

Yang-Mills equations and holomorphic structures
of vector bundles

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1. Main subject of Yang-Mills theory is to study
variational problem associated to connections on a principal
fibre bundle. Yang-Mills functional is defined over
a set of connections on a given bundle in terms of square
norm of curvature. Euler-Lagrange equation of this functional
is called Yang-Mills equation. With respect to this equation
we have two problems;

1. when are there solutions ?
2. what sort of geometric structure does the moduli space
(parameter space) of solutions admit ?

In this lecture we discuss these over a four-dimensional
manifold, especially over a complex 2-dim complex
manifold with positive definite metric.

2. Let P be a principal fibre bundle with gauge group
 $SU(n)$ over a compact oriented 4-dim manifold M with
metric h .

Definition. A connection on P is a system $A = \{A_\alpha, A_\beta, \dots\}$ where each A_α is an $SU(n)$ -valued 1-form defined
over a trivializing neighborhood U_α of P such that

these A_α 's satisfy over $U_\alpha \cap U_\beta \neq \emptyset$

$$(*) \quad A_\beta = g^{-1} \cdot dg + g^{-1} \cdot A_\alpha \cdot g.$$

Here $g = g_{\alpha\beta}$ denotes the transition function of P :
 $U_\alpha \cap U_\beta \xrightarrow{C^\infty} SU(n)$ and $\cdot$ means multiplication of matrices.

We have another global definition of connection, equivalent to the above. But we adopt the above for a convenience.

For each Lie algebra valued function on M the covariant derivative ∇_A is defined: $\nabla_A \phi = d\phi + [A, \phi] = d\phi + A \cdot \phi - \phi \cdot A$ and we have also the covariant exterior derivative $d_A \psi = d\psi + [A \wedge \psi] = d\psi + A \wedge \psi + (-1)^{p+1} \psi \wedge A$ (ψ is a p -form).

A 2-form $F(A) = dA + A \wedge A$ is said curvature form of A . The curvature form takes values in $\mathcal{M}(n)$. In local expression $F(A)$ is represented by $F(A) =$

$$\frac{1}{2} \sum_{\mu, \nu} F_{\mu\nu} dx^\mu \wedge dx^\nu \quad (F_{\mu\nu} = -F_{\nu\mu}), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

We have an identity $d_A \nabla_A \phi = [F(A), \phi]$, called Ricci formula which will be used later to define an elliptic complex.

Definition. Over the set of connections $\mathcal{E}_P$ Yang-Mills functional $\mathcal{YM}$ is defined by

$$\mathcal{YM}(A) = \frac{1}{2} \int_M |F(A)|^2 dv,$$

where $|F(A)|^2 = \frac{1}{2} \sum h^{\mu\sigma} h^{\nu\tau} (-\text{Tr } F_{\mu\nu} \cdot F_{\sigma\tau})$ ($(h^{\sigma\tau})$ is the inverse matrix of the metric $(h_{\mu\nu})$).

Its Euler-Lagrange equation is written locally by

$$\sum_{\sigma, \tau} h^{\sigma\tau} \nabla_{\sigma} F_{\tau\mu} = \sum h^{\sigma\tau} (D_{\sigma} F_{\tau\mu} + [A_{\sigma}, F_{\tau\mu}]) = 0$$

where D_{σ} denotes the covariant derivative with respect to Levi-Civita connection of M .

Definition. A connection is called Yang-Mills if it is a critical point of $\mathcal{YM}$, that is, it is a solution to the Yang-Mills equation.

We have assumed that M is oriented and 4-dimensional. Then we have the operator $*$ which is an involutive endomorphism of the bundle of 2-forms Λ^2 . Hence Λ^2 splits into $\Lambda_+^2 + \Lambda_-^2$ corresponding to eigenvalues. A 2-form α is said (anti-)self-dual if $\alpha \in \Lambda_+^2$ (or Λ_-^2). Curvature form splits also into $F(A) = F_+ + F_-$.

Definition. A connection is said self-dual (or anti-self-dual) if it is a solution of self-dual equation $F_- = 0$ (or anti-self-dual equation $F_+ = 0$).

