

On the regularity of a germ of analytic mapping

J. Cl. TOUGERON

Let (X, o) be a germ of analytic space (reduced and of pure dimension n) at the origin of $\mathbb{C}^N$; let $F: (\mathbb{C}^N, o) \rightarrow (\mathbb{C}^p, o)$ a germ of analytic mapping and $f = F|_X$ the restriction of F to X . We denote $\text{sing } F$ the singular set of F , i.e. the germ of points $x \in \mathbb{C}^N$ such that $dF(x)$ has a rank $< r(F)$, $r(F)$ meaning the generic rank of F . Many results on F or f are true and well known when $\text{sing } F = \emptyset$ or when F is flat. In this paper, we give examples where these results can be extended with an hypothesis on the codimension of $\text{sing } F$.

1) If the rank of F is constant ($= r$), F admits a factorisation $(\mathbb{C}^N, o) \xrightarrow{h} (\mathbb{C}^r, o) \xrightarrow{g} (\mathbb{C}^p, o)$, where h is a submersion and g an immersion. In the general situation, we associate to F a differential form Ω_F of degree r ; if the codimension of $\text{sing } \Omega_F$ in $\mathbb{C}^N$ is ≥ 3 and if Ω_F is decomposable, there exists a factorisation by a generic submersion and a generic immersion. If $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$ and if there exists a formal factorisation $F = \bar{g} \circ \bar{h}$, then there exists an analytic factorisation which approximates the formal one. These results are an easy consequence of Malgrange's Frobenius theorem.

2) If s is the generic rank of f , there does not exist in general a factorisation $(X, o) \rightarrow (Y, o) \xrightarrow{i} (\mathbb{C}^p, o)$, where (Y, o) is an analytic germ, reduced and of pure dimension s at the origin of $\mathbb{C}^p$ and i is the canonical injection. Nevertheless, this is true if F is a flat morphism and if $\text{codim}_{\mathbb{C}^N} X = \text{codim}_{\mathbb{C}^p} f(X)$. We prove analogous results when (X, o) is a complete intersection, an hypothesis about the codimension of $\text{sing } F$ taking the place of the flatness.

3) At last, let $y = (y_1, \dots, y_p)$ (resp. $x = (x_1, \dots, x_N)$) a system of coordinates at the origin of $\mathbb{C}^p$ (resp. $\mathbb{C}^N$) and let $\bar{N}$ a sub-module of $\mathbb{C}[[y]]^q$. Let us suppose that $(\bar{N} \circ F) \subset \mathbb{C}[[x]]$ is generated on $\mathbb{C}[[x]]$ by elements of $\mathbb{C}\{x\}^q$ ($\mathbb{C}\{x\}$ is the ring of convergent series in x); then, if F is flat, $\bar{N}$ is also analytic, i.e. is generated on $\mathbb{C}[[y]]$ by convergent series. The same is true when hypothesis about the codimension of $\text{sing } F$ take the place of the flatness.

1 - A factorisation theorem.

Let $r = r(F)$ be the generic rank of $F : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$ and let $\Lambda^r\{x\}$ be the free modulus on $\mathbb{C}\{x\}$ composed with germs at $0 \in \mathbb{C}^N$ of holomorphic differential forms of degree r .

Lemma 1.1 : *There exists a differential form $\Omega_F \in \Lambda^r\{x\}$, $r = r(F)$, unic modulo multiplication by invertible elements of $\mathbb{C}\{x\}$, such that :*

(1) $\text{codim}_{\mathbb{C}^N} \text{sing } \Omega_F \geq 2$.

(2) $\forall I = (i_1, \dots, i_r), 1 \leq i_1 < \dots < i_r \leq p$, there exists $\theta_I \in \mathbb{C}\{x\}$ such that $dF_I = \theta_I \cdot \Omega_F$.
($dF_I = dF_{i_1} \wedge \dots \wedge dF_{i_r}$ and $\text{sing } \Omega_F = \{x; \Omega_F(x) = 0\}$).

Proof : For every I such that $dF_I \neq 0$, we can write $dF_I = \theta'_I \cdot \Omega_I$ where $\theta'_I \in \mathbb{C}\{x\}$ and Ω_I is a form such that $\text{codim}_{\mathbb{C}^N} \text{sing } \Omega_I \geq 2$. Let I, J be such that $dF_I \neq 0, dF_J \neq 0$; the generic rank of F being r , we have $\Omega_J = \alpha \cdot \Omega_I$ with α meromorphic at the origin of $\mathbb{C}^N$; but α is holomorphic in $\mathbb{C}^N \setminus \text{sing } \Omega_I$, so $\alpha \in \mathbb{C}\{x\}$. Permuting I and J , we see that α is invertible and the lemma follows.

Let Θ_F be the ideal generated by all the θ_I in $\mathbb{C}\{x\}$ and let us denote $V(\Theta_F)$ the germ of zeros of Θ_F ; obviously :

$$\text{sing } F = V(\Theta_F) \cup \text{sing } \Omega_F.$$

If $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$ and if $F = g \circ h$, where $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^r, 0)$ is a generic submersion and $g : (\mathbb{C}^r, 0) \rightarrow (\mathbb{C}^p, 0)$ is a generic immersion, then we may choose $\Omega_F = d h_1 \wedge \dots \wedge d h_r$; Θ_F is then the ideal of $\mathbb{C}\{x\}$ generated by all the determinants of order r of the matrix $(dg) \circ h$.

