

SOME ERGODIC PROPERTIES OF A COMPLEX

CONTINUED FRACTION ALGORITHM

by

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Recently R. Kaneiwa, J. Tamura and the author of this paper [1] proved, by making use of a certain kind of continued fraction algorithm for complex numbers, a theorem of Perron on complex Diophantine approximations [2]: For any complex number θ not belonging to the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$ there exist infinitely many integral elements p, q in $\mathbb{Q}(\sqrt{-3})$ such that

$$\left| \theta - \frac{p}{q} \right| < \frac{1}{4\sqrt{13} |q|^2}.$$

If $\theta = \frac{1}{2}(\zeta + \sqrt{\zeta^2 + 4})$, where $\zeta = \frac{1}{2}(1 + \sqrt{-3})$, the constant $4\sqrt{13}$ can not be improved.

In this paper we investigate some ergodic properties of the complex continued fraction transformation defined as the remainders of the algorithm in [1].

1. Definition of the algorithm

Every complex number z can be uniquely written in the form $z = u\zeta + v\bar{\zeta}$, where u and v are real and $\bar{w}$ is the complex conjugate of a complex number w . We put

$$z = [u]\zeta + [v]\bar{\zeta},$$

where, in the right-hand side, $[x]$ is the largest rational integer not exceeding a real number x . Note that if z is real then $[z]$ becomes the ordinary Gauss's symbol. Now we define a continued fraction algorithm (*) as follows;

$$(*) \begin{cases} T^n z = \frac{1}{T^{n-1} z} - \left[\frac{1}{T^{n-1} z} \right] & (n \geq 1), \quad T^0 z = z - [z], \\ a_n = a_n(z) = \left[\frac{1}{T^{n-1} z} \right] & (n \geq 1), \quad a_0 = a_0(z) = [z]. \end{cases}$$

These procedures terminate, i.e. $T^n z = 0$ for some $n \geq 0$, if and only if z belongs to $Q(\sqrt{-3})$. Hence every complex number z can be expanded in the form

$$z = a_0 + \frac{1}{a_1} + \dots + \frac{1}{a_n + T^n z} \quad (n \geq 0), \quad (1)$$

provided $T^k z \neq 0$ for all $k < n$.

Let Z_ζ be the ring of all integers in $Q(\sqrt{-3})$ and let N_ζ be the subset of Z_ζ defined by

$$N_\zeta = \{u\zeta + v\bar{\zeta}; u, v \text{ non-negative integers with } u+v \geq 1\}.$$

We put

$$D = \{u\zeta + v\bar{\zeta}; u, v \geq 0, u+v \geq 0\},$$

$$X = \{u\zeta + v\bar{\zeta}; 0 \leq u, v < 1\},$$

and

$$Y = D \setminus \{z; z^{-1} \in X\}.$$

Thus the remainder $T^n z$, in the algorithm (*), is the n th power of the transformation T of X onto itself defined by

$$Tz = \frac{1}{z} - \left[\frac{1}{z} \right] \quad (z \in X),$$

which is an extension of the real continued fraction transformation

$$Tx = \frac{1}{x} - \left[\frac{1}{x} \right] \quad (x \in [0, 1)).$$

By definition we have

$$|z| \leq \frac{2\sqrt{3}}{3} \quad (z \in Y), \quad (2)$$

$$|z| \geq \frac{\sqrt{3}}{2} \quad (z \in D \setminus X), \quad (3)$$

and

$$\begin{aligned} \{a_0(z); z \in C\} &= Z_\zeta, \\ \{a_n(z); z \in C\} &= N_\zeta \subset D \setminus X \quad (n \geq 1) \end{aligned} \quad (4)$$

where C is the set of all complex numbers.

Every finite continued fraction

$$\sqrt{\frac{1}{z_1}} + \sqrt{\frac{1}{z_2}} + \dots + \sqrt{\frac{1}{z_n}}$$

whose partial denominators $z_1, z_2, \dots, z_n$ belong to $D \setminus \{0\}$ is well-defined, since the fractions $z_n^{-1}, z_{n-1} + z_n^{-1}, \dots$ are different from zero. (Note that if $z \in D \setminus \{0\}$ then $z^{-1} \in D \setminus \{0\}$ and that if $z, w \in D \setminus \{0\}$ then $z+w \in D \setminus \{0\}$.) Let, more precisely, $z_1, z_2, \dots, z_n \in D \setminus X$. Then $z_n^{-1} \in Y \setminus \{0\}$ and so $z_{n-1} + z_n^{-1} \in D \setminus X$. Repeating this process we get

$$z_1 + \sqrt{\frac{1}{z_2}} + \dots + \sqrt{\frac{1}{z_n}} \in D \setminus X \quad (5)$$

and

$$\sqrt{\frac{1}{z_1}} + \sqrt{\frac{1}{z_2}} + \dots + \sqrt{\frac{1}{z_n}} \in Y \setminus \{0\}. \quad (6)$$

Let $a_0, a_1, \dots$ be any sequence of integers in $Q(\sqrt{-3})$ such that $a_n \in N_\zeta$ ($n \geq 1$). Every finite continued fraction

$$a_0 + \sqrt{\frac{1}{a_1}} + \dots + \sqrt{\frac{1}{a_n}}$$

has a canonical representation p_n/q_n ($p_n, q_n \in \mathbb{Z}_5$), called n th approximant, in the form of an ordinary fraction. Especially if the sequence $a_0, a_1, \dots$ is given by the algorithm (*) we call the fraction p_n/q_n the n th approximant of z . Thus from the general theory of finite continued fractions we have the following formulae (7)-(10): (For the proofs see [3].)