From Chern-Weil homomorphism theorem y_m has the lowest bounds;

$$y_m(A) = \frac{1}{2} \int_M \{ |F_+|^2 + |F_-|^2 \} dv \geq \frac{1}{2} \left| \int_M \{ |F_+|^2 - |F_-|^2 \} dv \right|$$

for all $A \in \mathcal{E}_P$ and the integral of the right hand side represents $(-c) c_2(P)[M]$ for a universal constant $c > 0$ and the second Chern number $c_2(P)[M]$. Then the equality " $=$ " holds if and only if A is self-dual or anti-self-dual according to $c_2(P)[M] \leq 0$ or ≥ 0 .

Thus we have

Proposition 2.1. Every self-dual connection (or anti-self-dual connection) minimizes y_m . Hence it is a Yang-Mills connection.

Note. The Yang-Mills equation is a non-linear second order equation with respect to unknown connection. But the (anti-) self-duality equation is of first order.

For Problem 1 $c_2(P)[M] \geq 0$ is a necessary condition for P to admit an anti-self-dual connection.

Conversely we have

Theorem (Taubes [11]). If $H_+^2(M) = \{ \text{harmonic 2-form which is anti-self-dual} \}$ vanishes, then P with $c_2(P)[M] \geq 0$ admits an irreducible anti-self-dual connection provided that a principal fibre bundle over S^4 with identical Chern number carries an irreducible anti-self-dual connection.

The irreducibility of connection will be defined at § 4. It is seen that a connection is irreducible if and only if the gauge group can not be compressed into $S(U(n_1) \times U(n_2))$, ($n_1 + n_2 = n$).

With respect to Problem 2 we have by fundamental ideas of Atiyah-Hitchin-Singer [2] and also of Donaldson [3] the moduli space of anti-self-dual connections carries a structure of real analytic set. Donaldson applied the structure to 4-dimensional topology.

3. Assume that the base space M is a complex surface with complex coordinates z^1, z^2 and of a Kähler metric $h = \sum_{\mu, \nu} h_{\mu\bar{\nu}}(z^1, z^2) dz^\mu \cdot d\bar{z}^\nu$. Then the space M carries an orientation induced naturally from the complex structure.

Fact(Atiyah[1], Itoh[6]) A connection over a complex surface is anti-self-dual if and only if $F(A)$ is of type $(1,1)$ and primitive, that is, $F(A)$ is written by $F(A) = \sum_{\mu, \nu} F_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu$ and satisfies

$$\sum_{\sigma, \tau} h^{\sigma\bar{\tau}} F_{\sigma\bar{\tau}} = 0.$$

This fact is a starting point of research of the moduli space $\mathcal{M}_a$ over a complex surface.

Koiso generalizes this anti-self-duality to higher dimensional complex manifolds [9].

Theorem(Kobayashi[8]). If the bundle P admits an anti-self-dual connection, then the associated vector bundle $E = P \times_{\rho} \mathbb{C}^n$ must be a (semi-)stable holomorphic vector bundle in the sense of Mumford and Takemoto.

Recently Donaldson proved in his preprint [4] that over an algebraic surface the stability is indeed a sufficient condition.

Over a Kähler surface the following is an answer to Problem 2.

Theorem[7]. The moduli space of irreducible anti-self-dual connections admits over a compact Kähler surface a structure of complex analytic set, that is, it is a zero point set of local several holomorphic functions.

Remarks. 1. An anti-self-dual connection is just an Einstein Hermitian structure of the associated vector bundle in the sense of Kobayashi.

2. Over an algebraic surface the moduli of stable holomorphic structure of a smooth vector bundle is a quasi-projective variety (Maruyama[10]).

Example. In the case of $M = P_2(\mathbb{C})$, $SU(2)$ and $c_2(P) = k (> 1)$ the moduli space is a complex manifold without singularity of $\dim_{\mathbb{C}} 4k - 3$.

The theorem is first conjectured by Atiyah.

A sketch of proof of Theorem will be given in § 5.