Our result is a corollary of the singular Frobenius's theorem :

Theorem 1.2 (Malgrange, [3]) : Let $\omega_1, \dots, \omega_r$ be in $\Lambda^1\{x\}$ and let us put $\Omega = \omega_1 \wedge \dots \wedge \omega_r$. We suppose that for $i = 1, \dots, r$, $d\omega_i \wedge \Omega = 0$. Then :

(1) If $\text{codim}_{\mathbb{C}^N} \text{sing } \Omega \geq 3$, the system $\{\omega_1, \dots, \omega_r\}$ is integrable, i.e. there exist $f_1, \dots, f_r \in \mathbb{C}\{x\}$ such that :

$$(\omega_1, \dots, \omega_r) \cdot \mathbb{C}\{x\} = (df_1, \dots, df_r) \cdot \mathbb{C}\{x\}.$$

(2) If $\text{codim}_{\mathbb{C}^N} \text{sing } \Omega \geq 2$ and if the system $\{\omega_1, \dots, \omega_r\}$ is formally integrable

(i.e. there are formal series $\bar{f}_1, \dots, \bar{f}_r \in \mathbb{C}[[x]]$ such that $(\omega_1, \dots, \omega_r) \cdot \mathbb{C}[[x]] = (d\bar{f}_1, \dots, d\bar{f}_r) \cdot \mathbb{C}[[x]]$), then the system is integrable.

We use also the following result (cf [3] or [4]) :

Lemma 1.3 : Let $h : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^r, 0)$ be a germ of holomorphic mapping such that $r = r(h)$ and such that $\text{codim}_{\mathbb{C}^N} \text{sing } h \geq 2$. Then, if $f : (\mathbb{C}^N, 0) \rightarrow \mathbb{C}$ verifies $df \wedge dh_1 \wedge \dots \wedge dh_r = 0$, we have $f = g \circ h$, with $g : (\mathbb{C}^r, 0) \rightarrow \mathbb{C}$ analytic.

Proposition 1.4 :

(1) Let us suppose that $\text{codim}_{\mathbb{C}^N} \text{sing } \Omega_F \geq 3$ and let us suppose that Ω_F is decomposable, i.e. $\Omega_F = \omega_1 \wedge \dots \wedge \omega_r$, with $\omega_i \in \Lambda^1\{x\}$. Then, there exists a factorisation $F = g \circ h$, where $h : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^r, 0)$ and $g : (\mathbb{C}^r, 0) \rightarrow (\mathbb{C}^p, 0)$ are analytic.

(2) Conversely, if F admits such a factorisation and if $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$, Ω_F is decomposable.

Proof : The system $(\omega_1, \dots, \omega_r)$ is locally integrable in $\mathbb{C}^N \setminus V(\Theta_F)$, because $dF_I = \theta_I \cdot \Omega_F$ and so $d\omega_i \wedge \omega_1 \wedge \dots \wedge \omega_r = 0$ for $i = 1, \dots, r$. By theorem 1.2, we may suppose that $\omega_i = dh_i$, $i = 1, \dots, r$, with $h_i \in \mathbb{C}\{x\}$, $h_i(0) = 0$. At last, for every $g = 1, \dots, p$, $dF_j \wedge dh_1 \wedge \dots \wedge dh_r = 0$ and so $F_j = g_j(h_1, \dots, h_r)$ with g_j analytic, by lemma 1.3. The converse (2) is obvious.

Proposition 1.5 : *Let us suppose that $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$ and let us suppose that F admits a formal factorisation $F = \bar{g} \circ \bar{h}$ ($\bar{h} : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^r, 0)$ and $\bar{g} : (\mathbb{C}^r, 0) \rightarrow (\mathbb{C}^p, 0)$). Then F admits an analytic factorisation $F = g \circ h$ and we may choose g and h as closely as we wish to $\bar{g}$ and $\bar{h}$.*

Proof : From the hypothesis, Ω_F admits a formal decomposition :

$\Omega_F = \bar{\lambda} \cdot d\bar{h}_1 \wedge \dots \wedge d\bar{h}_r$, with $\bar{\lambda} \in \mathbb{C}[[x]]$ and $\bar{\lambda}(0) \neq 0$. By Artin's approximation theorem [1], Ω_F is decomposable, i.e. $\Omega_F = \omega_1 \wedge \dots \wedge \omega_r$ with $\omega_i \in \Lambda^1\{x\}$ and the system $\{\omega_1, \dots, \omega_r\}$ is formally integrable. By the part (2) of theorem 1.2, the system is integrable and we conclude as in the proof of 1.4.

Proposition 1.6 : *Let $\underline{F} : \mathbb{C}^N \supset U \rightarrow \mathbb{C}^p$ be an holomorphic mapping with generic rank r ; we suppose that the set of singular points of $\underline{F}$ has codimension ≥ 3 . Then, the set Γ of points $x \in U$ such that the germ $\underline{F}_x : (U, x) \rightarrow (\mathbb{C}^p, \underline{F}(x))$ is factorisable in the sense of 1.4, is the complement of a closed analytic subset of U .*

Proof : The result being of local nature, we may suppose that there exists $\underline{\Omega} \in \Lambda^r(U)$ such that $\forall x \in U$, the germ $\underline{\Omega}_x$ induced by $\underline{\Omega}$ in x , is a differential form $\Omega_{\underline{F}_x}$. By 1.4, the point x belongs to Γ if and only if the equation :

$\underline{\Omega}_x = \omega_1 \wedge \dots \wedge \omega_r$ admits an holomorphic solution. The proposition results from a general theorem about the solutions of a system of analytic equations depending analytically of a parameter (cf [6]).

Remark 1.7 : Let us suppose that $F : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^p, 0)$ admits a factorisation by $(\mathbb{C}^r, 0)$, with $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$. Then this factorisation is unic, in the following sense : if $F = g \circ h$, $F = g' \circ h'$ are two factorisations, there is a unic analytic

difféomorphism $\tau : (\mathbb{C}^r, 0) \xrightarrow{\sim} (\mathbb{C}^r, 0)$ such that the following diagram is commutative :

$$\begin{array}{ccccc}
 & & (\mathbb{C}^r, 0) & & \\
 & \nearrow h & \downarrow \wr \tau & \nwarrow g & \\
 (\mathbb{C}^N, 0) & & & & (\mathbb{C}^p, 0) \\
 & \searrow h' & & \nearrow g' & \\
 & & (\mathbb{C}^r, 0) & &
 \end{array}$$