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2} \quad (n \geq 1), \quad (7)$$

$$\frac{1}{a_n} + \frac{1}{a_{n-1}} + \dots + \frac{1}{a_1} = \frac{q_{n-1}}{q_n} \quad (n \geq 1), \quad (8)$$

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \quad (n \geq 0), \quad (9)$$

where $p_{-1} = 1, q_{-1} = 0, p_0 = a_0, q_0 = 1$. Further if p_n/q_n is the n th approximant of z , then

$$z - \frac{p_n}{q_n} = (-1)^n \left(a_{n+1} + T^{n+1} z + \frac{q_{n-1}}{q_n} \right)^{-1} \frac{1}{q_n^2}. \quad (10)$$

LEMMA 1. (R. Kaneiwa, I. Shiokawa and J. Tamura [2])

Let $a_0, a_1, \dots$ be any infinite sequences of integers in $\mathbb{Q}(\sqrt{-3})$ such that $a_n \in \mathbb{N}$ ($n \geq 1$) and let p_n/q_n be the n th approximant. Then we have

$$q_n \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.$$

For completeness we prove this lemma.

Proof. Suppose, on the contrary, that $q_n \not\rightarrow \infty$ as $n \rightarrow \infty$. So we can choose an infinite subsequence $\{q_{n_j}\}_{j=1}^{\infty}$ such that $|q_{n_j}| < M$ for all $n \geq 1$, where M is a constant independent of j . But from (2) and (6) we have

$$\left| \frac{p_n}{q_n} \right| < |a_0| + \frac{2\sqrt{3}}{3}$$

and so

$$\left| p_{n_j} \right| < (|a_0| + \frac{2\sqrt{3}}{3}) M,$$

where the right-hand side is also independent of j . It follows from these inequalities that $p_{n_j}/q_{n_j} = p_{n_k}/q_{n_k}$ for some j and k with $j < k$, since the ring of all integers in $\mathbb{Q}(\sqrt{-3})$ is discrete. Hence we have

$$\left[\frac{1}{a_{n_j+1}} \right] + \left[\frac{1}{a_{n_j+2}} \right] + \dots + \left[\frac{1}{a_{n_k}} \right] = 0,$$

which contradicts (7).

LEMMA 2. (ibid.) Let z be any complex number not belonging to $\mathbb{Q}(\sqrt{-3})$ and let p_n/q_n be its n th approximant. Then we have

$$z = \lim_{n \rightarrow \infty} \frac{p_n}{q_n}.$$

Proof. By (10) as well as (3), (5), (6), (8) we have

$$\left| z - \frac{p_n}{q_n} \right| < \frac{2\sqrt{3}}{3} |q_n|^{-2}$$

which tend to zero as $n \rightarrow \infty$.

LEMMA 3. (ibid.) With the same notations as in Lemma 1, the n th approximant p_n/q_n converges to some complex number which belongs to $b_0 + Y$.

Proof. Similar to that of Lemma 2.

Now, by means of Lemma 2, every complex number z can be expanded in a regular continued fraction whose

partial denominators $a_n(z)$ are integers in $Q(\sqrt{-3})$;

$$z = a_0(z) + \frac{1}{a_1(z)} + \frac{1}{a_2(z)} + \dots$$

This complex continued fraction expansion is a natural extension of the ordinary real one, since both algorithms coincide when z is real.

2. Admissible sequences and fundamental cells

We put

$$A^{(n)} = \{a_1(z) \dots a_n(z) ; z \in X\} \quad (1 \leq n \leq \infty)$$

Sequences belonging to $A^{(n)}$ ($1 \leq n \leq \infty$) will be called admissible. (Note that Lemma 3 suggests the existence of non-admissible sequences.) By definition if $a_1 \dots a_n \in A^{(n)}$ then $a_1 \dots a_{n-1} \in A^{(n-1)}$ and $a_2 \dots a_n \in A^{(n-1)}$. If $a_1 a_2 \dots \in A^{(\infty)}$ then $a_1 \dots a_n \in A^{(n)}$ for all $n \geq 1$. And the conjugate $\bar{a}_1 \bar{a}_2 \dots$ of some (finite or infinite) admissible sequence is also admissible.

