

Normal affine subalgebras of a polynomial ring

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Introduction. Let $R := \mathbb{C}[x_1, \dots, x_n]$ be a polynomial ring in n -variables over the complex number field $\mathbb{C}$. A cofinite subalgebra of R is a $\mathbb{C}$ -subalgebra A of R such that R is an A -module of finite type. We consider exclusively a normal cofinite subalgebra A of R . The following results are known by far:

1. A is finitely generated over $\mathbb{C}$, and all invertible element of A are constants, i.e., $A^* = \mathbb{C}^*$.
2. Let X be the normal affine variety defined by A . Then $H_i(X; \mathbb{Z})$ is finite for all $i \geq 1$, X is simply connected and Pic is trivial (Gurjar [3] and Kumar [4]). Therefore A is factorial if X is nonsingular.
3. If $n = 2$ and A is regular, A is then a polynomial ring in two variables over $\mathbb{C}$ (Miyanishi [6]).
4. Suppose $n = 2$. Then X has at worst quotient singularities (Brieskorn [1]). Moreover, if X is affine-ruled, i.e., X , by definition, contains a non-empty Zariski open set of the form $U_0 \times \mathbb{C}$, X has at worst cyclic quotient singularities (Miyanishi [6]).

We complement these results with the following:

THEOREM. Let A be a normal cofinite subalgebra of $\mathbb{C}[x_1, x_2]$ and let $X := \text{Spec } A$. Then either $X \simeq \mathbb{C}^2$ or $X \simeq \mathbb{C}^2/G$, where G is a small finite subgroup of $\text{GL}(2, \mathbb{C})$. If A is factorial, X is isomorphic to a hypersurface in $\mathbb{C}^3$ defined by $x_1^2 + x_2^3 + x_3^5 = 0$.

The theorem holds true even if we replace the ground field $\mathbb{C}$

by an algebraically closed field of characteristic $p \neq 2, 3, 5$. Moreover, the result is viewed as a global version of the following result of Brieskorn [1]:

Among two-dimensional normal singular analytic local rings, a factorial one is isomorphic to $\mathbb{C}\{x, y, z\}/(x^2 + y^3 + z^5)$.

1. Proof of Theorem: Nonsingular case.

Let X be a nonsingular algebraic surface defined over $\mathbb{C}$. Then there exists an open immersion of X into a nonsingular projective surface V such that $D := V - X$ consists of nonsingular curves which cross each other normally. Let K_V be the canonical divisor and denote by the same letter D the reduced effective divisor such that $\text{Supp } D = V - X$. Then we say that X has (logarithmic) Kodaira dimension $\kappa(X) = -\infty$ if $|n(D + K_V)| = \emptyset$ for every $n > 0$. Then the property $\kappa(X) = -\infty$ is independent of the choice of an immersion $X \hookrightarrow V$.

In proving the theorem, the following characterization of $\mathbb{C}^2$ plays a crucial role:

Let $X = \text{Spec } A$ be a two-dimensional affine surface defined over $\mathbb{C}$. Then $X \cong \mathbb{C}^2$ if and only if the following three conditions are satisfied:

(i) A is factorial, (ii) $A^* = \mathbb{C}^*$, (iii) X is affine-ruled.
When X is nonsingular, the condition (iii) is equivalent to (iii)' $\kappa(X) = -\infty$.

(See Miyanishi [5; 6].)

Let X now be the same as in the theorem. Suppose X is nonsingular. Then $A := \Gamma(X, \mathcal{O}_X)$ is factorial by virtue of a result of Gurjar-Kumar, and $A^* = \mathbb{C}^*$ because A is a $\mathbb{C}$ -subalgebra of

$R := \mathbb{C}[x_1, x_2]$. Moreover, since there is a finite morphism $\theta: \mathbb{C}^2 \rightarrow X$, we have $\kappa(X) = -\infty$. Then $X \cong \mathbb{C}^2$ by virtue of the above-mentioned characterization of $\mathbb{C}^2$.