1.8. Special Cases

(1.8.1.) Let us suppose that $V(\Theta_F) = \emptyset$; for instance, let us suppose that $\theta_{(1, \dots, r)}(0) = 0$. Then we may choose $\Omega_F = dF_1 \wedge \dots \wedge dF_r$ and if $j > r$, we get $F_j = g_j(F_1, \dots, F_r)$, with g_j analytic. So $F = g \circ h$, where g is the immersion $\mathbb{C}^r \ni (z_1, \dots, z_r) \rightarrow (z_1, \dots, z_r ; g_{r+1}(z), \dots, g_p(z))$. The converse is obvious and we get an equivalence :

$(V(\Theta_F) = \emptyset) \Leftrightarrow \text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2 \text{ and there exists a factorisation } F = g \circ h, \text{ where } g : (\mathbb{C}^r, 0) \rightarrow (\mathbb{C}^p, 0) \text{ is an immersion.}$

(1.8.2) Let us suppose that $\text{sing } \Omega_F = \emptyset$; the form Ω_F is generically decomposable and non singular and so, by remark 1.9, it is decomposable, and we may apply 1.4. We get that $F = g \circ h$ where h is a submersion and the converse is obvious :

$(\text{sing } \Omega_F = \emptyset) \Leftrightarrow \text{There exists a factorisation } F = g \circ h \text{ where } h : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^r, 0) \text{ is a submersion.}$

(1.8.3) Let us suppose that the rank of F at 0 is $r-1$. Then Ω_F is decomposable ; indeed, with a convenient choice of coordinates, we may suppose that $F_1 = x_1, \dots, F_{r-1} = x_{r-1}$ and so $\Omega_F = dx_1 \wedge \dots \wedge dx_{r-1} \wedge \omega$, and we may apply 1.4.

1.9 A decomposable form must verify obvious conditions. Let E be a vector space of dimension N on $\mathbb{C}$ and let $e_1, \dots, e_N$ be a basis of E . Let us consider the mapping :

$$\mathbb{C}^{Nr} \simeq E^r \ni (\omega_1, \dots, \omega_r) \rightarrow \Omega = \omega_1 \wedge \dots \wedge \omega_r \in \Lambda^r E \simeq \mathbb{C}^{\binom{N}{r}}.$$

Obviously, $\omega_1 \wedge \dots \wedge \omega_r = \omega'_1 \wedge \dots \wedge \omega'_r$ if and only if there exists a matrix $M \in GL(r, \mathbb{C})$ with determinant 1 such that

$$(\omega_1, \dots, \omega_r) M = (\omega'_1, \dots, \omega'_r).$$

Outside $\phi^{-1}(0)$ the mapping ϕ is a fibering with fiber of dimension r^2-1 and $\phi^{-1}(0)$ is the set of matrices $(\omega_1, \dots, \omega_r)$ with rank $< r$ and so $\phi^{-1}(0)$ is an algebraic variety in E^r of codimension $N-r+1$. The image of ϕ is an algebraic variety in $\Lambda^r E$, of dimension $Nr-r^2+1$, with an isolated singularity at the origin.

If $\Sigma \theta_I e_I$ ($e_I = e_{i_1} \wedge \dots \wedge e_{i_r}$) is the generic point of $\Lambda^r E$ and if U_I is the open set $\theta_I \neq 0$, then $\text{Im } \phi \cap U_I$ is regular and is the transverse intersection of $\binom{N}{r} - (Nr-r^2+1)$ algebraic hypersurfaces $F_{I,\alpha} = 0$, where $F_{I,\alpha}$ is homogeneous of degree r (if $r=2$, $\text{Im } \phi = \{\Omega; \Omega \wedge \Omega = 0\}$).

So, a decomposable differential form Ω must verify $\binom{N}{r} - (Nr-r^2+1)$ independent conditions. Conversely, if these conditions are full filled and if Ω is non singular, then Ω is decomposable. If Ω is singular and decomposable ($\Omega = \omega_1 \wedge \dots \wedge \omega_r$), every irreducible component of $\text{sing } \Omega = \{x; \text{rank } (\omega_1 \wedge \dots \wedge \omega_r) < r\}$ has codimension $\leq N-r+1$.

Another condition is the following one. Let $v(\Omega) = \inf_I v(\theta_I)$ be the infimum of the multiplicities at the origin of the θ_I ($\Omega = \sum_I \theta_I dx_I$) and let $\mathcal{J}(\Omega)$ be the ideal generated in $\mathbb{C}\{x\}$ by θ_I 's. If $v(\Omega) = s < r$ and if $\Omega = \omega_1 \wedge \dots \wedge \omega_r$, then $r-s$ forms ω_i are linearly independant at the origin and by choosing suitable coordinates :

$$\Omega = \left(\sum_I \theta_I^* dx_I \right) \wedge dx_{N-(r-s)+1} \wedge \dots \wedge dx_N$$

with $I = (i_1, \dots, i_s)$, $1 \leq i_1 < \dots < i_s \leq N-(r-s)$; so the minimal number of generators of $\mathcal{F}(\Omega)$ is $\leq \binom{N-(r-s)}{s}$.

For instance, if $N = 3$, $r = 2$, the conditions $F_{I,\alpha}(\Omega) = 0$ are vacant. A form Ω with an isolated singularity at the origin is not decomposable; the form $\Omega = xy dx dy + y^2 dy dz + z dx dz$, with the x -axis as a line of singularities, is not decomposable. Nevertheless, it is decomposable at every point outside the origin (if $x \neq 0$, $\Omega = (x dx - y dz) \wedge (y dy + \frac{z}{x} dz)$).

If $N = 4$, $r = 2$, there is one condition $F_{i,\alpha} = 0 : \theta_{12}\theta_{34} - \theta_{13}\theta_{24} + \theta_{14}\theta_{23} = 0$.

This condition is not sufficient, but I do not know if the hypotheses that Ω is decomposable at every point outside the origin, implies that Ω is decomposable.

