .*

If  $R$  is an Iwanaga-Gorenstein ring, we define a subcategory  $\text{CM}(R)$  of  $\text{mod } R$  as  $\text{CM}(R) = \{X \in \text{mod } R \mid \text{Ext}_R^i(X, R) = 0 \text{ (} i > 0)\}$ .

**Theorem 1.2 (Buchweitz [Bu])** *Assume  $R$  is Iwanaga-Gorenstein. The quotient category  $K^{-,b}/K^b$  is triangle equivalent to the stable module category  $\underline{\text{CM}}(R)$ .*

On the other hand, we observe the following.

**Lemma 1.3** *If  $R$  is Iwanaga-Gorenstein, the category  $K^{\infty,b}/K^{-,b}$  is equivalent to the stable module category  $\underline{\text{CM}}(R)$ .*

Naturally, the question arises: What is  $K^{\infty,b}/K^b$ ? Is it realizable as a stable module category? As an answer, we get the following Buchweitz-type theorem.

**Theorem 4.7** *Assume  $R$  is Iwanaga-Gorenstein. The quotient category  $K^{\infty,b}/K^b$  is triangle equivalent to the stable module category  $\underline{\text{CM}}(T_2(R))$  where  $T_2(R)$  is the ring of  $2 \times 2$  upper triangular matrices over  $R$ .*

## 2 The category $K^{\infty,b}$

For an object  $A$  of  $K^{\infty,b}$ , define objects  $X_A$  and  $T_A$  of  $K^{\infty,\emptyset}$  as follows.

Let  $l$  be the smallest integer such that  $H_l(A^*) \neq 0$ . Then  $\text{Cok } d_A^{l-1}$  is a maximal Cohen-Macaulay module. Define  $X_A \in K^{\infty,\emptyset}$  as

$$\tau_{\leq l} X_A = \tau_{\leq l} A$$

and

$$\cdots \rightarrow X_A^{l+1*} \rightarrow X_A^{l+2*} \rightarrow (\text{Cok } d_A^{l-1})^* \rightarrow 0$$

is exact. Then  $X_A$  is totally acyclic and  $\text{id}_{\text{Cok } d_A^{l-1}}$  induces a canonical chain map  $\xi_A : X_A \rightarrow A$  as  $\xi_A^i = \text{id}$  ( $i \leq l$ ).

Similarly, let  $r$  be the largest integer such that  $H^r(A) \neq 0$ . Then  $\text{Ker } d_A^r$  is a maximal Cohen-Macaulay module. Define  $T_A \in K^{\infty,\emptyset}$  as

$$\tau_{\geq r} X_A = \tau_{\geq r} A$$

and

$$\cdots \rightarrow T_A^{r-1} \rightarrow T_A^r \rightarrow (\text{Ker } d_A^r) \rightarrow 0$$

is exact. Then  $T_A$  is totally acyclic and  $\text{id}_{\text{Ker } d_A^r}$  induces a canonical chain map  $\zeta_A : A \rightarrow T_A$  as  $\zeta_A^i = \text{id}$  ( $i \geq r$ ).

Set a chain maps  $l_A : L_A \rightarrow A$  and  $r_{L_A} : L_A \rightarrow R_{L_A}$  as follows:

$$\begin{aligned} \tau_{\leq 0} L_A &= \tau_{\leq 0} X_A, \tau_{\geq 1} L_A = \tau_{\geq 1} A, \\ \tau_{\leq 0} l_A &= \tau_{\leq 0} \xi_A, \tau_{\geq 1} l_A = \tau_{\geq 1} \text{id}_A, \\ \tau_{\leq 0} R_{L_A} &= \tau_{\leq 0} L_A, \tau_{\geq 1} R_{L_A} = \tau_{\geq 1} T_{L_A}, \\ \tau_{\leq 0} r_{L_A} &= \tau_{\leq 0} \text{id}_{L_A}, \tau_{\geq 1} r_{L_A} = \tau_{\geq 1} \zeta_A \end{aligned}$$