2. Proof of Theorem: Singular case.

We shall assume below that X is singular. Set

$X' := X - \text{Sing } X$, $S = \mathbb{C}^2$, $\theta: S \rightarrow X$ the given finite morphism, $S' := \theta^{-1}(X')$, $\theta' := \theta|_{S'}: S' \rightarrow X'$, $q': Y' \rightarrow X'$ the topological universal covering space of X' .

Then $\kappa(S') = -\infty$, and θ' factors as

$$\theta': S' \xrightarrow{\pi'} Y' \xrightarrow{q'} X'.$$

Hence Y' is a nonsingular algebraic surface, and q' is a finite étale Galois covering with group G . Let

$$A := \Gamma(X', \underline{O}_{X'}) = \Gamma(X, \underline{O}_X),$$

$B :=$ the integral closure of A in the function field $\mathbb{C}(Y')$,

$$R := \mathbb{C}[x_1, x_2] = \Gamma(S, \underline{O}_S),$$

$$Y := \text{Spec } B,$$

$\pi: S \rightarrow Y$ and $q: Y \rightarrow X$: the finite morphisms induced by the canonical inclusions $B \subset R$ and $A \subset B$, respectively.

Then we know that:

- (i) $A = R \cap \mathbb{C}(X)$ and $B = R \cap \mathbb{C}(Y')$;
- (ii) Y is a normal affine surface defined over $\mathbb{C}$ such that Y' is an open set of Y with $\dim(Y - Y') \leq 0$;
- (iii) $\theta = q \cdot \pi$, $\pi' = \pi|_{S'}$, and $q' = q|_{Y'}$;
- (iv) G acts regularly on Y , and $X \cong Y/G$.

On the other hand, Y' is simply connected by the definition, Pic

is a torsion group because S' is a finite covering of Y' , and $\dim(Y-Y') \leq 0$. Therefore $\text{Pic } Y' = (0)$, and the divisor class group $\text{Cl}(Y)$ is trivial, i.e., B is factorial. Since $B \subset R$, we have $B^* = \mathbb{C}^*$. Hence if Y' is affine-ruled, so is Y , and $Y \cong \mathbb{C}^2$ by virtue of the characterization theorem of $\mathbb{C}^2$. The group G then becomes a finite subgroup of $\text{Aut } \mathbb{C}^2 = \text{Aut } \mathbb{C}[x_1, x_2]$, which is, up to conjugation, a finite subgroup of $\text{GL}(2, \mathbb{C})$. Let N be the normal subgroup of G consisting of all pseudo-reflections. Then $\mathbb{C}^2/N$ is isomorphic to $\mathbb{C}^2$, and $X \cong (Y/N)/(G/N) \cong \mathbb{C}^2/(G/N)$. Hence we may assume that G is small, i.e., G contains no pseudo-reflections.

Note that $\kappa(Y') = -\infty$ because S' is a finite covering of Y' and $\kappa(S') = -\infty$. We shall show that Y' is affine-ruled. By reductio absurdum, we assume that Y' is not affine-ruled. Then we have the following:

THEOREM (Tsunoda-Miyayishi [8]). There exist a Zariski open set U of Y' and a proper birational morphism $\phi : U \rightarrow Z$ from U onto a nonsingular algebraic surface Z defined over $\mathbb{C}$ such that:

- (i) Either $U = Y'$ or $Y'-U$ has pure dimension 1;
- (ii) Z is a Platonic $\mathbb{C}^*$ -fiber space.

A nonsingular algebraic surface Z is called a Platonic $\mathbb{C}^*$ -fiber space if there exists a surjective morphism $f : Z \rightarrow \mathbb{P}_{\mathbb{C}}^1$ such that general fibers of f are isomorphic to $\mathbb{C}^*$ and that f has exactly three singular fibers $\Delta_i = \mu_i \Gamma_i$ ($i = 0, 1, 2$; $\mu_0 \leq \mu_1 \leq \mu_2$) with $\Gamma_i \cong \mathbb{C}^*$, where $\{\mu_0, \mu_1, \mu_2\} = \{2, 2, n\}$ ($n \geq 2$), $\{2, 3, 3\}$, $\{2, 3, 4\}$ or $\{2, 3, 5\}$.