4. To define precisely the moduli space over a general 4-manifold we need notion of gauge transformations.

A bundle automorphism f of P (that is, $f: P \rightarrow P$ is a diffeomorphism and satisfies $f(ua) = f(u)a$)

which induces on M the identity transformation is called a gauge transformation. A gauge transformation can be

identified with a global section of the automorphism bundle

$G_P = P \times_{\text{conj}} \text{SU}(n)$. A gauge transformation operates

on a connection A to induce a new connection $f^*A = f^{-1}.df + f^{-1}.A.f$ with curvature form $F(f^*A) = f^{-1}.F(A).f$.

Therefore $\mathcal{Y}^m$ is gauge-invariant, because $|F(f^*A)|^2 = |F(A)|^2$ and an anti-self-dual connection is transformed

into new one which is also anti-self-dual. But in physical and geometrical meaning they represent the same thing.

So we have a quotient space of the set of anti-self-dual connections modulo gauge transformations and call it moduli space $\mathcal{M}_a$ of anti-self-dual connections:

$$\{ \text{anti-self-dual connections} \} \xrightarrow{\pi} \mathcal{M}_a.$$

The adjoint representation of the gauge group induces a vector bundle associated to P which we call the adjoint bundle and denote it by $\mathcal{G}_P$.

Almost everything of Yang-Mills theory must be discussed over this bundle because curvature forms are $\mathcal{G}_P$ -valued 2-forms and an infinitesimal gauge transformation is just a global section of $\mathcal{G}_P$ etc.

Now we assume that a connection A is anti-self-dual over a general compact oriented Riemannian 4-manifold M . By the aid of the Ricci formula we have the sequence:

$$(*) \quad 0 \longrightarrow \Omega^0(\mathcal{G}_P) \xrightarrow{\nabla_A} \Omega^1(\mathcal{G}_P) \xrightarrow{d_A^+ = P_+ \circ d_A} \Omega_+^2(\mathcal{G}_P) \longrightarrow 0$$

is an elliptic complex, that is, $d_A^+ \nabla_A = 0$ and the symbol sequence of this is exact at any covector $\xi \neq 0$.

Here $\Omega^k(\mathcal{G}_P)$ denotes the space of smooth $\mathcal{G}_P$ -valued k -forms ($k = 1, 2$) and $\Omega_+^2(\mathcal{G}_P)$ the space of $\mathcal{G}_P$ -valued self-dual 2-forms. Further P_+ is

the orthogonal projection of Λ^2 onto Λ_+^2 .

Cohomologies H^0 , H^1 and H_+^2 associated to $(*)$ are all finite dimensional. The Atiyah-Singer index theorem shows that the index of $(*)$, that is, $\dim H^0 - \dim H^1 + \dim H_+^2$ is given by $\int_M \{ \text{characteristic classes of } M \text{ and } P \}$

According to either $H^0 = 0$ or $H^0 \neq 0$ and either $H_+^2 = 0$ or $H_+^2 \neq 0$ we have four cases:

- (i) $H^0 = 0$ and $H_+^2 = 0$; the moduli space $\mathcal{M}_a$ is smooth at $[A] = \pi(A)$ and has dimension equal to $-\text{index } (*)$

- (ii) $H^0 = 0$ and $H_+^2 \neq 0$; $\mathcal{M}_a$ is at $[A]$ a zero point set of a C^ω -map Φ from a neighborhood of H^1 to H_+^2
- (iii) $H^0 \neq 0$ and $H_+^2 = 0$; $\mathcal{M}_a$ is written as a Γ_A -quotient of a 0-neighborhood of H^1
- (iv) $H^0 \neq 0$ and $H_+^2 \neq 0$; $\mathcal{M}_a$ has at $[A]$ a structure of Γ_A -quotient of zero point set of a C^ω -map Φ from a 0-neighborhood of H^1 to H_+^2

Here Γ_A is the isotropy group of A , that is, $\Gamma_A = \{ \text{gauge transformations } f; f^*A = A \}$.

