

Some remarks on generalized inverse $\ast$ -semigroups II¹

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Abstract

By using a concept of representations of generalized inverse $\ast$ -semigroups [2], we introduce a new partial product on a generalized inverse $\ast$ -semigroup. The purpose of this paper is to give a characterization of prehomomorphisms of generalized inverse $\ast$ -semigroups.

1 Introduction

A semigroup S with a unary operation $\ast : S \rightarrow S$ is called a *regular $\ast$ -semigroup* if it satisfies

$$(i) (x^\ast)^\ast = x; \quad (ii) (xy)^\ast = y^\ast x^\ast; \quad (iii) xx^\ast x = x.$$

Let S be a regular $\ast$ -semigroup. An idempotent e in S is called a *projection* if $e^\ast = e$. For a subset A of S , denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively.

Let S be a regular $\ast$ -semigroup. Define a relation $\leq$ on S as follows:

$$a \leq b \iff a = eb = bf \text{ for some } e, f \in P(S).$$

A regular $\ast$ -semigroup S is called a *generalized inverse $\ast$ -semigroup* if $E(S)$ satisfies the identity $xyzw = xzyw$. In this case, $E(S)$ forms a band.

Result 1.1. [1] *Let a and b be elements of S . Then the following conditions are equivalent:*

- (1) $a \leq b$,
- (2) $aa^\ast = ba^\ast$ and $a^\ast a = b^\ast a$,
- (3) $aa^\ast = ab^\ast$ and $a^\ast a = a^\ast b$,
- (4) $a = aa^\ast b = ba^\ast a$.

Moreover, if S is a generalized inverse $\ast$ -semigroup, the conditions above are equivalent to the following:

¹This paper is an abstract and the details will be published elsewhere.

(5) $a = eb = bf$ for some $e, f \in E(S)$.

Result 1.2. [1] *The relation $\leq$ on a regular $*$ -semigroup, defined above, is a partial order on S which preserves the unary operation. Moreover, if S is a generalized inverse $*$ -semigroup, $\leq$ is compatible.*

We call the partial order $\leq$, defined above, *the natural order on S* .

Let S and T be regular $*$ -semigroups. A mapping $\phi : S \rightarrow T$ is called a *prehomomorphism*, if it satisfies

$$(i) \quad (ab)\phi \leq (a\phi)(b\phi),$$

$$(ii) \quad (a\phi)^* = a^*\phi,$$

for all $a, b \in S$.

Result 1.3. [1] *Let ϕ be a prehomomorphism of a regular $*$ -semigroup S to a regular $*$ -semigroup T . Then we have the following:*

(1) *ϕ maps an idempotent of S to an idempotent of T , and so it maps a projection of S to a projection of T ,*

(2) *ϕ is isotone, that is, $a \leq b$ implies $a\phi \leq b\phi$,*

As a generalization of the Preston-Vagner representations, we obtain a representation of a generalized inverse $*$ -semigroup [2]. A non-empty set X with an equivalence relation σ is called a *transitive ι -set*, and denoted by $(X; \sigma)$. Let $(X; \sigma)$ be a transitive ι -set. A subset A of X is called an *ι -single subset* of $(X; \sigma)$ if there exists at most one element of A for each equivalence class induced by σ , that is, $x\sigma y$ ($x, y \in A$) implies $x = y$. Denote the set of all ι -single subsets of $(X; \sigma)$ by $\mathbf{T}$. A mapping α in $\mathcal{I}_X$, the symmetric inverse semigroup on X , is called a *partial one-to-one ι -mapping* on $(X; \sigma)$ if $d(\alpha), r(\alpha)$ are both ι -single subsets of $(X; \sigma)$, where $d(\alpha)$ and $r(\alpha)$ are the domain and the range of α , respectively. Denote the set of all partial one-to-one ι -mappings of $(X; \sigma)$ by $\mathcal{GI}_{(X; \sigma)}$.

For any ι -single subsets A and B of $(X; \sigma)$, define $\theta_{A, B}$ by

$$\theta_{A, B} = \{(a, b) \in A \times B : (a, b) \in \sigma\} = (A \times B) \cap \sigma.$$

Since a subset of an ι -single subset is also an ι -single subset, $\theta_{A, B} \in \mathcal{GI}_{(X; \sigma)}$. For any $\alpha, \beta \in \mathcal{GI}_{(X; \sigma)}$, define $\theta_{\alpha, \beta}$ by $\theta_{\alpha, \beta} = \theta_{r(\alpha), d(\beta)}$, and let $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in \mathcal{GI}_{(X; \sigma)}\}$, an indexed set of one-to-one partial functions. Now, define a multiplication $\circ$ and a unary operation $*$ on $\mathcal{GI}_{(X; \sigma)}$ as follows:

$$\alpha \circ \beta = \alpha \theta_{\alpha, \beta} \beta \quad \text{and} \quad \alpha^* = \alpha^{-1},$$