1.10 In this paragraph, we give some upper bounds for $\text{codim}_{\mathbb{C}^N} \text{sing } F$

(1.10.1). First, every irreducible component of $F^{-1}(0)$ has codimension $\leq r = r(F)$ (indeed, if F is the germ at 0 of $\underline{F} : U \rightarrow \mathbb{C}^p$, the generic codimension of the fiber $\underline{F}_\xi^{-1}(\underline{F}(\xi))$ is r and this codimension is a lower semi-continuous function of ξ);

after, $F^{-1}(0) \setminus \text{sing } F$ is a regular variety of codimension r . Accordingly :

$$r(F) \geq \text{codim}_{\mathbb{C}^N} F^{-1}(0) \geq \inf(r(F), \text{codim}_{\mathbb{C}^N} \text{sing } F)$$

(1.10.2) If $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$ and if $V(\Theta_F) \neq \emptyset$, there is an inclusion :

$$F^{-1}(0) \subset V(\Theta_F).$$

Accordingly, if $V(\Theta_F) \neq \emptyset$, we get $\text{codim}_{\mathbb{C}^N} V(\Theta_F) \leq r$, and so :

$$\text{codim}_{\mathbb{C}^N} \text{sing } F \leq r \quad (\text{we suppose } r \geq 1).$$

Indeed, if $F^{-1}(0) \not\subset V(\Theta_F)$ there exists an holomorphic curve $\mathbb{C} \ni t \rightarrow x(t) \in \mathbb{C}^N$ such that $x(0) = 0$ and $x(t) \in F^{-1}(0) \setminus V(\Theta_F)$ if $t \neq 0$. From (1.8.1), the morphism $\underline{F}_t : (\mathbb{C}^N, x(t)) \rightarrow (\mathbb{C}^p, 0)$ ($t \neq 0$ small enough) admits a factorisation through a

germ Σ_t of analytic variety of dimension r at the origin of $\mathbb{C}^p$. All Σ_t are equal to a Σ and $F = \underline{F}_0$ admits a factorisation through Σ . From (1.8.1), $V(\Theta_F) = \emptyset$, c.q.f.d.

(1.10. 3) Let us suppose that Ω_F is decomposable. From 1.9, if $\text{sing } \Omega_F \neq \emptyset$:

$$\text{codim}_{\mathbb{C}^N} \text{sing } \Omega_F \leq N-r+1.$$

Nevertheless, if $\text{sing } \Omega_F = \emptyset$, there is from (1.8.2) a factorisation $F = g \circ h$, where h is a submersion, and $V(\Theta_F)$ is the germ of zeros of the ideal $h^* \mathcal{J}$, where $\mathcal{J}$ is the ideal generated by all determinants of order r of the matrix $p \times r (dg_1, \dots, dg_p)$.

So, if $V(\Theta_F) \neq \emptyset$, $\text{codim}_{\mathbb{C}^N} V(\Theta_F) \leq p-r+1$.

Accordingly, if $\text{sing } F \neq \emptyset$ and if Ω_F is decomposable :

$$\text{codim}_{\mathbb{C}^N} \text{sing } F \leq \sup (p, n) - r + 1.$$

The codimension being lower semi-continuous :

Let us suppose that there exist points $x \in \text{sing } F$, as closely as we wish from the origin, such that $\Omega_{\underline{F}_x}$ is decomposable (this is true if $d_x F$ has rank $r-1$, cf

(1.8.3)). Then :

$$\text{codim}_{\mathbb{C}^N} \text{sing } F \leq \sup (p, n) - r + 1.$$

We have also the following remarks :

Let us suppose there exist points $x \in \text{sing } \Omega_F$, as closely as we wish from the origin, such that $\Omega_{\underline{F}_x}$ is decomposable ; then

$$\text{codim}_{\mathbb{C}^N} \text{sing } \Omega_F \leq n - r + 1$$

Let us suppose there exist points $x \in V(\Theta_F)$, as closely as we wish from the origin, such that $\Omega_{\underline{F}_x}$ is decomposable ; if $\text{codim}_{\mathbb{C}^N} \text{sing } \Omega_F \geq 3$, then :

$$\text{codim}_{\mathbb{C}^N} V(\Theta_F) \leq p - r + 1.$$

Remarks 1.11 : The proposition 1.5 is false in general if we suppose that the dimension of the space by which we factorise is not equal to $r = r(F)$. For example (cf [2]), there exists an analytic morphism $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^4, 0)$

with $F_1 = x_1$, $F_2 = x_1 x_2$, $F_3 = x_1 x_2 e^{x_2}$, $F_4 = \bar{\phi}(F_1, F_2, F_3)$, where $\bar{\phi}$ is formal and cannot be chosen analytic. Then $r(F) = 2$, $\ker F^* = 0$ and $\ker \hat{F}^*$ is generated by $y_4 - \bar{\phi}(y_1, y_2, y_3)$. So, there is a formal factorisation of $F: (\mathbb{C}^2, 0) \xrightarrow{\bar{h}} (\mathbb{C}^3, 0) \xrightarrow{\bar{g}} (\mathbb{C}^4, 0)$, where $\mathbb{C}^3 = \{y \in \mathbb{C}^4; y_4 = 0\}$ and $\bar{g}$ is the graph of $\bar{\phi}$; if $F: g \circ f$ is an analytic factorisation of F near of the preceding one, g is (as $\bar{g}$) an immersion and $\ker F^* \neq 0$, but this is false. In this example, $\text{codim}_{\mathbb{C}^2} \text{sing } F = 1$, but we may modify it, in such a way that $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$.

2 - On the regularity of a germ of analytic mapping.

In this paragraph and the following one, we suppose that F is a generic submersion, i.e. $r(F) = p$.

