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TORIC FGA ALGEBRAS

FLORIN AMBRO

FGA algebras are certain graded algebras introduced by Shokurov in his construction of 4-fold log flips [5]. Shokurov reduced the existence of log flips in dimension d to the log Minimal Model Program in dimension d−1 and the finite generation of **FGA** algebras in dimension d−1. Thus, the finite generation of **FGA** algebras would suffice for an inductive proof of the existence of log flips.

FGA algebras are known to be finitely generated in dimension one and two (Shokurov [5]). In this note we present the finite generation of **FGA** algebras in the toric case [1]. The key ingredient is a universal upper bound on the width of a convex body in terms of the number of lattice points it contains [4].

FGA algebras. Let $(X/\mathbf{S}, \mathbf{B})$ be the data consisting of a proper surjective morphism $\pi\colon X \to \mathbf{S}$ of complex algebraic varieties, and a log variety $(X, \mathbf{B})$ with Kawamata log terminal singularities such that $-(\mathrm{K}+\mathbf{B})$ is nef and big relative to π.

The simplest example of an **FGA** algebra is the graded $\mathbf{O}_S$-algebra

$$\mathbf{R}_{X/S}(D) = \bigoplus_{i=0}^{\infty} \pi_* \mathbf{O}_X(\mathrm{i}D),$$

for a Cartier divisor D on X. The finite generation of $\mathbf{R}_{X/S}(D)$ is a consequence of the Log Minimal Model Program in dimension $\dim(X)$.

In general, an **FGA** algebra is a graded subalgebra $\mathbf{L} \subseteq \mathbf{R}_X(D)$ satisfying an extra property called asymptotic saturation. To explain this, we need some preparation. The projective normalization of $\mathbf{L}$ is

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[2]The problem of the finite generation of FGA algebras was rendered obsolete by a recent result of Hacon and M$^{\mathrm{c}}$Kernan [3]. Using ideas of Siu, Kawamata and Nakayama on extensions of pluricanonical forms, they simplified Shokurov's reduction argument, showing that d-dimensional log flips exist if the (d − 1)-dimensional log Minimal Model Program holds for log varieties (X, B) whose boundary B has real coefficients.

also a graded subalgebra of $\mathbf{R}_{X/S}(D)$, and it has a presentation

$$\bar{\mathbf{L}} = \bigoplus_{i=0}^{\infty} \pi_{i*}\mathbf{O}_{X_i}(\mathrm{M}_i),$$

where $\mu_i \colon X_i \to X$ is a birational modification, $\pi_i = \pi \circ \mu_i$ and M_i is a π_i-free divisor on X_i. We may replace the X_i's by higher birational models, so that X_i is nonsingular and contains a simple normal crossings divisor which supports both $\mathrm{K}_{X_i} - \mu_i^*(\mathrm{K} + \mathbf{B})$ and M_i. Then $\mathbf{L}$ is called asymptotically saturated if

$$\pi_{i*}\mathbf{O}_{X_i}(\lceil \mathrm{K}_{X_i} - \mu_i^*(\mathrm{K} + \mathbf{B}) + \frac{\mathrm{j}}{\mathrm{i}}\mathrm{M}_i\rceil) \subseteq \pi_{j*}\mathbf{O}_{X_j}(\mathrm{M}_j), \forall \mathrm{i}, \mathrm{j} \geq 1.$$

Asymptotic saturation involves apriori infinitely many birational models of X. If $\dim(X) \leq 2$, its particular case $\mathrm{i} = \mathrm{j}$ bounds the singularities on X of the (non-complete) linear systems $|\mathbf{L}_i| \subset |\mathrm{i}D|$, and this in turn can be used to restate asymptotic saturation in terms of just one birational model of X. Finite generation follows then by a standard argument. It is an interesting open question whether a similar approach works for $\dim(X) \geq 3$.