Obviously  $C(l_A)$  and  $C(r_{L_A})$  belongs to  $\mathbf{K}^b$ , hence as an object of  $\mathbf{K}^{\infty, b}/\mathbf{K}^b$ ,  $A$  is isomorphic to the complex

$$R_{L_A} : \cdots \rightarrow X_A^{-1} \rightarrow X_A^0 \rightarrow T_A^1 \rightarrow T_A^2 \rightarrow \cdots$$

We may assume  $\lambda_A = H^0(\tau_{\leq 0} \xi_A \zeta_A) : \text{Cok } d_{X_A}^{-1} \rightarrow \text{Ker } d_{T_A}^1$  to be surjective by adding some split exact sequence of projective modules if necessary.

### 3 A functor to the category of morphisms

We define category  $\text{Mor}(R)$  as follows: objects of  $\text{Mor}(R)$  are the morphisms  $\alpha : X_\alpha \rightarrow T_\alpha$  of  $\text{Mod}(R)$ . For  $\alpha, \beta \in \text{Mor}(R)$ , we define the set of morphisms from  $\alpha$  to  $\beta$  as

$$\{(f_X, f_T) \in \text{Hom}_R(X_\alpha, X_\beta) \times \text{Hom}_R(T_\alpha, T_\beta) \mid f_T \alpha = \beta f_X\}.$$

And the subcategory  $\text{mor}_s^{CM}(R)$  of  $\text{Mor}(R)$  consists of the objects  $\alpha : X_\alpha \rightarrow T_\alpha$  of  $\text{CM}(R)$  that are surjective. The structure of  $\text{mor}_s^{CM}(R)$  is obtained by the next lemma.

**Lemma 3.1** *Let  $T_2(R)$  be the category of  $2 \times 2$  upper triangular matrices with entries in  $R$ . Then  $\text{Mod}(T_2(R))$  is equivalent to  $\text{Mor}(R)$ . And  $\text{mor}_s^{CM}(R)$  is equivalent to the category  $\text{CM}(T_2(R))$ .*

**proof.** An object  $f : X_f \rightarrow T_f$  of  $\text{Mor}(R)$  corresponds to an  $T_2(R)$ -module  $M_f = X_f \times T_f$  where  $(x \ t) \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = (xa \ f(x)b + tc)$ . This correspondence gives an equivalence between  $\text{CM}(T_2(R))$  and  $\text{mor}_i^{CM}(R)$  consisting of injective maps  $\alpha : X_\alpha \rightarrow T_\alpha$  with  $X_\alpha, T_\alpha, \text{Cok } f \in \text{CM}(R)$ . Obviously  $\text{mor}_i^{CM}(R)$  is equivalent to  $\text{mor}_s^{CM}(R)$ . (q.e.d.)

Thus  $\text{mor}_s^{CM}(R)$  is a Frobenius category together with projective-injective objects consisting of  $p \in \text{mor}_s^{CM}(R)$  that  $X_p$  and  $T_p$  are

projective modules. Hence the stable category  $\underline{\text{mor}}_s^{CM}(R)$  is triangulated.

We shall construct a functor between  $K^{\infty,b}/K^b$  and  $\underline{\text{mor}}_s^{CM}(R)$ . Let  $\alpha : X_\alpha \rightarrow T_\alpha$  be an object of  $\text{mor}_s^{CM}(R)$  and let  $F_{X_\alpha}$  and  $F_{T_\alpha}$  be acyclic projective complexes such that  $H^0(\tau_{\leq 0} F_{X_\alpha}) = X_\alpha$  and  $H^0(\tau_{\leq 0} F_{T_\alpha}) = T_\alpha$ . Set natural maps  $\rho : F_{X_\alpha}^0 \rightarrow X_\alpha$  and  $\epsilon : T_\alpha \rightarrow F_{T_\alpha}$ . Make a projective complex  $F_\alpha$  as

$$\tau_{\leq 0} F_\alpha = \tau_{\leq 0} F_{X_\alpha}, \quad \tau_{\geq 1} F_\alpha = \tau_{\geq 1} F_{T_\alpha}, \quad d_{F_\alpha} = \epsilon \alpha \rho.$$

**Lemma 3.2** 1) A morphism  $f \in \text{mor}_s^{CM}(R)(\alpha, \beta)$  induces a chain map  $F_f : F_\alpha \rightarrow F_\beta$ .