We call a connection to be irreducible if $\text{Ker } \nabla_A = 0$ and call a connection reducible when it is not irreducible.

Then by combining (i) — (iv) we have

Proposition 4.1. The moduli space $\{ [A] \in \mathcal{M}_a; A \text{ is irreducible} \}$ is a real analytic set and the whole moduli space is around reducible connection a Γ_A -quotient of a real analytic set.

Those properties of $\mathcal{M}_a$ stated above are based on the following facts:

Fact 1. Since $F(A + \alpha) = F(A) + d_A \alpha + \alpha \wedge \alpha$ ($\alpha \in \Omega^1(\mathfrak{g}_P)$), for a fixed anti-self-dual connection A a connection $A + \alpha$ is anti-self-dual if and only if $d_A^+ \alpha + P_+(\alpha \wedge \alpha) = 0$.

Fact 2. Because $\text{Ker } d_A^* = \{ \alpha \in \Omega^1(\mathcal{G}_P); d_A^* \alpha = 0 \}$ is transversal in $\mathcal{E}_P$ to the orbit through A of gauge transformations, where d_A^* is the formal adjoint of d_A , $\mathcal{M}_a$ has a neighborhood homeomorphic to a slice $\mathcal{S}_A = \{ \alpha \in \Omega^1(\mathcal{G}_P); d_A^* \alpha = 0, d_A^+ \alpha + P_+(\alpha \wedge \alpha) = 0 \}$ or a Γ_A -quotient $\mathcal{S}_A / \Gamma_A$ according to the irreducibility of A .

Fact 3. We combine these facts to derive so-called Kuranishi's map from the slice to H^1 . Define a map $\Phi; \Omega^1(\mathcal{G}_P) \longrightarrow \Omega^1(\mathcal{G}_P)$, $\Phi(\alpha) = \alpha + (d_A^+)^* \circ G_A(P_+ \alpha \wedge \alpha)$ where G_A is the Green operator of the Laplacian $d_A^+ \circ (d_A^+)^*$. Since $\frac{d}{d\alpha} \Phi|_{\alpha=0} = \text{identity}$, Φ has an inverse over

a neighborhood of 0. We give a map $\psi; \{ \beta \in \text{Ker } d_A^*; \|\beta\|_{L_k^2} < \varepsilon \} \longrightarrow H^2$ by $\psi(\beta) = -$ harmonic part of $P_+ \alpha \wedge \alpha$ for $\alpha = \Phi^{-1}(\beta)$ to assert that $\Phi; \mathcal{S}_A \longrightarrow \{ \beta \in H^1; \|\beta\|_{L^2} < \varepsilon, \psi(\beta) = 0 \}$ is a homeomorphism.

5. Now we assume again in this section that the base space is Kähler. In order to give the moduli space $\mathcal{M}_a$ a complex manifold structure we need finer considerations than the discussion of § 4. For this purpose we complexify every real structures, $SU(n)$ to $SL(n; \mathbb{C})$, the principal fibre bundle, the adjoint bundle, gauge transformations etc, except but the transition functions of the bundle. A connection A then splits into the $(1,0)$ -part A' and the $(0,1)$ -part A'' so that A'' satisfies

$$A''_{\beta} = g^{-1} \cdot \bar{\partial} g + g^{-1} \cdot A''_{\alpha}.$$

Definition. A system $\{\tilde{A}_\alpha, \tilde{A}_\beta, \dots\}$ of locally defined $\mathcal{L}(n; \mathbb{C})$ -valued $(0,1)$ -forms compatible with the transition functions of P ,

$$\tilde{A}_\beta = g^{-1} \cdot \bar{\partial} g + g^{-1} \cdot \tilde{A}_\alpha \cdot g \quad \text{on } U_\alpha \cap U_\beta,$$

is called a $(0,1)$ -connection.