In the toric case, we do not know if the above approach works. Instead, we interpret toric asymptotic saturation only in terms of the limit $\lim_{i\to\infty} \frac{1}{i}\mathrm{M}_i$, which can be regarded as (the support function of) a convex set $\square$. The finite generation of toric **FGA** algebras becomes then a criterion for $\square$ to be rational and polyhedral.

Convex sets. Let N be a lattice, with dual lattice M, let $N_{\mathbb{R}}$ and $\mathrm{M}_{\mathbb{R}}$ be the scalars extensions, with induced pairing $\langle\cdot,\cdot\rangle \colon \mathrm{M}_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbf{R}$. For a function $\mathrm{h} \colon N_{\mathbb{R}} \to \mathbf{R}$, define

$$\square_h = \{\mathrm{m} \in \mathrm{M}_{\mathbb{R}}; \langle \mathrm{m}, \mathrm{e}\rangle \geq \mathrm{h}(\mathrm{e}), \forall \mathrm{e} \in N_{\mathbb{R}}\}$$

$$\overset{\circ}{\square}_h = \{\mathrm{m} \in \mathrm{M}_{\mathbb{R}}; \langle \mathrm{m}, \mathrm{e}\rangle > \mathrm{h}(\mathrm{e}), \forall \mathrm{e} \in N_{\mathbb{R}} \setminus 0\}.$$

For a non-empty compact convex set $\square \subset \mathrm{M}_{\mathbb{R}}$, its support function is

$$\mathrm{h}_\square \colon N_{\mathbb{R}} \to \mathbf{R}, \mathrm{h}_\square(\mathrm{e}) = \min_{m\in\square}\langle \mathrm{m}, \mathrm{e}\rangle.$$

The set of non-empty compact convex subsets of $\mathrm{M}_{\mathbb{R}}$ is in bijection with the set of functions $\mathrm{h} \colon N_{\mathbb{R}} \to \mathbf{R}$ satisfying

(i) positively homogeneous: $\mathrm{h}(\lambda \mathrm{e}) = \lambda \mathrm{h}(\mathrm{e})$ for $\lambda \geq 0, \mathrm{e} \in N_{\mathbb{R}}$.
(ii) upper convex: $\mathrm{h}(\mathrm{e}_1 + \mathrm{e}_2) \geq \mathrm{h}(\mathrm{e}_1) + \mathrm{h}(\mathrm{e}_2)$ for $\mathrm{e}_1, \mathrm{e}_2 \in N_{\mathbb{R}}$.

The correspondence is given by $\square \mapsto \mathrm{h}_\square$ and $\mathrm{h} \mapsto \square_h$.

Definition 1. A log discrepancy function is a positively homogeneous, continuous function $\psi \colon N_{\mathbb{R}} \to \mathbf{R}$ such that $\psi(\mathrm{e}) > 0$ for $\mathrm{e} \neq 0$, and $\{\mathrm{e} \in N_{\mathbb{R}}; \psi(\mathrm{e}) \leq 1\}$ is a compact set.

Toric **FGA** algebras. For simplicity, we assume that **S** is a point. Thus $X = \mathbf{T}_N \operatorname{emb}(\Delta)$ is a proper torus embedding, $\mathbf{B} = \sum_{e \in \Delta(1)} b_e V(e)$ is an invariant $\mathbb{Q}$-divisor and $K + \mathbf{B}$ is $\mathbb{Q}$-Cartier. This means that there exists a function $\psi \colon N_{\mathbb{R}} \to \mathbf{R}$ such that $\psi(e) = 1 - b_e$ for every $e \in \Delta(1)$, and ψ is Δ-linear. Since $(X, \mathbf{B})$ has Kawamata log terminal singularities, ψ is a log discrepancy function. The terminology is inspired by the following property: a primitive lattice point $e \in N$ defines a toric valuation v_e of X, and the log discrepancy of $(X, \mathbf{B})$ at v_e is exactly $\psi(e)$. Also, note that $-(K + \mathbf{B})$ is nef if and only if $-\psi$ is upper convex.