Since $U \subset Y' \subset Y$ and Y is affine, U does not contain any complete curve. Therefore $\phi : U \rightarrow Z$ is an isomorphism. We claim that $U = Y'$. Otherwise, since $Y' - U$ has pure dimension 1 and $\text{Pic } Y' = (0)$, there exists an element b of $B = \Gamma(Y', \mathcal{O}_{Y'})$ such that $\text{Supp } (b)_{0, Y'} = \text{Supp}(Y' - U)$. Hence b is invertible on U and $b \notin \mathbb{C}^*$. Meanwhile, there exists a completion W of U such that W is a normal projective surface, W has at worst quotient singularities and $W - U$ consists of two connected components, the one being a single irreducible curve and the other being a single quotient singular point (see Example 1 in the Section 3). Since $(b)_W$ has support on $W - U$, b must be a constant, i.e., $b \in \mathbb{C}^*$. This is a contradiction. Hence $U = Y'$. In order to complete the proof of the first assertion of the theorem we make use of the following:

THEOREM (Miyayishi [7]; Fujita [2]). Let $\tilde{U} \rightarrow U$ be the topological universal covering space of U . Then $\tilde{U}$ is an affine-ruled nonsingular algebraic surface. Moreover we have

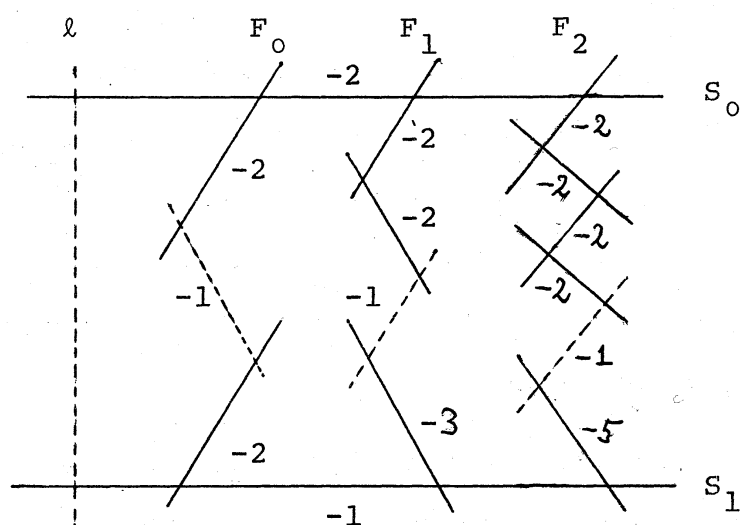
$$\pi_1(U) \cong \begin{cases} D_{2n} & \text{if } \{\mu_0, \mu_1, \mu_2\} = \{2, 2, n\} \\ A_4 & \text{if } \{\mu_0, \mu_1, \mu_2\} = \{2, 3, 3\} \\ S_4 & \text{if } \{\mu_0, \mu_1, \mu_2\} = \{2, 3, 4\} \\ A_5 & \text{if } \{\mu_0, \mu_1, \mu_2\} = \{2, 3, 5\} \end{cases},$$

where D_{2n} is a dihedral group of order $2n$, (see Example 2 of the Section 3).

However Y' is simply connected by the definition. This is apparently a contradiction. Thus Y' is affine-ruled, and we are done.

3. Examples.

(1) Let T be a hypersurface $x_1^2 + x_2^3 + x_3^5 = 0$ in $\mathbb{A}^3$ and let T' be the minimal resolution of the unique singular point $P := (0,0,0)$ of T . Then T' is embedded into a nonsingular projective surface V in such a way that, in the configuration below, the top solid lines represent the exceptional curves arising from the minimal resolution of singularity at P , and the bottom solid lines represent the curves attached to T' to compactify the surface T' :