A $(0,1)$ -connection $\tilde{A}$ induces the partial covariant derivative $\bar{\partial}_{\tilde{A}}: \Omega^0(\mathcal{O}_P^{\mathbb{C}}) \longrightarrow \Omega^{0,1}(\mathcal{O}_P^{\mathbb{C}}), \quad \phi \longmapsto \bar{\partial} \phi + [A, \phi]$ and also the covariant exterior derivative $\bar{\partial}_A: \Omega^{p,q}(\mathcal{O}_P^{\mathbb{C}}) \longrightarrow \Omega^{p,q+1}(\mathcal{O}_P^{\mathbb{C}})$ where $\mathcal{O}_P^{\mathbb{C}}$ denotes the complexification of $\mathcal{O}_P$.

Definition. A $(0,1)$ -connection $\tilde{A}$ is said holomorphic if its curvature $F(\tilde{A}) = \bar{\partial} \tilde{A} + \tilde{A} \wedge \tilde{A}$ vanishes.

Remark. Each holomorphic $(0,1)$ -connection induces by integrability condition a holomorphic structure on $\mathcal{O}_P^{\mathbb{C}}$ such that $\bar{\partial}_A$ is just $\bar{\partial}$ -operator (Atiyah-Hitchin-Singer [2] and Griffiths [5]).

Note. Holomorphic $(0,1)$ -connection can be also defined over a higher dimensional complex manifold.

In a way similar to the case of anti-self-dual connections we can define moduli space $\mathcal{M}_h$ of holomorphic $(0,1)$ -connections with respect to complex gauge transformations.

Since each anti-self-dual connection A has curvature form of type $(1,1)$, its $(0,1)$ -component A'' gives automatically a holomorphic $(0,1)$ -connection because $F(A'')$ is just the $(0,2)$ -component of full curvature form $F(A)$. Then we have a canonical map ϕ from $\mathcal{M}_a$ to $\mathcal{M}_h$ by assigning $[A'']$ to $[A]$.

When $\tilde{X}$ is holomorphic, the sequence $(**)$ is an elliptic complex

$$(**) \quad 0 \longrightarrow \Omega^0(\mathcal{O}_P^{\mathbb{C}}) \xrightarrow{\bar{\partial}_{\tilde{X}}} \Omega^{0,1}(\mathcal{O}_P^{\mathbb{C}}) \xrightarrow{\bar{\partial}_A} \Omega^{0,2}(\mathcal{O}_P^{\mathbb{C}}) \longrightarrow 0.$$

Proposition 5.1. The moduli space of irreducible holomorphic $(0,1)$ -connections is a complex analytic set.

Here a $(0,1)$ -connection is called irreducible when $\text{Ker} \{ \bar{\partial}_{\tilde{X}}: \Omega^0(\mathcal{O}_P^{\mathbb{C}}) \longrightarrow \Omega^{0,1}(\mathcal{O}_P^{\mathbb{C}}) \} = 0$. This proposition is shown by a way similar to the anti-self-dual case, while the complex analyticity is assured by the fact that the Kuranishi's map is in this case holomorphic.

Proposition 5.2. i) The spaces $\mathcal{M}_a$ and $\mathcal{M}_h$ of generic connections (that is, connections with $H^0 = 0$ and $H^2 = 0$, respectively) are smooth manifolds with same real dimension $2 \{ c_2(\mathcal{O}_P^{\mathbb{C}}) - \dim \text{SU}(n) \cdot p(M) \}$ where $p(M)$ is the arithmetic genus of M and ii) the canonical map ϕ is smooth and of maximal rank over

$$\mathcal{M}_{a,\text{gen}} = \{ [A] \in \mathcal{M}_a; A \text{ is generic} \}$$

Note. We use the moment map due to Donaldson [4] to conclude that ϕ is one-to-one over $\mathcal{M}_{a,ir} = \{ [A] \in \mathcal{M}_a; A \text{ is irreducible} \}$

As a consequence of these results the moduli space $\mathcal{M}_{a,ir}$ has a structure of complex analytic set which around a generic one is smooth and is singular at one with $H_+^2 = 0$. This is just the statement of Theorem.

We remark that around a reducible anti-self-dual connection the moduli space $\mathcal{M}_a$ is a Γ_A -quotient of a real analytic set.

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