For any $a_1 \dots a_n \in A^{(n)}$ we define

$$X_{a_1 \dots a_n} = \{z \in X ; a_k(z) = a_k, 1 \leq k \leq n\},$$

which will be called a fundamental cell of rank n . Thus we have

$$X = \bigcup_{a_1 \dots a_n \in A^{(n)}} X_{a_1 \dots a_n}$$

where $X_{a_1 \dots a_n} \cap X_{b_1 \dots b_n} = \emptyset$ if $a_k \neq b_k$ for some k with $1 \leq k \leq n$; i.e. the set of all fundamental cells of rank n form a partition of X . Besides, for any fixed infinite admissible sequence $a_1 a_2 \dots$ we find

$$X \supset X_{a_1} \supset \dots \supset X_{a_1 \dots a_{n-1}} \supset X_{a_1 \dots a_n}$$

and (by Lemma 2)

$$\text{diam}(X_{a_1 \dots a_n}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence every Lebesgue measurable subset of X may be approximated with any accuracy by finite unions of mutually disjoint fundamental cells.

For any given $a_1 \dots a_n \in A^{(n)}$ we define a function of z by

$$\psi_{a_1 \dots a_n}(z) = \left\lfloor \frac{1}{a_1} \right\rfloor + \dots + \left\lfloor \frac{1}{a_{n-1}} \right\rfloor + \left\lfloor \frac{1}{a_n + z} \right\rfloor$$

or (replacing a_n in (7) by $a_n + z$)

$$= \frac{p_n + p_{n-1}z}{q_n + q_{n-1}z} \quad (z \in X).$$

Because of the formula (9) the linear transformation

$\psi_{a_1 \dots a_n}$ has the inverse

$$(\psi_{a_1 \dots a_n})^{-1}(z) = \frac{p_n - q_n z}{-p_{n-1} + q_{n-1} z} \quad (z \in \psi_{a_1 \dots a_n}(X)).$$

But the equality (1) can be rewritten in the form

$$z = \psi_{a_1 \dots a_n}(T^n z) \quad (z \in X).$$

Hence for any fixed $a_1 \dots a_n \in A^{(n)}$ the n th power of T restricted on the cell $X_{a_1 \dots a_n}$ is identical with the inverse of $\psi_{a_1 \dots a_n}$;

$$T^n z = (\psi_{a_1 \dots a_n})^{-1}(z) \quad (z \in X_{a_1 \dots a_n}). \quad (11)$$

Especially we have for any $a_1 \dots a_n \in A^{(n)}$

$$X_{a_1 \dots a_n} = \psi_{a_1 \dots a_n}(T^n X_{a_1 \dots a_n}). \quad (12)$$

Now we need some notations : Put

$$U_1 = \left\{ z \in X ; \left| z + \frac{\sqrt{-3}}{3} \right| > \frac{\sqrt{3}}{3} \right\},$$

$$U_2 = \{ z \in X ; \operatorname{Im}(z) > 0 \},$$

$$U_3 = \{ z \in X ; \bar{z} \in U_1, \operatorname{Im}(z) > 0 \},$$

and define

$$U_{-j} = \{ \bar{z} ; z \in U_j \} \quad (j = 1, 2, 3).$$

Further we set $U_0 = X$ for notational convenience.

Considering the reciprocals $U_j^{-1} = \{ z ; z^{-1} \in U_j \}$

we obtain (see Fig. 1)

$$X = \psi_{\zeta}(U_1) \cup \psi_{\bar{\zeta}}(U_{-1}) \cup \left(\bigcup_{\substack{a \in \mathbb{N}_{\zeta} \\ a \neq \zeta, \bar{\zeta}}} \psi_a(X) \right), \quad (13.0)$$

$$U_1 = \psi_{\zeta}(U_{-3}) \cup \psi_{\bar{\zeta}}(U_{-1}) \cup \left(\bigcup_{k=1}^{\infty} \psi_{\zeta+k}(U_{-2}) \right) \cup \left(\bigcup_{\substack{a \in \mathbb{N}_{\zeta}, a \neq \bar{\zeta} \\ \operatorname{Im}(a) \leq 0}} \psi_a(X) \right), \quad (13.1)$$

$$U_2 = \psi_{\bar{\zeta}}(U_{-1}) \cup \left(\bigcup_{k=1}^{\infty} \psi_k(U_{-2}) \right) \cup \left(\bigcup_{\substack{a \in \mathbb{N}_{\zeta}, a \neq \bar{\zeta} \\ \operatorname{Im}(a) < 0}} \psi_a(X) \right), \quad (13.2)$$

and

$$U_3 = \psi_{\bar{\zeta}}(U_3) \cup \bigcup_{k=1}^{\infty} (\psi_k(U_{-2}) \cup \psi_{\bar{\zeta}+k}(U_2)). \quad (13.3)$$

Taking the complex conjugate of (13.1)–(13.3) we have

also the same relations for U_{-1} , U_{-2} and U_{-3} to which we assign (13.-1), (13.-2) and (13.-3) resp.