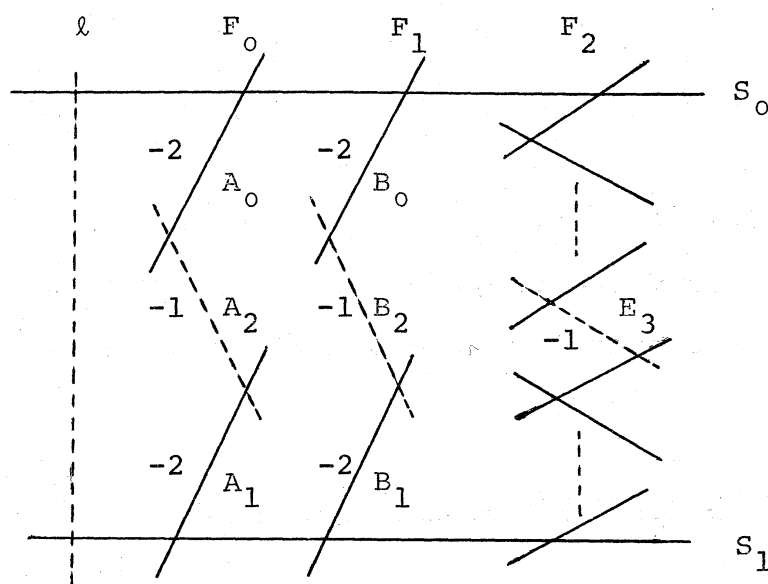


where "solid line" = a nonsingular rational curve, "broken line with weight -1 " = an exceptional curve of the first kind and ℓ is a fiber of f .

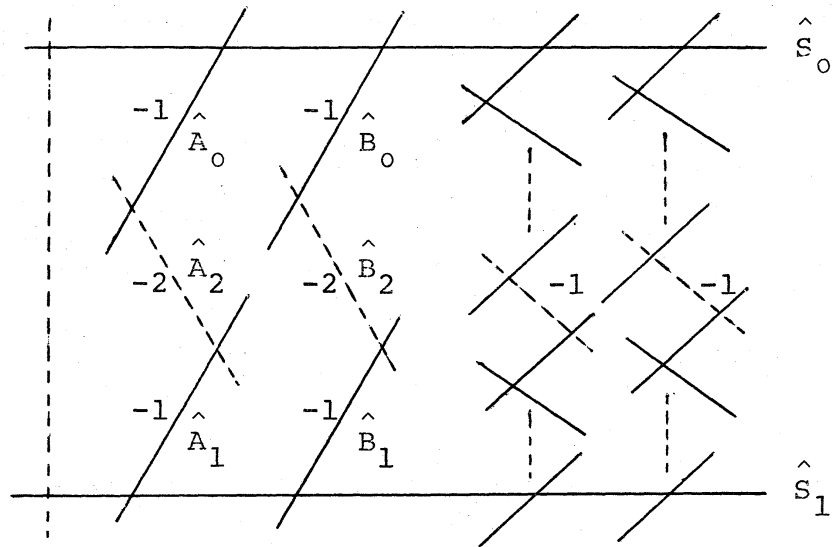
Moreover V has a structure of a $\mathbb{P}^1$ -fibration $f : V \rightarrow C$ over $C \cong \mathbb{P}^1_{\mathbb{C}}$ such that f has three singular fibers F_i ($i = 0, 1, 2$) and two cross-sections S_0, S_1 as indicated in the configuration. Let $U := T - \{P\}$. Then $U \cong V - (\text{all solid lines})$, and U is a Platonic $[*, 2, 3, 5]$ -fiber space with the triple $\{2, 3, 5\}$. The top solid lines contract down back to the quotient singular point P , and the bottom solid lines (sprouting from S_1) contract down to three cyclic quotient singular points. With these contractions performed, we obtain a normal projective surface W such that T is an open set

of W and $W-T$ consists of a single irreducible curve which is the proper transform of S_1 .