2.1. Let us suppose that $(X, 0)$ is irreducible and let us suppose that $f: F|X: (X, 0) \rightarrow \mathbb{C}^p$ has generic rank s . If $\mathcal{O}_X$ is the ring of germs of holomorphic functions on $(X, 0)$ and if $f^*: \mathbb{C}\{y\} \rightarrow \mathcal{O}_X$ is the homomorphism induced by f , there is inequalities :

$$s \leq s' = \dim(\hat{\mathcal{O}}_X / \ker \hat{f}^*) \leq s'' = \dim(\mathcal{O}_X / \ker f^*).$$

Let us recall the following result (Gabrielov, [2]) :

Theorem 2.2 : *If $s = s'$, we get $s = s' = s''$, i.e if the topological dimension of the image $f(X)$ is equal to its formal dimension, then it is also equal to its analytical dimension ; therefore : $\ker \hat{f}^* = \widehat{\ker f^*}$.*

The morphism f is regular (Gabrielov's definition) if $s = s' = s''$; if $(X, 0)$ is reduced, the morphism f is regular if it is regular in restriction to each irreducible component of $(X, 0)$. The morphism f is regular if f is finished or if $(X, 0)$ and f are algebraic ; here is another condition :

Proposition 2.3 : *Let us suppose that $(X,0)$ is irreducible ; the morphism f is regular under every following hypothesis :*

- (1) $r(f) = \text{codim}_X f^{-1}(0)$ (the inequality $r(f) \geq \text{codim}_X f^{-1}(0)$ is always true)
- (2) F is a flat morphism and $\text{codim}_{\mathbb{C}^N} X = \text{codim}_{\mathbb{C}^p} f(X)$.

Proof : (1) Let $\Sigma \subset \mathbb{C}^N$ be a generic plane of codimension $n-s$ ($s = r(f)$) passing through the origin ; then, every irreducible component $X_{\Sigma,i}$ of $X \cap \Sigma$ has codimension $(n-s) + (N-n) = N-s$ in $\mathbb{C}^N$, so has dimension s and $X_{\Sigma,i} \cap f^{-1}(0) = (0)$.

If $g = f|_{X_{\Sigma,i}}$, $g : X_{\Sigma,i} \rightarrow \mathbb{C}^p$ is a finite morphism, and its rank is s .

The kernel of $g^* : \mathbb{C}\{y\} \rightarrow \mathcal{O}_{X_{\Sigma,i}}$ is a prime ideal $\mathfrak{p}_{\Sigma,i}$ such that $\mathbb{C}\{y\}/\mathfrak{p}_{\Sigma,i}$ has dimension s . Generically, $X_{\Sigma,i}$ contains points x as closely as we wish to the origin, which are regular for $X_{\Sigma,i}$ and X with :

$$\text{rank } d_x \underline{f} = \text{rank } d_x \underline{g} = s$$

(the notations $\underline{X}, \underline{g}$ etc... mean sets, functions etc., the germs of which at the origin being $X, g...$). If $\varphi \in \mathfrak{p}_{\Sigma,i}$, $\varphi \circ f$ is null on $X_{\Sigma,i}$ and $\varphi \circ f$ is null on $\underline{X}$ in the neighborhood of every x . Therefore, $\varphi \circ f = 0$ and $\mathfrak{p}_{\Sigma,i} \subset \ker f^*$. The inverse inclusion is obvious because $\mathbb{C}\{y\}/\mathfrak{p}_{\Sigma,i}$ has dimension s , and the morphism f is regular.

(2) The morphism F being flat, $\text{codim}_{\mathbb{C}^N} F^{-1}(0) = p$, so $\text{codim}_{\mathbb{C}^N} f^{-1}(0) \geq p$ and $\text{codim}_X f^{-1}(0) = \text{codim}_{\mathbb{C}^N} f^{-1}(0) - \text{codim}_{\mathbb{C}^N} X \geq p - \text{codim}_{\mathbb{C}^p} f(X) = r(f)$.

Therefore $r(f) = \text{codim}_X f^{-1}(0)$ and the result is a consequence of (1).

Example 2.4 : Let $\varphi_1(x_n), \dots, \varphi_{n-1}(x_n) \in \mathbb{C}\{x_n\}$ be germs, algebraically independent on $\mathbb{C}$, such that $\varphi'_1(0) = \dots = \varphi'_{n-1}(0) = 1$. Let us consider the morphism f :

$$(\mathbb{C}^n, 0) \ni (x_1, \dots, x_n) \rightarrow (x_1, \dots, x_{n-1}, x_1 \varphi_1(x_n), \dots, x_{n-1} \varphi_{n-1}(x_n)) \in \mathbb{C}^{2n-2}.$$

Then $r(f) = n$ and $\text{sing } f = f^{-1}(0)$ is the x_n -axis ; so $\text{codim}_{\mathbb{C}^N} f^{-1}(0) = n-1$;

besides, the rank of df at 0 is $n-1$. The morphism f is not regular if $n > 2$; more precisely, $\ker \hat{f}^* = 0$. Indeed, let $g \in \mathbb{C}[[y]]$ be such that :

$$g(x_1, \dots, x_{n-1}, x_1 \varphi_1(x_n), \dots, x_{n-1} \varphi_{n-1}(x_n)) = 0.$$

If $g = \sum_{v=1}^{\infty} g_v$ is the decomposition of g in homogeneous polynomials, and if

$$x_1 = t \xi_1, \dots, x_{n-1} = t \xi_{n-1} :$$

$$\sum_{v=1}^{\infty} t^v g_v(\xi_1, \dots, \xi_{n-1}, \xi_1 \varphi_1(x_n), \dots, \xi_{n-1} \varphi_{n-1}(x_n)) = 0$$

so $g_v(\xi_1, \dots, \xi_{n-1}, \xi_1 \varphi_1(x_n), \dots, \xi_{n-1} \varphi_{n-1}(x_n)) = 0$, i.e. $g_v = 0, \forall v$.

This example, a variant of Osgood's example, shows that it is difficult to improve 2.3. Nevertheless, in 2.3 (2), we may replace the hypothesis of flatness on F by a condition of regularity on F .

Remark 2.5 : If $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq p$, the morphism F is flat. Indeed, by (1.10.1), $p = r(F) = \text{codim}_{\mathbb{C}^N} F^{-1}(0)$ and this means exactly that F is flat. Here is an example where $\text{codim}_{\mathbb{C}^N} \text{sing } F = p-1$ and F is not flat ; $F : \mathbb{C}^{2p-2} \rightarrow \mathbb{C}^p$ is defined by

$$F_1(x) = x_1 ; \dots ; F_{p-1}(x) = x_{p-1} ; F_p(x) = x_1 x_p + x_2 x_{p+1} + \dots + x_{p-1} x_{2p-2}.$$

Proposition 2.6 : Let us suppose that $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 3$, where $F : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^p, 0)$ is a generic submersion. If $(X, 0)$ is a germ of hypersurface at the origin of $\mathbb{C}^N$ such that $\text{codim}_{\mathbb{C}^p} f(X) = 1$ ($= \text{codim}_{\mathbb{C}^N} X$), then $f = F|_X$ is regular ($X = F^{-1}(Y)$, where Y is a germ of hypersurface at the origin of $\mathbb{C}^p$).