2) For morphisms  $f \in \text{mor}_s^{CM}(R)(\alpha, \beta)$  and  $g \in \text{mor}_s^{CM}(R)(\beta, \gamma)$ ,  $F_{gf} = F_g F_f$ .

3) An exact sequence  $0 \rightarrow \alpha \xrightarrow{f} \beta \xrightarrow{g} \gamma \rightarrow 0$  in  $\text{mor}_s^{CM}(R)$  induces an exact sequence  $0 \rightarrow F_\alpha \xrightarrow{F_f} F_\beta \xrightarrow{F_g} F_\gamma \rightarrow 0$  in  $C^{\infty,b}$ .

4) An object  $p$  of  $\text{mor}_s^{CM}(R)$  is projective if and only if  $F_p$  is a bounded complex.

**Lemma 3.3** The operation  $F$  gives a functor  $\text{mor}_s^{CM}(R) \rightarrow K^{\infty,b}$ . And  $F$  induces a functor  $\underline{F} : \underline{\text{mor}}_s^{CM}(R) \rightarrow K^{\infty,b}/K^b$ .

**Proposition 3.4** The functor  $\underline{F} : \underline{\text{mor}}_s^{CM}(R) \rightarrow K^{\infty,b}/K^b$  is triangulated.

We state that  $\underline{F}$  is a triangle equivalence. It leads us to the answer to our question together with Lemma 3.1.

**Theorem 3.5** The category  $K^{\infty,b}/K^b$  is triangle equivalent to  $\underline{\text{mor}}_s^{CM}(R)$ .

To prove the equivalence of  $\underline{F}$ , we have already seen that  $\underline{F}$  is dense from the previous section. For the proof of fully faithfulness, we use some torsion structure in our category.

## 4 Stable t-structures

**Definition 4.1** ([Mil]) For full subcategories  $\mathcal{U}$  and  $\mathcal{V}$  of a triangulate category  $\mathcal{C}$ ,  $(\mathcal{U}, \mathcal{V})$  is called a stable t-structure in  $\mathcal{C}$  provided that

- $\mathcal{U}$  and  $\mathcal{V}$  are stable for translations.
- $\text{Hom}_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$ .
- For every  $X \in \mathcal{C}$ , there exists a triangle  $U \rightarrow X \rightarrow V \rightarrow \Sigma U$  with  $U \in \mathcal{U}$  and  $V \in \mathcal{V}$ .

**Proposition 4.2** ([BBD], [Mi3]) *Let  $\mathcal{C}$  be a triangulated category. The following hold.*

- 1 *Let  $(\mathcal{U}, \mathcal{V})$  be a stable  $t$ -structure in  $\mathcal{C}$ ,  $i_* : \mathcal{U} \rightarrow \mathcal{C}$  and  $j_* : \mathcal{V} \rightarrow \mathcal{C}$  the canonical embeddings. Then there are a right adjoint  $i^! : \mathcal{C} \rightarrow \mathcal{U}$  of  $i_*$  and a left adjoint  $j^* : \mathcal{C} \rightarrow \mathcal{V}$  of  $j_*$  which satisfy the following.*

$$(a) \ j^* i_* = 0, \ i^! j_* = 0.$$

- (b) *The adjunction arrows  $i_* i^! \rightarrow 1_{\mathcal{C}}$  and  $1_{\mathcal{C}} \rightarrow j_* j^*$  imply a triangle  $i_* i^! X \rightarrow X \rightarrow j_* j^* X \rightarrow \Sigma i_* i^! X$  for any  $X \in \mathcal{C}$ .*