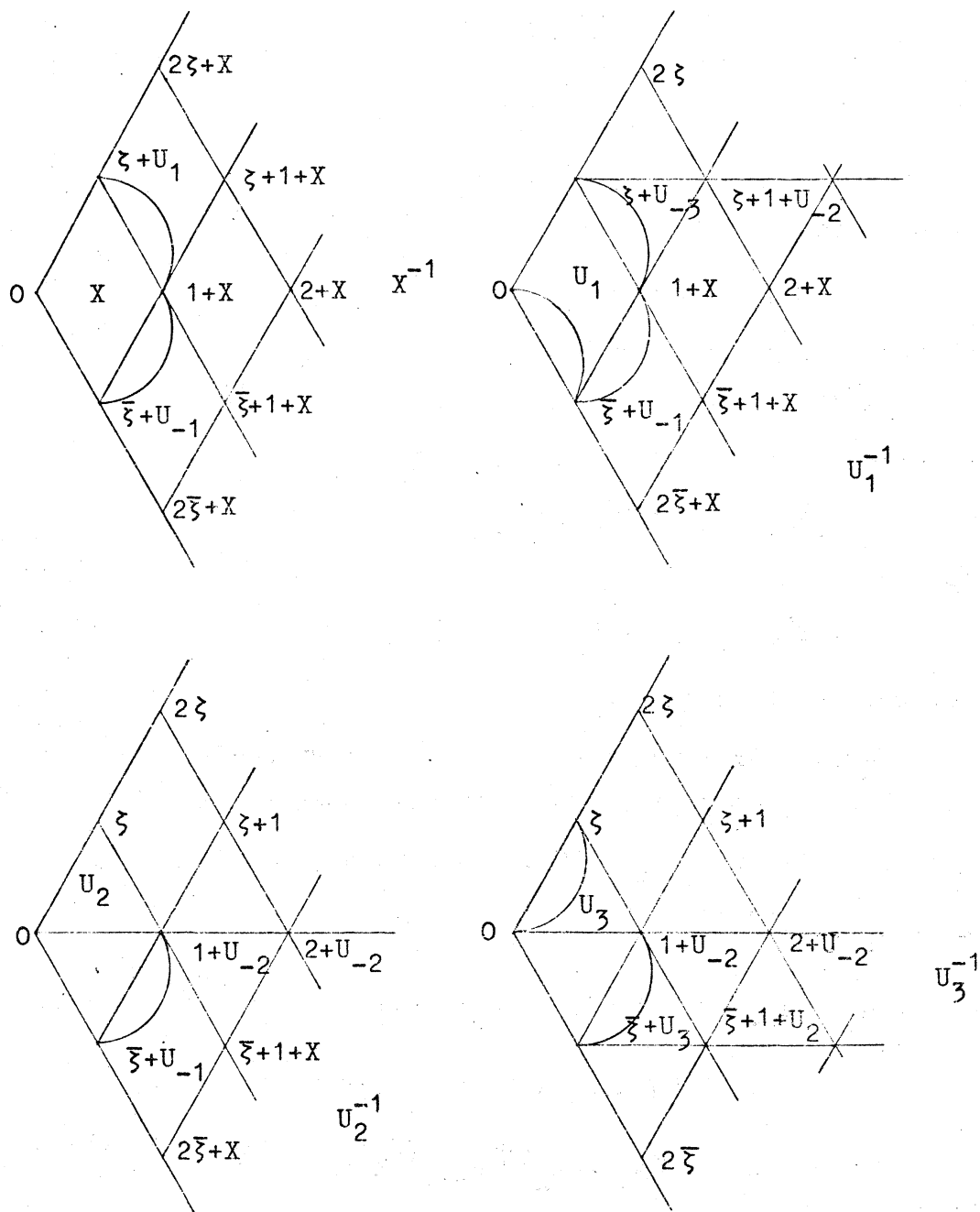


Fig. 1

In any case U_j can be written in the form

$$U_j = \bigcup_{a \in M_j} \psi_a(U_k) \quad (14)$$

where M_j is a subset of N_ζ and k ($-3 \leq k \leq 3$) are chosen uniquely according as j and a . In addition, we note that

$$\psi_a(X) \cap \psi_b(X) = \phi \quad (15)$$

whenever $a \neq b$ ($a, b \in N_\zeta$).

LEMMA 4. Let $n \geq 1$ and let $a_1 \dots a_n \in A^{(n)}$. Then we have

$$X_{a_1 \dots a_n} = \psi_{a_1 \dots a_n}(U_j) \quad (16)$$

and so

$$T^n X_{a_1 \dots a_n} = U_j \quad (17)$$

for some j ($-3 \leq j \leq 3$).

Proof. By induction on n . First we prove (16).