Proof : Let $\varphi = 0$ be a reduced equation of X ; the condition on the generic rank of f means that at each regular point of X , $d_x \varphi$ is a linear combination of $d_x F_i$, i.e. :

$$(*) \quad d\varphi = \varphi \cdot \omega + \sum_{i=1}^p \varphi_i dF_i$$

with $\omega \in \Lambda^1\{x\}$ and $\varphi_i \in \mathbb{C}\{x\}$.

$$\text{So} \quad d\omega = - \sum_{i=1}^p d\left(\frac{\varphi_i}{\varphi}\right) \wedge dF_i;$$

and $d\omega \wedge dF_1 \wedge \dots \wedge dF_p = 0$. By 2.9, the hypothesis $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 3$ implies that $\omega = d\Psi \bmod(dF)$, where (dF) is the submodule of $\Lambda^1\{x\}$ generated by $dF_1, \dots, dF_p$. From (*):

$$d(\varphi e^{-\Psi}) \in (dF)$$

and from lemma 1.3 : $\varphi = e^\Psi \cdot (\theta \circ F)$, with $\theta \in \mathbb{C}\{y\}$.

If Y is the hypersurface with reduced equation $\theta = 0$, then $X = f^{-1}(Y)$.

Corollary 2.7 : *If $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 3$ and if Y is a germ of irreducible hypersurface at the origin of $\mathbb{C}^p$, $X = F^{-1}(Y)$ is also irreducible (indeed, if $X = X' \cup X''$ is a proper decomposition of X , we may apply to X' and X'' the previous reasoning, and $X' = F^{-1}(Y')$, $X'' = F^{-1}(Y'')$; so $Y = Y' \cup Y''$ is a proper decomposition of Y which is not irreducible).*

Corollary 2.8 : *Let us suppose that $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 3$ and that Ω_F is decomposable (we do not suppose that F is a generic submersion). If $\text{codim}_{\mathbb{C}^N} V(\Theta_F) \geq r = r(F)$, F and the restriction of F to every hypersurface $(X, 0)$ of $(\mathbb{C}^N, 0)$, are regular morphisms.*

Proof : By 1.4, there is a factorisation, $F = g \circ h$, where h is a generic submersion and g is a generic finite immersion. So F is regular ; if X is a germ of irreducible hypersurface at the origin of $\mathbb{C}^N$, either the rank of $h|_X$ is equal to r and $f = F|_X$ is regular ; or this rank is $r-1$, but then we may apply 2.6 and again f is regular.

2.9. In the proof of 2.6, we used a very particular case of the following result (cf [3] or [4]). Let us suppose that $\text{codim}_{\mathbb{C}^N} \text{sing } F > q$

($F : (\mathbb{C}^v, 0) \rightarrow (\mathbb{C}^p, 0)$ is a generic submersion) and let $1 \leq s \leq r \leq q$ be integers.

We put :

$$\Lambda_F^{r,s}\{x\} = \{\omega \in \Lambda^r\{x\} ; \omega \wedge dF_{i_1} \wedge \dots \wedge dF_{i_{p-s+1}} = 0$$

for every $1 \leq i_1 < i_2 < \dots < i_{p-s+1} \leq p\}$ =

$$\{\omega \in \Lambda^r\{x\} ; \omega = \sum_{j_1 < \dots < j_s} \theta_{j_1} \dots \theta_{j_s} \wedge dF_{j_1} \wedge \dots \wedge dF_{j_s}$$

with $\theta_{j_1} \dots \theta_{j_s} \in \Lambda^{r-s}\{x\}$.

This last equality is an easy consequence of the division lemma (Saito, [8]) stated below. If we write $\Lambda^{r,s}(F) = \Lambda^r\{x\} / \Lambda_F^{r,s}\{x\}$ then d induces a morphism :

$$\Lambda^{r,s}(F) \rightarrow \Lambda^{r+1,s}(F) ;$$

there is an exact sequence :

$$\Lambda^{s-1}\{x\} \xrightarrow{d} \Lambda^{s,s}(F) \xrightarrow{d} \Lambda^{s+1,s}(F) \xrightarrow{d} \dots \xrightarrow{d} \Lambda^{q,s}(F)$$

and the kernel of the first d is the submodule of $\Lambda^{s-1}\{x\}$ generated by the images of $F^* : \Lambda^{s-1}\{y\} \rightarrow \Lambda^{s-1}\{x\}$ and $d : \Lambda^{s-2}\{x\} \rightarrow \Lambda^{s-1}\{x\}$. In particular, if $q = 2$, there is an exact sequence $0 \rightarrow \mathbb{C}\{y\} \xrightarrow{F^*} \mathbb{C}\{x\} \xrightarrow{d} \Lambda^{1,1}(F) \xrightarrow{d} \Lambda^{2,1}(F)$; this sequence is used in the proof of 2.6.

The division lemma says that, if $\text{codim}_{\mathbb{C}^N} \text{sing } F > q$ and if $\omega \in \Lambda^q\{x\}$ is such that $\omega \wedge dF_1 \wedge \dots \wedge dF_p = 0$, then $\omega = \sum_{i=1}^p \theta_i \wedge dF_i$, with $\theta_i \in \Lambda^{q-1}\{x\}$.