*In this case,  $j^*$  (resp.,  $i^!$ ) implies the triangulated equivalence  $\mathcal{C}/\mathcal{U} \simeq \mathcal{V}$  (resp.,  $\mathcal{C}/\mathcal{V} \simeq \mathcal{U}$ ).*

- 2 *If  $\{\mathcal{C}, \mathcal{C}''; j^*, j_*\}$  (resp.,  $\{\mathcal{C}, \mathcal{C}''; j_!, j^*\}$ ) is a localization (resp., a colocalization) of  $\mathcal{C}$ , that is,  $j_*$  (resp.,  $i_*$ ) is a fully faithful right (resp., left) adjoint of  $i^!$ , then  $(\text{Ker } j^*, \text{Im } j_*)$  (resp.,  $(\text{Im } j_!, \text{Ker } j^*)$ ) is a stable  $t$ -structure. In this case, the adjunction arrow  $1_{\mathcal{C}} \rightarrow j_* j^*$  (resp.,  $j_! j^* \rightarrow 1_{\mathcal{C}}$ ) implies triangles*

$$\begin{aligned} U &\rightarrow X \rightarrow j_* j^* X \rightarrow \Sigma U \\ (\text{resp., } j_! j^* X &\rightarrow X \rightarrow V \rightarrow \Sigma j_! j^* X) \end{aligned}$$

*with  $U \in \text{Ker } j^*$ ,  $j_* j^* X \in \text{Im } j_*$  (resp.,  $j_! j^* X \in \text{Im } j_!$ ,  $V \in \text{Ker } j^*$ ) for all  $X \in \mathcal{C}$ .*

**Definition 4.3** *Let  $\mathcal{D}_1, \mathcal{D}_2$  be triangulated categories. Let  $(\mathcal{U}_i, \mathcal{V}_i)$  be stable  $t$ -structures in  $\mathcal{D}_i$  ( $i = 1, 2$ ). A triangle functor  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  sends a stable  $t$ -structure  $(\mathcal{U}_1, \mathcal{V}_1)$  to a stable  $t$ -structure  $(\mathcal{U}_2, \mathcal{V}_2)$  if  $F(\mathcal{U}_1) \subset \mathcal{U}_2$  and  $F(\mathcal{V}_1) \subset \mathcal{V}_2$ .*

**Lemma 4.4** *If a triangle functor  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  sends a stable  $t$ -structure  $(\mathcal{U}_1, \mathcal{V}_1)$  in  $\mathcal{D}_1$  to a stable  $t$ -structure  $(\mathcal{U}_2, \mathcal{V}_2)$  of  $\mathcal{D}_1$ . Then we have the following:*

- 1 *If  $F|_{\mathcal{U}_1}$  is full (faithful), then  $\text{Hom}_{\mathcal{D}_1}(U, X) \rightarrow \text{Hom}_{\mathcal{D}_2}(FU, FX)$  is surjective (injective) for  $U \in \mathcal{U}_1$  and  $X \in \mathcal{D}_1$ .*
- 2 *If  $F|_{\mathcal{V}_1}$  is full (faithful), then  $\text{Hom}_{\mathcal{D}_1}(X, V) \rightarrow \text{Hom}_{\mathcal{D}_2}(FX, FV)$  is surjective (injective) for  $X \in \mathcal{D}_1$  and  $V \in \mathcal{V}_1$ .*
- 3 *If  $F$  is fully faithful and  $F|_{\mathcal{U}_1}$  and  $F|_{\mathcal{V}_1}$  are equivalences, then  $F$  is an equivalence.*