Any invariant normal algebra is of the form

$$\mathbf{L} = \bigoplus_{i=0}^{\infty} \Big(\bigoplus_{m \in M \cap \square_i} \mathbf{C}\chi^m \Big)$$

where $(\square_i)_{i \geq 0}$ is a sequence of lattice convex polytopes in $M_{\mathbb{R}}$ satisfying the following properties:

- $\square_0 = \{0\}$.
- $\square_i + \square_j \subset \square_{i+j}$ for $i, j \geq 1$.
- $\square := \bigcup_{i \geq 1} \frac{1}{i} \square_i$ is a bounded convex set.

The algebra is finitely generated if and only if $\square_i = \square$ for some i. For an example of such a sequence, fix a compact convex set $\square \subset M_{\mathbb{R}}$, and let $\square_i$ be the convex hull of $M \cap i\square$.

Since limits of upper convex functions converge uniformly on compact sets, the asymptotic saturation of $\mathbf{L}$ with respect to $(X, \mathbf{B})$ is equivalent to the following *Diophantine* property

$$M \cap \overset{\circ}{\square}_{jh-\psi} \subset \square_j, \forall j \geq 1.$$

Then $\mathbf{L}$ is finitely generated if and only if $\square$ is a rational polytope, i.e. the convex hull of finitely many rational points, and this follows from the following result:

Theorem 2. *Let ψ be a log discrepancy function and let $\square \subset M_{\mathbb{R}}$ be a non-empty compact convex set with support function h. Assume that the following properties hold:*

(i) *$M \cap \overset{\circ}{\square}_{jh-\psi} \subset j\square$ for every $j \geq 1$.*
(ii) *$h - \psi$ is upper convex.*

Then $\square$ is a rational polytope.

Theorem 2 is proved by induction on dimension. Consider the unit sphere $\mathbf{S}(N_{\mathbb{R}})$ with respect to some norm on $N_{\mathbb{R}}$. Using *Diophantine Approximation* [2], we show that every point $e \in \mathbf{S}(N_{\mathbb{R}})$ is contained in the relative interior of a rational polyhedral cone σ_e on which $h_\square$

is rational and linear with respect to some fan with support σ_e. If $\dim(\sigma_e) = \dim(N)$ for every e, one can use the compactness of $\mathbf{S}(N_{\mathbb{R}})$ to finish the proof. Otherwise, we increment the dimension of σ_e, using restriction to an appropriate face of $\square$ and induction on dimension. The key idea behind the induction step is to bound the pairs $(\square, \psi)$ appearing in Theorem 2, assuming that $\square$ is a rational polytope. The general case is a combination of the following two examples:

1) If $\square$ is a rational polytope of maximal dimension, property (i) is equivalent to

$$\Delta_{\square}(1) \subseteq \{\mathrm{e} \in N_{\mathbb{R}}; \psi(\mathrm{e}) \leq 1\}.$$

Here $\Delta_{\square}$ is the ample fan in N associated to $\square$, defined by

$$\mathbf{T}_N \operatorname{emb}(\Delta_{\square}) = \operatorname{Proj}(\bigoplus_{i=0}^{\infty} \bigoplus_{m \in M \cap i\square} \mathbf{C}\chi^m),$$

and $\Delta_{\square}(1)$ is the set of primitive lattice vectors on the one dimensional cones of Δ. It follows that $\Delta_{\square}$ belongs to a finite family of fans, by the compactness of the level sets of ψ.

2) If $\square = \{0\}$, property (i) is equivalent to

$$\mathrm{M} \cap \overset{\circ}{\square}_{-\psi} = \{0\}.$$

By (ii) and Kannan-Lovász's effective bound [4] of the width of a convex set in terms of the number of lattice points it contains, there exists $\mathrm{e} \in N \setminus 0$ such that $\psi(\mathrm{e}) + \psi(-\mathrm{e}) \leq \mathbf{C}$, where $\mathbf{C}$ is a positive constant depending on the dimension of the lattice N only.

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