¹It is called a $\vee$ -prehomomorphism in [4]

where the multiplication of the right side of the first equality is that of $\mathcal{I}_X$. Denote $(\mathcal{GI}_{(X;\sigma)}, \circ, *)$ by $\mathcal{GI}_{(X;\sigma)}(\mathcal{M})$ or simply by $\mathcal{GI}_{(X;\sigma)}$. In this paper, we use $\mathcal{GI}_{(X;\sigma)}$ rather than $\mathcal{GI}_{(X;\sigma)}(\mathcal{M})$.

Result 1.4. [2] *For a transitive ι -set $(X; \sigma)$, we have the following:*

- (1) *The $*$ -groupoid $\mathcal{GI}_{(X;\sigma)}$, defined above, is a generalized inverse $*$ -semigroup. Moreover, any generalized inverse $*$ -semigroup can be embedded (up to $*$ -isomorphism) in $\mathcal{GI}_{(X;\sigma)}$ on some transitive ι -set $(X; \sigma)$.*
- (2) *$E(\mathcal{GI}_{(X;\sigma)}) = \mathcal{M}$ and $P(\mathcal{LI}_{(X;\sigma)}) = \{1_A : A \text{ is an } \iota\text{-single subset of } (X; \sigma)\}$.*
- (3) *If σ is the identity relation on X , then $\mathcal{GI}_{(X;\sigma)}$ is the symmetric inverse semigroup $\mathcal{I}_X$ on X .*

2 Characterization of prehomomorphisms

Let S be a generalized inverse $*$ -semigroup. For any element $a \in S$, aa^* and a^*a by $d(a)$ and $r(a)$, respectively. Define a new partial product $\cdot$ on S as follows:

$$a \cdot b = \begin{cases} ab & \text{if } r(a) = d(a^*abb^*) \text{ and } d(b) = r(a^*abb^*) \\ \text{undefined} & \text{otherwise} \end{cases}$$

The partial product $\cdot$ is called a *restricted product* of S .

Lemma 2.1. *Let a and b be elements of a generalized inverse $*$ -semigroup S .*

- (1) *$a \cdot b$ is defined if and only if $a^*a = a^*abb^*a^*a$ and $bb^* = bb^*a^*abb^*$.*
- (2) *If $a \cdot b$ is defined, then $d(a \cdot b) = d(a)$ and $r(a \cdot b) = r(b)$.*

The following lemma is a basic property of the restricted product of S .

Lemma 2.2. *Let S be a generalized inverse $*$ -semigroup.*

- (1) *Let x be an element of S and e a projection of S such that $e \leq x^*x$. Then $a = xe$ is the unique element in S such that $a \leq x$ and $a^*a = e$.*
- (2) *Let x be an element of S and e a projection of S such that $e \leq xx^*$. Then $a = ex$ is the unique element in S such that $a \leq x$ and $aa^* = e$.*
- (3) *For any elements $x, y \in S$, $xy = a \cdot b$ where $a = xe$, $b = fy$, $e = x^*xyy^*x^*x$ and $f = yy^*x^*xyy^*$.*

Lemma 2.3. *Let $\phi : S \rightarrow T$ be a prehomomorphism of a generalized inverse $*$ -semigroup S to a generalized inverse $*$ -semigroup T , and a, b elements of S .*

- (1) *$(aa^*)\phi = (a\phi)(a\phi)^*$ and $(a^*a)\phi = (a\phi)^*(a\phi)$.*

- (2) If $a \cdot b$ is defined, then $a\phi \cdot b\phi$ is defined and $((a \cdot b)\phi)^*(a \cdot b)\phi = (a\phi \cdot b\phi)^*(a\phi \cdot b\phi)$.
- (3) By the lemma above, $ab = (ae) \cdot (fb)$ where $e = a^*abb^*a^*a$ and $f = bb^*a^*abb^*$. If ϕ satisfies that $(gh)\phi = (g\phi)(h\phi)$ for any $g, h \in E(S)$, then $(ae)\phi = (a\phi)(e\phi)$ and $(fb)\phi = (f\phi)(b\phi)$.

Now, we have the main theorem.

Theorem 2.4. Let S and T be generalized inverse $*$ -semigroups and $\phi : S \rightarrow T$ a mapping.

- (1) ϕ is a prehomomorphism if and only if it preserves the restricted product and the natural order.
- (2) ϕ is a homomorphism if and only if it is a prehomomorphism which satisfies $(ef)\phi = (e\phi)(f\phi)$ for all $e, f \in E(S)$.

References

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