**Corollary 4.5** *Let  $\mathcal{D}_1, \mathcal{D}_2$  be triangulated categories. Let  $(\mathcal{U}_n, \mathcal{V}_n)$  and  $(\mathcal{V}_n, \mathcal{W}_n)$  be stable  $t$ -structures in  $\mathcal{D}_n$  ( $n = 1, 2$ ). Assume a triangle functor  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  sends  $(\mathcal{U}_1, \mathcal{V}_1)$  and  $(\mathcal{V}_1, \mathcal{W}_1)$  to  $(\mathcal{U}_2, \mathcal{V}_2)$  and  $(\mathcal{V}_2, \mathcal{W}_2)$  respectively. If  $F|_{\mathcal{U}_1}$ ,  $F|_{\mathcal{V}_1}$  and  $F|_{\mathcal{W}_1}$  are fully faithful (equivalent), so is  $F$ .*

**proof** For given  $X, Y \in \mathcal{D}_1$ , there are triangles

$$U_X \rightarrow X \rightarrow V^X \rightarrow \Sigma U_X$$

$$U_Y \rightarrow Y \rightarrow V^Y \rightarrow \Sigma U_Y$$

where  $U_X, U_Y \in \mathcal{U}_1$  and  $V^X, V^Y \in \mathcal{V}_1$ . These triangles induce a diagram of abelian groups with exact rows and columns:

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{D}_1}(V^X, U_Y) & \rightarrow & \mathrm{Hom}_{\mathcal{D}_1}(X, U_Y) & \rightarrow & \mathrm{Hom}_{\mathcal{D}_1}(U_X, U_Y) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}_1}(V^X, Y) & \rightarrow & \mathrm{Hom}_{\mathcal{D}_1}(X, Y) & \rightarrow & \mathrm{Hom}_{\mathcal{D}_1}(U_X, Y) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{D}_1}(V^X, V^Y) & \rightarrow & \mathrm{Hom}_{\mathcal{D}_1}(X, V^Y) & \rightarrow & \mathrm{Hom}_{\mathcal{D}_1}(U_X, V^Y) \end{array}$$

Since  $\mathrm{Hom}_{\mathcal{D}_1}(U, Z) \rightarrow \mathrm{Hom}_{\mathcal{D}_2}(FU, FZ)$ ,  $\mathrm{Hom}_{\mathcal{D}_1}(Z, V) \rightarrow \mathrm{Hom}_{\mathcal{D}_2}(FZ, FV)$ , and  $\mathrm{Hom}_{\mathcal{D}_1}(V, Z) \rightarrow \mathrm{Hom}_{\mathcal{D}_2}(FV, FZ)$  are bijective by Lemma 4.4 1) 2), so is

$\mathrm{Hom}_{\mathcal{D}_1}(X, Y) \rightarrow \mathrm{Hom}_{\mathcal{D}_2}(FX, FY)$  from five-lemma. In the case that  $F|_{\mathcal{U}_1}$ ,  $F|_{\mathcal{V}_1}$  and  $F|_{\mathcal{W}_1}$  are equivalences,  $F$  is dense by Lemma 4.4 3). (q.e.d.)

**Proposition 4.6** *Let  $\mathcal{D}_1, \mathcal{D}_2$  be triangulated categories. Let  $(\mathcal{U}_i, \mathcal{V}_i)$ ,  $(\mathcal{V}_i, \mathcal{W}_i)$  and  $(\mathcal{W}_i, \mathcal{U}_i)$  be stable  $t$ -structures in  $\mathcal{D}_i$  ( $i = 1, 2$ ). Assume a triangle functor  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  sends stable  $t$ -structures  $(\mathcal{U}_1, \mathcal{V}_1)$ ,  $(\mathcal{V}_1, \mathcal{W}_1)$  and  $(\mathcal{W}_1, \mathcal{U}_1)$  to  $(\mathcal{U}_2, \mathcal{V}_2)$ ,  $(\mathcal{V}_2, \mathcal{W}_2)$  and  $(\mathcal{W}_2, \mathcal{U}_2)$  respectively. If  $F|_{\mathcal{U}}$  is fully faithful (equivalent)  $k$ , so is  $F$ .*