If $n = 1$ (16) follows from (13.0). Suppose that (16) hold for all $a_1 \dots a_n \in A^{(n)}$. Then we have for any

$$a_1 \dots a_{n+1} \in A^{(n+1)}$$

$$\begin{aligned} X_{a_1 \dots a_{n+1}} &= \{ z \in X_{a_1 \dots a_n}; a_{n+1}(z) = a_1(T^n z) = a_{n+1} \} \\ &= \{ \psi_{a_1 \dots a_n}(w); w \in U_j, a_1(w) = a_{n+1} \} \\ &= \psi_{a_1 \dots a_n}(\psi_{a_{n+1}}(U_k)), \quad (\text{by (14), (15)}) \\ &= \psi_{a_1 \dots a_{n+1}}(U_k), \end{aligned}$$

where j is defined by $U_j = T^n X_{a_1 \dots a_n}$ and k chosen uniquely in (14). Now (17) follows from (12) and (16).

Let E be any subset of X . Then by Lemma 4 we have for any $a_1 \dots a_n \in A^{(n)}$

$$\begin{aligned} T^{-n}E &= \{ z \in X ; T^n z \in E \} \\ &= \bigcup_{a_1 \dots a_n \in A^{(n)}} \{ z \in X_{a_1 \dots a_n} ; T^n z \in E \cap U_j \} \\ &= \bigcup_{a_1 \dots a_n \in A^{(n)}} \psi_{a_1 \dots a_n}(E \cap U_j), \quad U_j = T^n X_{a_1 \dots a_n} \end{aligned} \quad (18)$$

3. Estimates of the Lebesgue measure

Let m be the Lebesgue measure on the complex plane and let $\mathcal{B}$ be the σ -field of all measurable subsets of X . Then we have for any $a_1 \dots a_n \in A^{(n)}$ and $E \in \mathcal{B}$

$$m(\psi_{a_1 \dots a_n}(E)) = \iint_E |\psi'_{a_1 \dots a_n}(z)|^2 dx dy, \quad z = x + iy. \quad (19)$$

But using (9) we find

$$\psi_{a_1 \dots a_n}(z) = (-1)^n (q_n + q_{n-1}z)^{-2}$$

and so

$$|\psi'_{a_1 \dots a_n}(z)|^2 = |q_n|^{-4} \left| 1 + \frac{q_{n-1}}{q_n} z \right|^{-4} \quad (20)$$

Hence we have

$$3^{-4} < |q_n|^4 |\psi'_{a_1 \dots a_n}(z)|^2 < 3^4 \quad (21)$$

and

$$3^{-4} < |q_n|^{-4} |(\psi_{a_1 \dots a_n}^{-1})'(z)|^2 < 3^4, \quad (22)$$

since (from (2), (3), (6), (8))

$$3^{-1} < \frac{\sqrt{3}}{2} \leq \left| 1 + \frac{q_n}{q_n} z \right| \leq 1 + \frac{2\sqrt{3}}{3} < 3.$$

Taking account of the fact that $3^{-2} < m(U_j) < 1$ ($-3 \leq j \leq 3$) we have from (19) and (21)

$$3^{-6} < |q_n|^4 m(X_{a_1 \dots a_n}) < 3^4 \quad (a_1 \dots a_n \in A^{(n)}) \quad (23)$$

We write

$$S(n) = \sum_{a_1 \dots a_n \in A^{(n)}} |q_n|^{-4}$$

Then for any $n \geq 1$ we have

$$3^{-5} < S(n) < 3^6 \quad (24)$$

Indeed it follows from (23) that

$$3^4 S(n) > \sum_{A^{(n)}} m(X_{a_1 \dots a_n}) = m(X) > 3^{-1}$$

and

$$3^{-6} S(n) < m(X) < 1.$$

By means of Lemma 4 the set $A^{(n)}$ of all admissible sequences can naturally be divided into seven subsets ; we put

$$A_j^{(n)} = \{ a_1 \dots a_n \in A^{(n)} ; T^n X_{a_1 \dots a_n} = U_j \} \quad (-3 \leq j \leq 3),$$

then we have

$$A^{(n)} = \bigcup_{j=-3}^3 A_j^{(n)}.$$

By (13.j) $(-3 \leq j \leq 3)$ we have for any $n \geq 1$ the following relations;

$$\begin{aligned} A_0^{(n)} = & \{ a_1 \dots a_n \in A^{(n)} ; a_1 \dots a_{n-1} \in A_0^{(n-1)}, a_n \neq \zeta, \bar{\zeta} ; \\ & \text{or } a_1 \dots a_{n-1} \in A_1^{(n-1)}, a_n \neq \bar{\zeta}, \operatorname{Im}(a_n) \leq 0 ; \\ & \text{or } a_1 \dots a_{n-1} \in A_{-1}^{(n-1)}, a_n \neq \zeta, \operatorname{Im}(a_n) \geq 0 ; \\ & \text{or } a_1 \dots a_{n-1} \in A_2^{(n-1)}, a_n \neq \bar{\zeta}, \operatorname{Im}(a_n) < 0 ; \\ & \text{or } a_1 \dots a_{n-1} \in A_{-2}^{(n-1)}, a_n \neq \zeta, \operatorname{Im}(a_n) > 0 \} \end{aligned} \quad (25.0)$$

