

AN EXPLICIT DESCRIPTION OF THE RELATIVE $\mathrm{SL}(4, \mathbb{C})$ -CHARACTER VARIETY OF THE PROJECTIVE LINE

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Introduction

- n : a positive integer.
- $\mu = (\mu^1, \dots, \mu^k)$: a k -tuple of partitions of n .
- $(C_1, \dots, C_k)$: a k -tuple of semisimple conjugacy classes such that $C_i \subset \mathrm{SL}(n, \mathbb{C})$ and the multiplicities of eigenvalues of matrices in C_i are given by $\mu^i = (\mu_1^i, \mu_2^i, \dots)$.

Character variety of the projective line

$$\mathcal{R}_{n,k}^\mu := \{(M_1, \dots, M_k) \in C_1 \times \dots \times C_k \mid M_1 \dots M_k = \mathrm{Id}\} / \mathrm{SL}(n, \mathbb{C}).$$

Suppose that $(C_1, \dots, C_k)$ is generic. Then, we have

1. $\mathcal{R}_{n,k}^\mu$ is non-singular;
2. $\mathcal{R}_{n,k}^\mu$ has a holomorphic symplectic structure.

The list of the cases where $\dim \mathcal{R}_{n,k}^\mu = 2$

$\mu = ((11), (11), (11), (11))$	[1]
$\mu = ((111), (111), (111))$	[3]
$\mu = ((1111), (1111), (22))$	main result
$\mu = ((111111), (222), (33))$	unknown.

Problem

Give explicit descriptions of these character varieties. Moreover, give compactifications of these character varieties.

A conjecture due to Simpson (Motivation)

There exists a non-singular compactification of $\mathcal{R}_{n,k}^\mu$ such that the boundary complex is a simplicial decomposition of sphere $S^{\dim \mathcal{R}_{n,k}^\mu - 1}$.

$$((11), (11), (11), (11)) \text{ and } ((111), (111), (111))$$

Invariants of the $\mathrm{SL}(2, \mathbb{C})$ -action.

$$x_i := \mathrm{Tr}(M_i M_j) \quad (i, j, k) = \text{a cyclic permutation of } (1, 2, 3), \\ a_i := \mathrm{Tr} M_i \quad (i = 1, 2, 3), \quad a_1 := \mathrm{Tr}(M_3 M_2 M_1).$$

Relation.

$$x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(a) x_1 - \theta_2(a) x_2 - \theta_3(a) x_3 + \theta_1(a)$$

where

$$\theta_i(a) = a_{\alpha i} + a_{\beta i} \quad (i, j, k) = \text{a cyclic permutation of } (1, 2, 3), \\ \theta_i(a) = a_1 a_2 a_3 a_1 + a_1^2 + a_2^2 + a_3^2 + a_1^2 - 4.$$

Compactification.

Put $x_1 := x/w, x_2 := y/w, x_3 := z/w$. Then, we obtain the following homogeneous polynomial

$$xyz + x^2 w + y^2 w + z^2 w \\ - \theta_1(a) x w^2 - \theta_2(a) y w^2 - \theta_3(a) z w^2 + \theta_1(a) w^3 = 0.$$

Fricke–Klein, et al

The compactification $\overline{\mathcal{R}}_{2,1}^\mu$ is a del Pezzo surface of degree 3. The divisor at infinity is a triangle.

In a similar way as in the case $((11), (11), (11), (11))$, we have

Lawton

The compactification $\overline{\mathcal{R}}_{3,3}^\mu$ is a del Pezzo surface of degree 3. The divisor at infinity is a nodal rational curve.



$$((1111), (1111), (22))$$

Normalization of matrices.

$$\left(\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}, M_3 \right)$$

where $M_3^{-1} = M_1 M_2$.

- $\mathbf{b} = (b_1, \dots, b_4)$: the eigenvalues of matrices of C_2 .
- c_1, c_2 : the eigenvalues of matrices of C_3 .

Relations

$$\begin{aligned} \det(\lambda \mathrm{Id} - M_2) &= \lambda^4 + f_3(\mathbf{b}) \lambda^3 + f_2(\mathbf{b}) \lambda^2 + f_1(\mathbf{b}) \lambda + 1, \\ (M_3 - c_1^{-1})(M_3 - c_2^{-1}) &= \mathbf{0}, \\ \mathrm{Tr}(M_3) &= -2c_1^{-1} - 2c_2^{-1}. \end{aligned} \quad (1)$$

Invariants of the torus action.

$$\begin{aligned} s_{ii} &:= x_{ii}, \\ s_{ij} &:= x_{ij} x_{ji} \quad (i < j), \\ s_{ijk} &:= x_{ij} x_{jk} x_{ki} \quad (i < j, i < k, j \neq k), \\ s_{ijl} &:= x_{ij} x_{jk} x_{kl} x_{li} \quad (i < j, i < k, i < l, j \neq k, k \neq l, l \neq j). \end{aligned}$$

By the equations (1), we have the equations of $s_{ii}, s_{ij}, s_{ijk},$ and s_{ijl} . We may solve these equations for the variables by

$$s_{231}, s_{311}, \text{ and } s_{111}.$$

By the following equations

$$s_{ijl} s_{ijk} = s_{ij} s_{jk} s_{il},$$

we have an equation of $s_{231}, s_{311},$ and s_{111} . The equation defines the character variety $\mathcal{R}_1$ in $\mathbb{C}^3$.

Compactification.

Put $s_{231} := w/x^3, s_{311} := z/x^2, s_{111} := y/x$. Then, we have the following polynomial

$$f_{1,3}^\mu := Aw^2 + \left(z f_1^{(1)}(x, y) + f_3^{(1)}(x, y) \right) w \\ + \left(Bz^3 + z^2 f_2^{(2)}(x, y) + z f_1^{(2)}(x, y) + f_6^{(2)}(x, y) \right)$$

where $f_i^{(j)}$ is a homogeneous polynomial of degree i . We denote by $\overline{\mathcal{R}}_{1,3}^\mu$ the locus defined by $f_{1,3}^\mu$ in $\mathbb{P}(1, 1, 2, 3)$.

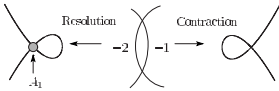
$$\begin{aligned} \phi: \overline{\mathcal{R}}_{1,3}^\mu &\xrightarrow{2:1} \mathbb{P}(1, 1, 2) \\ [x : y : z : w] &\mapsto [x : y : z]. \end{aligned}$$

Singularity (of type A_1).

$$[x : y : z] = \left[0 : 1 : \frac{(a_1 - a_4)(a_1 - a_2)}{(a_2 - a_3)(a_3 - a_1)} \right] \in \mathbb{P}(1, 1, 2).$$

The main result

The compactification $\overline{\mathcal{R}}_{1,3}^\mu$ is a singular del Pezzo surface of degree 1. The surface has a singularity of type A_1 . The divisor at infinity is a nodal rational curve passing through the singularity.

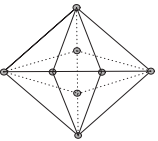


Higher dimensional case

Suppose that $n = 2, k = 5$, and $\mu = ((11), (11), (11), (11), (11))$. Then $\dim \mathcal{R}_{2,5}^\mu = 4$.

We obtain the boundary complex which is a simplicial decomposition of sphere S^3 [2].

The subcomplex associated to S^2 in S^3 :



References

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