**Proposition 4.7** *Let  $R$  be a coherent ring. Then we have the following.*

- $(K^{-,b}, K^{\infty, \emptyset})$  is a stable  $t$ -structure of  $K^{\infty, b}$ . Hence  $(K^{-,b}/K^b, K^{\infty, \emptyset})$  is a stable  $t$ -structure of  $K^{\infty, b}/K^b$ .
- $(K^{+,b}/K^b, K^{-,b}/K^b)$  is a stable  $t$ -structure of  $K^{\infty, b}/K^b$ .
- If  $R$  is Iwanaga-Gorenstein, then  $(K^{\infty, \emptyset}/K^b, K^{+,b}/K^b)$  is a stable  $t$ -structure of  $K^{\infty, b}/K^b$ .

Let  $R$  be an Iwanaga-Gorenstein ring. Let  $\underline{\mathcal{CM}}_0$  (resp.,  $\underline{\mathcal{CM}}_1$ ,  $\underline{\mathcal{CM}}_p$ ) be the full subcategory of  $\underline{\mathrm{mor}}_s^{CM}(R)$  consisting of objects of the form  $X \rightarrow 0$  (resp.,  $S \xrightarrow{\equiv} S$ ,  $P \rightarrow T$ , with  $P$  being projective).

**Proposition 4.8** *The following are stable  $t$ -structures of  $\underline{\mathrm{mor}}_s^{CM}(R)$ .*

$$(\underline{\mathcal{CM}}_0, \underline{\mathcal{CM}}_1), (\underline{\mathcal{CM}}_p, \underline{\mathcal{CM}}_0), (\underline{\mathcal{CM}}_1, \underline{\mathcal{CM}}_p).$$

**Proposition 4.9** *The triangulated functor  $F$  induces equivalences*

$$\begin{aligned} \underline{F}|_{\underline{\mathcal{CM}}_0} : \underline{\mathcal{CM}}_0 &\rightarrow K^{-,b}/K^b, \\ \underline{F}|_{\underline{\mathcal{CM}}_1} : \underline{\mathcal{CM}}_1 &\rightarrow K^{\infty, \emptyset}, \\ \text{and } \underline{F}|_{\underline{\mathcal{CM}}_p} : \underline{\mathcal{CM}}_p &\rightarrow K^{+,b}/K^b. \end{aligned}$$

**proof.** It is well-known that  $\underline{\text{CM}}(R)$  is equivalent to  $K^{\infty, \emptyset}$ , by the correspondence  $P : X \mapsto P_X^\bullet$  where  $P_X^\bullet$  is the complete resolution of  $X$ ;  $P_X^{-1} \rightarrow P_X^0 \rightarrow X \rightarrow 0$  and  $0 \rightarrow X \rightarrow P_X^1 \rightarrow P_X^2$  are exact. The functor  $\underline{F} : \underline{\text{CM}}_1 \rightarrow \underline{\text{CM}}(R) : (X \xrightarrow{\sim} X) \mapsto X$ , hence is an equivalence. (q.e.d.)

**The proof of Theorem 3.5.** We easily see  $\underline{F}(\underline{\text{CM}}_0) \subset K^{-,b}/K^b$ ,  $\underline{F}(\underline{\text{CM}}_1) \subset K^{\infty, \emptyset}$ , and  $\underline{F}(\underline{\text{CM}}_p) \subset K^{+,b}/K^b$ . Propositions 4.6 and 4.9 imply the equivalence of  $\underline{F}$ .

Together with lemma 3.1, we obtain a Buchweitz-type theorem:

**Theorem 4.10** *If  $R$  is Iwanaga-Gorenstein, then  $K^{\infty, b}/K^b$  is triangle equivalent to  $\underline{\text{CM}}(T_2(R))$ .*

## 5 Polygon of Recollements

**Definition 5.1** *Let  $\mathcal{C}$  be a triangulated categories and let  $U_1, U_2, \dots, U_n$  be full subcategories of  $\mathcal{C}$ .  $(U_1, U_2, \dots, U_n)$  is an  $n$ -gon of recollements in  $\mathcal{C}$  if  $(U_1, U_2), (U_2, U_3), \dots, (U_{n-1}, U_n), (U_n, U_1)$  are stable  $t$ -structures in  $\mathcal{C}$ .*