$$A_1^{(n)} = \{ a_1 \dots a_n \in A^{(n)}; a_1 \dots a_{n-1} \in A_0^{(n-1)} \cup A_{-1}^{(n-1)} \cup A_{-2}^{(n-1)}, \\ a_n = \zeta \} \quad (25.1)$$

$$A_2^{(n)} = \{ a_1 \dots a_n \in A^{(n)}; a_1 \dots a_{n-1} \in A_{-1}^{(n-1)} \cup A_{-3}^{(n-1)}, a_n - \bar{\zeta} \in N; \\ \text{or } a_1 \dots a_{n-1} \in A_{-2}^{(n-1)} \cup A_{-3}^{(n-1)}, \\ a_n \in N \} \quad (25.2)$$

$$A_3^{(n)} = \{ a_1 \dots a_n \in A^{(n)}; a_1 \dots a_{n-1} \in A_{-1}^{(n-1)} \cup A_3^{(n-1)}, \\ a_n = \bar{\zeta} \} \quad (25.3)$$

and

$$A_{-j}^{(n)} = \{ a_1 \dots a_n; a_1 \dots a_n \in A_j^{(n)} \} \quad (j=1,2,3) \quad (25.-j)$$

where N is the set of all positive integers.

We write

$$S_j(n) = \sum_{a_1 \dots a_n \in A_j^{(n)}} |q_n|^{-4} \quad (-3 \leq j \leq 3).$$

Thus we have

$$S_j(n) = S_{-j}(n) \quad (-3 \leq j \leq 3). \quad (26)$$

and

$$S(n) = \sum_{j=-3}^3 S_j(n). \quad (27)$$

LEMMA 5. For any $n \geq 3$ we have

$$S_j(n) > 3^{-12} \quad (-3 \leq j \leq 3)$$

Proof. From (7) and the inequality

$$3^{-1} |a_n| < \left| a_n + \frac{q_{n-2}}{q_{n-1}} \right| < 3 |a_n|$$

we find for any $a_1 \dots a_n \in A^{(n)}$

$$3^{-1} |a_n| |q_{n-1}| < |q_n| < 3 |a_n| |q_{n-1}|. \quad (28)$$

By (25.0), (26) and (28) we have

$$\begin{aligned}
 3 S_0(n) &> \sum_{a \in \mathbb{N}_z \setminus \{\xi, \bar{\xi}\}} |a|^{-4} \sum_{a_1 \dots a_{n-1} \in A_0^{(n-1)}} |q_{n-1}|^{-4} \\
 &+ 2 \sum_{\substack{a \in \mathbb{N}_z \setminus \{\xi\} \\ \text{Im}(a) \leq 0}} |a|^{-4} \sum_{a_1 \dots a_{n-1} \in A_1^{(n-1)}} |q_{n-1}|^{-4} \\
 &+ 2 \sum_{\substack{a \in \mathbb{N}_z \setminus \{\xi\} \\ \text{Im}(a) < 0}} |a|^{-4} \sum_{a_1 \dots a_{n-1} \in A_2^{(n-1)}} |q_{n-1}|^{-4} \\
 &> S_0(n-1) + 2S_1(n-1) + 2|\bar{\xi} + 1|^{-4} S_2(n-1) \\
 &> 3^{-2}(S_0(n-1) + S_1(n-1) + S_2(n-1)).
 \end{aligned}$$

Hence we have

$$S_0(n) > 3^{-3}(S_0(n-1) + S_1(n-1) + S_2(n-1)).$$

In the same way we obtain

$$S_1(n) > 3^{-1}(S_0(n-1) + S_1(n-1) + S_2(n-1)),$$

$$S_2(n) > 3^{-3}(S_1(n-1) + S_2(n-1) + S_3(n-1)),$$

and

$$S_3(n) > 3^{-1}(S_1(n-1) + S_3(n-1))$$

(using (25.1) - (25.3).) It follows from these inequalities with (24), (26), (27) that

$$S_0(n) > 3^{-6} \sum_{j=0}^3 S_j(n-2) > 3^{-7} S(n) > 3^{-12}.$$

Similary we have for any $n \geq 3$

$$S_j(n) > 3^{-12} \quad (-3 \leq j \leq 3).$$

4. Invariant measure and ergodicity

THEOREM 1. Let E be any measurable subset of X such that $T^{-1}E = E$. Then $m(E) = 0$ or 1 .