2.10. It would be interesting to extend 2.6 to complete intersections. If a complete intersection $(X, 0)$ of codimension k at the origin of $\mathbb{C}^N$ is defined by a reduced system of equations $\varphi_1 = \dots = \varphi_k = 0$ and if $\text{codim}_{\mathbb{C}^p} F(X) = \text{codim}_{\mathbb{C}^N} X$,

then

$$d\varphi = \Omega \cdot \varphi + \sum_{i=1}^p \theta_i \cdot dF_i$$

where $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_k \end{pmatrix}$, Ω is a $k \times k$ matrix with coefficients in $\Lambda^1\{x\}$, θ_i is a column

vector with coefficients in $\mathbb{C}\{x\}$. So :

$$(d\Omega - \Omega \wedge \Omega) \varphi = 0 \quad \text{mod } (dF_i).$$

If we may choose Ω such that $d\Omega - \Omega \wedge \Omega = 0 \quad \text{mod } (dF_i)$ and if $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 3$, then by the arguments of the proof of 2.9 :

$$\Omega = dM.M^{-1} \quad \text{mod } (dF_i)$$

where M is an invertible $k \times k$ matrix with coefficients in $\mathbb{C}\{x\}$, so

$$d(M^{-1} \varphi) = 0 \quad \text{mod } (dF_i)$$

By 2.9, $M^{-1} \varphi = \theta \circ F$, where $\theta = \begin{pmatrix} \theta_1 \\ \theta_k \end{pmatrix}$, $\theta_i \in \mathbb{C}\{y\}$, $\theta_i(0) = 0$, and $X = F^{-1}(Y)$

where Y is a complete intersection.

Therefore, the main problem is finding conditions on X and F such that the integrability condition $d\Omega - \Omega \wedge \Omega = 0 \quad \text{mod } (dF_i)$ is verified by a suitable Ω .

3 - A criteria of analyticity for modulus.

If A is a (commutative and unitary) ring without divisors of zero, we denote by $[A]$ the quotient field of A . A modulus $\mathcal{M}$ on A , of finite type, is without torsion if $a \in A \setminus \{0\}$, $m \in \mathcal{M} \setminus \{0\}$ implies $a.m \neq 0$. This means also that $\mathcal{M}$ is isomorphic to a submodule of A^r , where $r = \dim_{[A]} \mathcal{M} \otimes_A [A]$ is the generic rank of $\mathcal{M}$.

If $(X, 0)$ is a germ of analytic space, we denote by $\hat{\mathcal{O}}_X$ the completion of the ring $\mathcal{O}_X$ of analytic germs on X . We shall use the following result (Tougeron, [7]):

Theorem 3.1 : Let $f : (X,0) \rightarrow (Y,0)$ be a generic analytic submersion between two irreducible germs of analytic spaces (so $f^* : \hat{\mathcal{O}}_Y \rightarrow \hat{\mathcal{O}}_X$ is injective). Then :

$$\{\varphi \in [\hat{\mathcal{O}}_Y] \text{ and } \varphi \circ f \in [\mathcal{O}_X]\} \Rightarrow \varphi \in [\mathcal{O}_Y].$$

A submodule $\mathcal{N}$ of $\hat{\mathcal{O}}_Y^q$ is analytic if it is generated on $\hat{\mathcal{O}}_Y$ by elements of $\mathcal{O}_Y^q$.

Corollary 3.2 : Under the hypothesis of 3.1, if $\mathcal{M} = \hat{\mathcal{O}}_Y^q / \mathcal{N}$ is without torsion and if the vector space generated by $\mathcal{N} \circ f$ in $[\hat{\mathcal{O}}_X]^q$ is analytic (i.e. is generated by vectors with coefficients in $\mathcal{O}_X$), then $\mathcal{N}$ is analytic.

Proof : Let $\varphi_1, \dots, \varphi_s \in \mathcal{N}$ be such that $\varphi_1 \wedge \dots \wedge \varphi_s \neq 0$ and $r =$ generic rank of $\mathcal{M} = q-s$. Then $\varphi \in \mathcal{N} \Leftrightarrow \varphi \wedge \varphi_1 \wedge \dots \wedge \varphi_s = 0$ (because $\mathcal{M}$ is without torsion). Let us put $\varphi_1 \wedge \dots \wedge \varphi_s = \sum_I \theta_I e_I$ where $e_I = e_{i_1} \wedge \dots \wedge e_{i_s}$ is the canonical basis of $\hat{\mathcal{O}}_Y^q$.

Let us suppose that $\theta_{I_0} \neq 0$; the modulus generated by $\mathcal{N} \circ f$ being analytic, each $(\theta_I / \theta_{I_0}) \circ f$ is analytic, so by 3.1 θ_I / θ_{I_0} is analytic for every I . Therefore $\mathcal{N}$ is analytic, c.q.f.d.

This corollary admits the following extension :

Proposition 3.3 : Let $f : (X,0) \rightarrow (Y,0)$ be a morphism between two irreducible germs of analytic spaces and let us suppose that the germ of points $x \in X$ such that f_x is not flat has codimension v in X . Let $\mathcal{M} = \hat{\mathcal{O}}_Y^q / \mathcal{N}$ be a modulus such that the two following conditions are full filled :

- every prime ideal associated to $\mathcal{M}$ has height $< v$,
- the submodule $f^* \mathcal{N}$ generated by $\mathcal{N} \circ f$ in $\hat{\mathcal{O}}_X^q$ is analytic. Then $\mathcal{N}$ is analytic.

Let us recall that if $\mathcal{M}$ is a modulus on a ring $A = \hat{\mathcal{O}}_Y$ and if $\mathfrak{p}$ is a prime ideal of A , then $\mathfrak{p}$ is associated to $\mathcal{M}$ if there is an injective map : $A/\mathfrak{p} \hookrightarrow \mathcal{M}$. The modulus $\mathcal{M}$ is coprimary if $a \in A \setminus \mathfrak{p} \Rightarrow (\mathcal{M} \ni m \rightarrow a.m \in \mathcal{M})$ is injective and $a \in \mathfrak{p} \Rightarrow (\mathcal{M} \ni m \rightarrow a.m \in \mathcal{M})$ is nilpotent. If $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are the prime ideals associated to $\mathcal{M}$ then there exist submodules $\mathcal{N}_i$ of $\mathcal{M}$, with $\mathcal{M}/\mathcal{N}_i$ $\mathfrak{p}_i$ -coprimary, such that $\bigcap_{i=1}^k \mathcal{N}_i = 0$ (cf [5]).