The  $n$ -gon of recollements is not bizarre. The property of the category  $\mathcal{C}$  often naturally induces an  $n$ -gon of recollements. For instance, let  $k$  be a field, and let  $\mathcal{C}$  be a  $m/n$ -Calabi-Yau category. That is,  $\mathcal{C}$  is a  $k$ -linear triangulated category with a Serre functor  $S : \mathcal{C} \rightarrow \mathcal{C}$ , which satisfies  $\text{Hom}_{\mathcal{C}}(X, Y) \cong \text{D Hom}_{\mathcal{C}}(Y, SX)$ . and  $\Sigma^m \cong S^n$ . Then we have the following.

**Proposition 5.2** *For any functorially finite thick subcategory  $\mathcal{U}$  of  $\mathcal{C}$ ,  $(\mathcal{U}, \mathcal{U}^\perp, S\mathcal{U}, (S\mathcal{U})^\perp, \dots, S^{n-1}\mathcal{U}, (S^{n-1}\mathcal{U})^\perp)$  is a  $2n$ -gon of recollements in  $\mathcal{C}$ .*

Since  $\text{D}^b(\text{mod } T_n(k))$  is an  $n-1/n+1$ -Calabi-Yau category, Applying the previous lemma, we have the following.

**Lemma 5.3** *The bounded derived category  $\text{D}^b(\text{mod } T_n(k))$  of  $\text{mod } T_n(k)$  has a  $2(n+1)$ -gon of recollements @  $(\mathcal{U}, \mathcal{U}^\perp, S\mathcal{U}, (S\mathcal{U})^\perp, \dots, S^n\mathcal{U}, (S^n\mathcal{U})^\perp)$  where  $\mathcal{U}$  is a functorially finite thick subcategory.*

*In the case  $n = 2n'$  is even, there is an  $(n+1)$ -gon of recollements  $(\mathcal{U}, S^{n'+1}\mathcal{U}, S\mathcal{U}, S^{n'+2}\mathcal{U}, \dots, S^{n'-2}\mathcal{U}, S^{n-1}\mathcal{U}, S^{n'-1}\mathcal{U})$  by a suitable choice of  $\mathcal{U}$ .*

This induces the following, which includes our case.

**Corollary 5.4** *Let  $R$  be an Iwanaga-gorenstein ring. Then,  $\underline{\text{CM}}(T_n(R))$  has a  $2(n+1)$ -gon of recollements. If  $n$  is even, then  $\underline{\text{CM}}(T_n(R))$  has an  $(n+1)$ -gon of recollements.*

Finally, we give a definition of "recollement" appeared in the title.

**Definition 5.5** ([BBD]) *A nine-tuple  $\{C', C, C''; j^*, j_*, j^!, s_!, s^*, s_*\}$  consisting of triangulated categories and functors*

$$\begin{array}{ccccc} & \xleftarrow{j^*} & & \xleftarrow{s_!} & \\ C' & \xrightarrow{j_*} & C & \xrightarrow{s^*} & C'' \\ & \xleftarrow{j^!} & & \xleftarrow{s_*} & \end{array}$$

*is called a recollement if it satisfies the following:*

- $j_*$ ,  $s_!$ , and  $s_*$  are fully faithful.
- $(j^*, j_*)$ ,  $(j_*, j^!)$ ,  $(s_!, s^*)$ , and  $(s^*, s_*)$  are adjoint pairs.
- $j^*s_! = 0$ ,  $s^*j_* = 0$ , and  $j^!s_* = 0$ .
- For each object  $C$  of  $C$  has triangles

$$\begin{aligned} j_*j^!C &\rightarrow C \rightarrow s_!s^*C \rightarrow \Sigma j_*j^!C, \\ s_*s^*C &\rightarrow C \rightarrow j_*j^*C \rightarrow \Sigma s_*s^*C. \end{aligned}$$

As a recollement implies two consecutive stable t-structures as a localization does one stable t-structure.