Proof. We assume that $m(E) > 0$. By (17) and (18) we find for any $a_1 \dots a_n \in A^{(n)}$

$$\begin{aligned}
E \cap X_{a_1 \dots a_n} &= T^{-n}E \cap \psi_{a_1 \dots a_n}(U_j) \\
&= \psi_{a_1 \dots a_n}(E \cap U_j), \quad U_j = T^n X_{a_1 \dots a_n}.
\end{aligned}$$

From this as well as (19), (21), and (23) we have

$$\begin{aligned}
m(E \cap X_{a_1 \dots a_n}) &\geq 3^{-4} |q_n|^{-4} m(E \cap U_j) \\
&\geq 3^{-8} m(X_{a_1 \dots a_n}) \min \{m(E \cap U_3), m(E \cap U_{-3})\}. \quad (29)
\end{aligned}$$

But (13.3) and (18) implies that

$$E \cap U_3 = T^{-1}E \cap U_3 \supset \psi_{\bar{\xi}+1}(E \cap U_2) \cup \psi(E \cap U_{-2}).$$

Beside for any measurable subset F of U_2 we have by (19) and (20)

$$\begin{aligned}
m(\psi_1(F)) &= \iint_F |1+z|^{-4} dx dy \\
&> \iint_F |\bar{\xi}+1+z|^{-4} dx dy = m(\psi_{\bar{\xi}+1}(F)).
\end{aligned}$$

Hence

$$\begin{aligned}
m(E \cap U_3) &> m(\psi_{\bar{\xi}+1}(E \cap U_2)) + m(\psi_{\bar{\xi}+1}(E \cap U_{-2})) \\
&= m(\psi_{\bar{\xi}+1}(E)) > 3^{-4} |\bar{\xi}+1|^{-4} m(E) = 3^{-6} m(E). \quad (30)
\end{aligned}$$

Similary we have

$$m(E \cap U_{-3}) \geq 3^{-6} m(E) \quad (31)$$

By (29), (30) and (31) the inequality

$$m(E \cap F) \geq 3^{-14} m(E) m(F). \quad (32)$$

hold for all fundamental cell F , and so for any measurable set F in X . Thus, putting $F = X \setminus E$ in (32), we have

$$m(E) m(X \setminus E) = 0$$

which implies $m(E) = 1$.

THEOREM 2. There exists an unique, T -invariant probability measure μ equivalent to Lebesgue measure such that the inequalities

$$3^{-15} \frac{m(E)}{m(X)} \leq \mu(E) \leq 3^{10} \frac{m(E)}{m(X)}, \quad (33)$$

hold for all $E \in \mathcal{B}$.

Proof. To prove the existence it is enough to show that the inequalities

$$3^{-15} m(E) \leq m(T^{-n}E) \leq 3^{10} m(E), \quad (n \geq 0) \quad (34)$$

hold for all $E \in \mathcal{B}$. (see F. Schweiger [5] §6-§7). By (18) (19), (21) and (24) we have

$$\begin{aligned} m(T^{-n}E) &\leq \sum_{A(n)} m(\psi_{a_1 \dots a_n}(E)) \\ &\leq 3^4 m(E) S(n) \leq 3^{10} m(E) \end{aligned}$$

To prove the left-hand side inequalities in (34), we suppose first that $E \subset U_3$. Then, by (18), (19), (21) and Lemma 5, we have

$$\begin{aligned} m(T^{-n}E) &\geq \sum_{j=0}^3 \sum_{A_j(n)} m(\psi_{a_1 \dots a_n}(E)) \\ &\geq 3^{-4} m(E) \sum_{j=0}^3 S_j(n) \geq 3^{-15} m(E), \end{aligned}$$

as required. Similary for any $E \subset U_2 \setminus U_3$

$$m(T^{-n}E) = \sum_{j=0}^2 \sum_{A_j(n)} m(\psi_{a_1 \dots a_n}(E)) \geq 3^{-15} m(E).$$

Besides the left-hand side of the inequalities (34) is also true for any subset E of U_{-2} or $U_{-2} \cup U_{-3}$. As a result (34) holds for any subset E of X , since

$$E = (E \cap U_3) \cup (E \cap (U_2 \setminus U_3)) \cup (E \cap U_{-2}) \cup (E \cap (U_{-2} \setminus U_{-3})).$$

By Theorem 1 the T -invariant probability measure is uniquely given by the limit

$$\mu(E) = \frac{1}{m(X)} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} m(T^{-k}E), \quad E \in \mathcal{B}. \quad (35)$$

(see also F. Schweiger [5].) And so (33) follows from (34) and (35).

THEOREM 3. T is ergodic with respect to μ ; i.e. for any $f \in L^1(X)$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k z) = \int_X f(z) d\mu, \quad \text{a.e.}$$

Proof. Follows from Theorem 1, 2 and Birkhoff's individual ergodic theorem.

As an application of Theorem 3, we have

$$\lim_{n \rightarrow \infty} (a_1(z) + \dots + a_n(z))^{\frac{1}{n}} = e^\alpha, \quad \text{a.e.}$$

where

$$\alpha = \int_X \log a_1(z) d\mu.$$

(Note that $f(z) = \log a_1(z) \in L^1(X)$, since the series $\sum_{a \in N_\zeta} a^{-4} \log a$ is convergent.)