(2) Consider a Platonic $\mathbb{C}^*$ -fiber space U with the triple $\{2,2,n\}$. In general, it can be embedded into a nonsingular projective surface V , whose boundary $V-U$ has the configuration as shown in the following picture:



Moreover, V has a structure of a $\mathbb{P}^1$ -fibration $f : V \rightarrow C$ over $C \cong \mathbb{P}_{\mathbb{C}}^1$ for which S_0 and S_1 are cross-sections and F_0, F_1 and F_2 are the only singular fibers. Let D be the reduced effective divisor on V supported by all solid lines. Then $D + K_V \sim \ell - (A_2 + B_2 + E_3)$, where ℓ is a fiber of f . Hence $2(E_3 + D + K_V) \sim A_0 + A_1 + B_0 + B_1$. This implies that there exists a finite double covering $\alpha : \hat{V} \rightarrow V$ ramified only over $A_0 + A_1 + B_0 + B_1$. The surface $\hat{V}$ has a $\mathbb{P}^1$ -fibration $\hat{f} : \hat{V} \rightarrow \hat{C} \cong \mathbb{P}_{\mathbb{C}}^1$ which is induced by the $\mathbb{P}^1$ -fibration $f : V \rightarrow C$ and has the configuration given in the next page. Indeed, $\hat{V}$ is the normalization of $V \times_C \hat{C}$, where $\hat{C} \rightarrow C$ is the double covering ramified over the points $f(F_0)$ and $f(F_1)$.



Now contracting $\hat{A}_0, \hat{A}_1, \hat{B}_0$ and $\hat{B}_1$, we are reduced to the case where $\hat{f}$ has only two singular fibers of the same form as F_2 . It is now a good exercise to show that $\hat{V}$ - (all solid lines) is affine-ruled.

4. Proof of Theorem: The second assertion.

We start with the following situation:

$G \subset GL(2, \mathbb{C})$: a small finite subgroup,

$X := \mathbb{C}^2/G$: a singular normal affine surface,

$P :=$ the unique singular point of X which is the image of the point of origin $(0,0)$ of $\mathbb{C}^2$,

$A := \Gamma(X, \underline{O}_X)$.

Then A is factorial if and only if $\text{Pic}(X - \{P\}) = (0)$. A line bundle L on $X - \{P\}$ is constructed from a multiplicative character χ of G in the following way: Let L be a line bundle on $X - \{P\}$ and let $\theta : \mathbb{C}^2 \rightarrow X$ be the (finite) quotient morphism. Then θ^*L is a trivial line bundle on $\mathbb{C}^2 - \{0\}$. The action of G on θ^*L is given by

$$(x, t) \in (\mathbb{C}^2 - \{0\}) \times \mathbb{C} \longmapsto ({}^g x, \chi(g, x)t) \in (\mathbb{C}^2 - \{0\}) \times \mathbb{C},$$

where $g \in G$ and $\chi(g, x) \in \mathbb{C}^*$. Moreover, we have

$$\chi(gg'; x) = \chi(g; {}^{g'} x) \chi(g'; x) \quad \text{for } g, g' \in G.$$

If $g \in G$ is fixed, then we have

$$\chi(g; x) \in \Gamma(\mathbb{C}^2 - \{0\}, \underline{0}^*) = \mathbb{C}[x_1, x_2]^* = \mathbb{C}^*.$$

Hence $\chi(g, x)$ is independent of x . Write $\chi(g) = \chi(g; x)$. Then $\chi : G \rightarrow \mathbb{C}^*$ is a multiplicative character. Conversely, a multiplicative character χ of G defines a line bundle $L_\chi := (\mathbb{C}^2 - \{0\}) \times \mathbb{C} / G$ with respect to the action of G as described above. Thus we have a 1 - 1 correspondence between

$$L \in \text{Pic}(X - \{P\}) \longleftrightarrow \chi \in \hat{G}.$$

Then we have:

$$\begin{aligned} \text{Pic}(X - \{P\}) = (0) &\iff \hat{G} = \{1\} \\ &\iff G \text{ is a binary icosahedral} \\ &\quad \text{group in } \text{SL}(2, \mathbb{C}) \\ &\iff X \text{ is isomorphic to a} \\ &\quad \text{hypersurface } x_1^2 + x_2^3 + x_3^5 = 0 \\ &\quad \text{in } \mathbb{C}^3. \end{aligned}$$

This completes the proof of the theorem.

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