If $\mathcal{M} = A^q/\mathcal{N}$ is $\mathfrak{p}$ -coprimary, we define a sequence $\mathcal{N}_0 = \mathcal{N} \subset \mathcal{N}_1 \subset \dots \subset \mathcal{N}_s \subset \mathcal{N}_{s+1} = A^q$ of submodules of A^q such that for every $i = 0, \dots, s$: $\mathcal{N}_{i+1} \setminus \{\xi \in A^q ; \mathfrak{p} \cdot \xi \subset \mathcal{N}_i\}$. Then, for every i : $\mathcal{N}_{i+1}/\mathcal{N}_i$ is a modulus on $A/\mathfrak{p}$ without torsion. We prove first :

Lemma 3.4 : *With the hypothesis of 3.3, let us suppose that $\mathcal{M}$ is $\mathfrak{p}$ -coprimary ; then 3.3 is true.*

Proof : First , we observe that $\mathfrak{p}$ is analytic. In fact, as a consequence of the flatness, the prime ideals of height $< v$ associated to $\hat{\mathcal{O}}_X^q / f^* \mathcal{N}$ are exactly the prime ideals of height $< v$ associated to $\hat{\mathcal{O}}_X / f^* \mathfrak{p}$; $f^* \mathcal{N}$ being analytic, these prime ideals are analytic, and there exist a prime ideal $\mathfrak{p}'$ of $\mathcal{O}_X$ such that $\hat{\mathfrak{p}}'$ is a minimal prime ideal containing $f^* \mathfrak{p}$. If X' is the germ of analytic set ($\subset X$) defined by $\mathfrak{p}'$, then $\mathfrak{p}$ is the kernel of the morphism $\hat{f}^* : \hat{\mathcal{O}}_Y \rightarrow \hat{\mathcal{O}}_X / \hat{\mathfrak{p}}'$ and by Gabrielov's theorem, $\mathfrak{p}$ is analytic.

After, we prove by induction on $i = s, s-1, \dots, 0$ that $\mathcal{N}_i$ is analytic. If $\mathcal{N}_{i+1}$ is analytic, let $g_1, \dots, g_h$ be an analytic system of generators of $\mathcal{N}_{i+1}$ and let us write $\mathcal{N}_{i+1} = \hat{\mathcal{O}}_Y^h / \mathfrak{R}_{i+1}$ where $\mathfrak{R}_{i+1}$ is the modulus of relations between $g_1, \dots, g_h$. Then $\mathcal{N}_{i+1} / \mathcal{N}_i = \hat{\mathcal{O}}_Y^h / \mathfrak{R}'_{i+1}$, $\mathfrak{R}'_{i+1} \supset \mathfrak{R}_{i+1}$; by the flatness of f , $f^* \mathfrak{R}'_{i+1}$ is analytic at the generic point of X' ; by 3.2, $\mathfrak{R}'_{i+1}$ is analytic and so $\mathcal{N}_i$ is analytic.

Remark : In the previous proof we don't use the complete assertion that $f^*\mathcal{N}$ is analytic ; we only use that $f^*\mathcal{N}$ is analytic at the generic point of X' .

Proof of 3.3 : Let $\mathfrak{P}_1, \dots, \mathfrak{P}_k$ be the minimal prime ideals associated to $\mathcal{N} = \hat{\mathcal{O}}_Y^q / \mathcal{N}$. As in the proof of 3.4, let $\mathfrak{P}'_i$ be a minimal prime ideal of $\hat{\mathcal{O}}_X$ containing $f^*\mathfrak{P}_i$; then $\mathfrak{P}'_i$ is analytic. If X'_i is the germ of analytic set defined by $\mathfrak{P}'_i$, then $\mathfrak{P}_i$ is the kernel of the morphism $\hat{f}^* : \hat{\mathcal{O}}_Y \rightarrow \hat{\mathcal{O}}_X / \mathfrak{P}'_i$ and so, by Gabrielov's theorem, $\mathfrak{P}_i$ is analytic. Let $\mathcal{N}_i$ be a submodule of $\hat{\mathcal{O}}_Y^q$ such that $\bigcap_{i=1}^k \mathcal{N}_i = \mathcal{N}$ and $\hat{\mathcal{O}}_Y^q / \mathcal{N}_i$ is $\mathfrak{P}_i$ -coprimary.

Let $\mathfrak{P}_1, \dots, \mathfrak{P}_{k'}$ be the minimal prime ideals in the family $\{\mathfrak{P}_1, \dots, \mathfrak{P}_k\}$; then, by flatness, $f^*\mathcal{N}_i$ is analytic at the generic point of X'_i , for $i = 1, \dots, k'$. By 3.4 and the remark, $\mathcal{N}_i$ is analytic if $i \leq k'$.

There is an injection :

$$(*) \quad \left(\bigcap_{i \leq k'} \mathcal{N}_i \right) / \left(\bigcap \mathcal{N}_i \right) \rightarrow \hat{\mathcal{O}}_Y^q / \bigcap_{i > k'} \mathcal{N}_i .$$

Let $g_1, \dots, g_h$ be a system of analytic generators of $\bigcap_{i \leq k'} \mathcal{N}_i$ and let us write

$\hat{\mathcal{O}}_Y^h / \mathfrak{R} = \bigcap_{i \leq k'} \mathcal{N}_i$ where $\mathfrak{R}$ is the modulus of relations between the g_i . Then

$\left(\bigcap_{i \leq k'} \mathcal{N}_i \right) / \left(\bigcap \mathcal{N}_i \right) \sim \hat{\mathcal{O}}_Y^h / \mathfrak{R}'$ where $\mathfrak{R}' \supset \mathfrak{R}$. The prime ideal associated to $\hat{\mathcal{O}}_Y^h / \mathfrak{R}'$

are among $\mathfrak{P}_{k'+1}, \dots, \mathfrak{P}_k$, because of the injection (*) and by flatness $f^*\mathfrak{R}'$ is analytic outside an analytic set of codimension v .

Therefore, we may prove the result by induction on the number k of prime ideals $\mathfrak{P}_i$. By the induction hypothesis, $\mathfrak{R}'$ is analytic and so $\mathcal{N}$ is analytic.

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