5. Exactness

A measure-preserving transformation T on a normalized measure space $(X, \mathcal{B}, \mu)$ is said to be exact if

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B} = \{\emptyset, X\},$$

or equivalently, if for every set E of positive measure with the measurable images $TE, T^2E, \dots$ the relation

$$\lim_{n \rightarrow \infty} \mu(T^n E) = 1 \quad (36)$$

holds. (see V.A. Rohlin [4])

THEOREM 4. The transformation T is exact.

The proof requires the following

LEMMA 6. Let $\varepsilon > 0$ and let E be any measurable set such that

$$\mu(U_j \setminus E) < \varepsilon$$

for some j ($-3 \leq j \leq 3$). Then

$$\mu(TE) > 1 - 3^{31} \varepsilon.$$

Proof of Lemma 6. It is clearly enough to consider only the case $j = \pm 3$. We may assume further that $j = 3$, since the following arguments are available for the conjugate case $j = -3$. First, by (13.3), we note that

$$\psi_1(U_{-2}) \cup \psi_{\bar{3}+1}(U_2) \subset U_3. \quad (37)$$

But, by (33) and (22) with $a_n = 1$, we have

$$\begin{aligned} \mu(T(\psi_1(U_{-2}) \setminus E)) &\leq 3^{10} m(X)^{-1} m(T(\psi_1(U_{-2}) \setminus E)) \\ &\leq 3^{14} m(X)^{-1} m(\psi_1(U_{-2}) \setminus E) \leq 3^{29} \mu(\psi_1(U_{-2}) \setminus E). \end{aligned} \quad (38)$$

In the same way, (using (22) with $a_n = \bar{3}+1$)

$$\mu(T(\psi_{\bar{3}+1}(U_2) \setminus E)) \leq 3^{31} \mu(\psi_{\bar{3}+1}(U_2) \setminus E). \quad (39)$$

Hence it follows from (37), (38), and (39) that

$$\begin{aligned} &\mu(T((\psi_1(U_{-2}) \cup \psi_{\bar{3}+1}(U_2)) \setminus E)) \\ &\leq 3^{31} \mu((\psi_1(U_{-2}) \cup \psi_{\bar{3}+1}(U_2)) \setminus E) \\ &\leq 3^{31} \mu(U_3 \setminus E) \leq 3^{31} \varepsilon. \end{aligned} \quad (40)$$

Therefore, by (37), (40) and (13.3), we obtain

$$\begin{aligned}\mu(TE) &\geq \mu(T((\psi_1(U_{-2}) \cup \psi_{\bar{\xi}+1}(U_2)) \cap E)) \\ &\geq \mu(T(\psi_1(U_{-2}) \cup \psi_{\bar{\xi}+1}(U_2))) - \mu(T((\psi_1(U_{-2}) \cup \psi_{\bar{\xi}+1}(U_2)) \setminus E)) \\ &> 1 - 3^{31}\varepsilon.\end{aligned}$$

Proof of Theorem 3. We prove (36). Let $E \in \mathcal{B}$ given arbitrary. (Note that, by the definition of T , $E \in \mathcal{B}$ if and only if $TE \in \mathcal{B}$.) Let $\varepsilon > 0$. Then there exists a fundamental interval $F = X_{a_1 \dots a_n}$ such that

$$m(F \setminus E) < 3^{-50}\varepsilon m(F). \quad (41)$$

Otherwise, the inequality

$$m(F \setminus E) \geq 3^{-50}\varepsilon m(F)$$

holds for all fundamental interval F , and so it holds also for arbitrary measurable set F . Putting $F = E$ we have $m(F) = 0$; a contradiction.

Now by Lemma 4, (33), (11), (22), (23) and (41)

$$\begin{aligned}\mu(T^n F \setminus T^n E) &\leq \mu(T^n(F \setminus E)) \\ &\leq 3^{11} m(T^n(F \setminus E)) \leq 3^{15} |q_n|^4 m(F \setminus E) \\ &\leq 3^{19} m(F)^{-1} m(F \setminus E) < 3^{-31}\varepsilon.\end{aligned} \quad (42)$$

Noticing that $T^n F = U_j$ for some j by Lemma 4, we have from (42) and Lemma 6

$$\mu(T^{n+1} E) > 1 - \varepsilon.$$

Since $\mu(E)$, $\mu(TE)$, $\mu(T^n E)$, ... is increasing, the relation (36) is proved.

As a general property of exact transformations (see V.A. Rohlin [4]) we have

Corollary. The transformation T is mixing of all degrees. In particular T is strongly mixing; i.e. for

any $E, F \in \mathcal{B}$ we have

$$\lim_{n \rightarrow \infty} \mu(T^{-n}E \cap F) = \mu(E) \mu(F).$$

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