**Proposition 5.6** ([BBD], [Mi1]) *1) If  $(\mathcal{U}, \mathcal{V})$  and  $(\mathcal{V}, \mathcal{W})$  are stable t-structures of  $C$ , then the canonical embedding  $j_* : \mathcal{V} \rightarrow C$  produces a recollement*

$$\begin{array}{ccccc} & \xleftarrow{j^*} & & \xleftarrow{s_!} & \\ \mathcal{V} & \xrightarrow{j_*} & C & \xrightarrow{s^*} & C/\mathcal{V} \\ & \xleftarrow{j^!} & & \xleftarrow{s_*} & \end{array}$$

*2) If  $\{C', C, C''; j^*, j_*, j^!, s_!, s^*, s_*\}$  is a recollement, then  $(\text{Im } j_*, \text{Im } s_*)$  and  $(\text{Im } s_!, \text{Im } j_*)$  are stable t-structures.*

## References

- [BBD] A. A. Beilinson, J. Bernstein and P. Deligne, Faisceaux Pervers, Astérisque **100** (1982).
- [Be] A. Beligiannis, The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co-)stabilization, Comm. Algebra **28** (2000), 4547-4596.
- [Bu] R.O. Buchweitz, Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings, Unpublished manuscript (1987), 155 pp.

- [Iw] Y. Iwanaga, On rings with finite self-injective dimension II, *Tsukuba J. Math.*, Vol. 4 (1980), 107-113.
- [IKM] O. Iyama, K. Kato, J. Miyachi, Recollement of homotopy categories and Cohen-Macaulay modules, in preparation.
- [IK] S. Iyengar, H. Krause, Acyclicity versus total acyclicity for complexes over noetherian rings, *Documenta Math.*, 11 (2006), 207-240.
- [Ke] B. Keller, Deriving DG categories, *Ann. Sci. École Norm. Sup. (4)* 27 (1994), 637-102.
- [Fr] J. Franke, On the Brown representability theorem for triangulated categories. *Topology* 40 (2001), no. 4, 667-680.
- [KV] B. Keller, D. Vossieck, Sous les catégories dérivées, *C. R. Acad. Sci. Paris*, 305, Série I, 1987, 225-228.
- [Kr] H. Krause, The stable derived category of a noetherian scheme, *Compos. Math.*, 141 (2005), 1128-1162.
- [LAM] Leovigildo Alonso Tarrío, Ana Jeremías López, María José Souto Salorio, Localization in categories of complexes and unbounded resolutions, *Canad. J. Math.* 52 (2000), no. 2, 225-247.
- [Mi1] J. Miyachi, Localization of Triangulated Categories and Derived Categories, *J. Algebra* 141 (1991), 463-483.
- [Mi2] J. Miyachi, Duality for Derived Categories and Cotilting Bimodules, *J. Algebra* 185 (1996), 583 - 603.
- [Mi3] J. Miyachi, Recollement and tilting complexes, *Journal of Pure and Applied Algebra*, 183 (2003), 245-273.
- [My] Y. Miyashita, Tilting modules of finite projective dimension. *Math. Z.* 193 (1986), no. 1, 113-146.
- [Rd1] J. Rickard, Morita Theory for Derived Categories, *J. London Math. Soc.* 39 (1989), 436-456.
- [Sp] N. Spaltenstein, Resolutions of Unbounded Complexes, *Composition Math.* 65 (1988), 121-154.
- [Ve] J. Verdier, "Catéories Déivées, état 0", pp. 262-311, *Lecture Notes in Math.* 569, Springer-Verlag, Berlin, 1977.

K. Kato: Graduate School of Science, Osaka Prefecture University,  
 1-1 Gakuen-cho, Nakaku, Sakai, Osaka 599-8531, JAPAN  
 kiriko@mi.s.osakafu-u.ac.jp