

# GENERAL ELEMENTS OF IDEALS IN LOCAL RINGS

David Rees (Exeter, England)

In many situations arising in the theory of local rings, it is necessary to make use of elements  $x_1, \dots, x_s$  of ideals  $\mathcal{O}_1, \dots, \mathcal{O}_s$  which are sufficiently general in some sense, depending on the particular situation involved. The purpose of this lecture is to describe a general set-up in which such general elements can be defined which satisfy the required conditions in most such situations and to give an illustration of its application.

We suppose that  $(Q, \mathfrak{m}, k)$  is a local ring of dimension  $d$ . We first construct the general extension  $Q_g$  of  $Q$ . Let  $x_1, x_2, \dots$  be a countable sequence of indeterminates over  $Q$ . Then  $Q_g$  is the localisation of  $Q[x_1, x_2, \dots]$  at the prime ideal  $\mathfrak{m}[x_1, x_2, \dots]$ . It follows from a general result of Grothendieck that  $Q_g$  is noetherian (alternatively one can prove that if  $\mathcal{O}$  is a finitely generated ideal of  $Q_g$ , then  $\bigcap_{n=1}^{\infty} (\mathcal{O} + \mathfrak{m}_g^n) = \mathcal{O}$ , and then, observing that the completion of  $Q_g$  is noetherian, use the above to show that if  $\mathcal{O}\overline{Q}_g = \mathcal{O}'\overline{Q}_g$  where  $\mathcal{O}'$  is a finitely generated ideal of  $Q_g$  contained in  $\mathcal{O}$ , then  $\mathcal{O} = \mathcal{O}'$ .)

Now suppose that  $\mathcal{O}_1, \dots, \mathcal{O}_s$  are ideals of  $Q$ , and that  $\mathcal{O}_1$  has a basis  $a_{i1}, \dots, a_{im_1}$ . Write  $M_1 = \mathfrak{m}_1 + \dots + \mathfrak{m}_s$ . Then we term  $x_1, \dots, x_s$  an independent set of general elements of  $\mathcal{O}_1, \dots, \mathcal{O}_s$  if there exists an automorphism  $T$  of  $Q_g$  over  $Q$  such that

$$T(x_i) = \sum_{j=1}^{m_i} X_{M_{i-1}+j} a_{ij} \quad (i = 1, \dots, s).$$

It is a simple matter to prove that this definition is independent of the choice of bases of  $\mathcal{O}_1, \dots, \mathcal{O}_s$ . It also follows that the ideal  $(x_1, \dots, x_s) \cap Q$  of  $Q$  and the  $Q$ -algebra  $Q_g/(x_1, \dots, x_s)$  (to within isomorphism as a  $Q$ -algebra) depend only on the ideals  $\mathcal{O}_1, \dots, \mathcal{O}_s$ . I will only consider the first in the case when the ideals  $\mathcal{O}_1, \dots, \mathcal{O}_s$  are all equal to  $\mathcal{O}$ . Let  $a(\mathcal{O})$  denote the analytic spread of  $\mathcal{O}$ , and  $v(\mathcal{O})$  the minimal number of generators of  $\mathcal{O}$ . Then

- i) if  $s < a(\mathcal{O})$ , the ideal  $(x_1, \dots, x_s) \cap Q$  is nilpotent;
- ii) if  $s = a(\mathcal{O})$ ,  $(x_1, \dots, x_s)$  is a reduction of  $\mathcal{O}Q_g$  and hence  $(x_1, \dots, x_s) \supseteq \mathcal{O}^n Q_g$  for  $n$  large, and hence  $(x_1, \dots, x_s) \cap Q$  contains a power of  $\mathcal{O}$ ;
- iii) if  $s \geq v(\mathcal{O})$ , we have  $(x_1, \dots, x_s) \cap Q = \mathcal{O}$ .

Now we consider the second. In this case we will be concerned with the case when  $s = d-1$  or  $d$ , and the ideals  $\mathcal{O}_1, \dots, \mathcal{O}_s$  are all  $\mathcal{M}$ -primary. Let  $N$  be any integer and define  $Q_N$  to be the ring  $Q[Y_1, \dots, Y_N]$  localised at  $\mathcal{M}[Y_1, \dots, Y_N]$ ,  $Y_1, \dots, Y_N$  being indeterminates over  $Q$ . If we replace  $Y_i$  by  $X_i$ , it is clear that we can consider  $Q_N$  as a subring of  $Q_g$ . Now suppose that  $\mathcal{O}$  is any ideal of  $Q_g$ . Then for some  $N$ ,  $\mathcal{O}$  is generated by elements of the sub-ring  $Q_N$  of  $Q_g$  and therefore  $\mathcal{O} = (\mathcal{O} \cap Q_N)Q_g$ . Now we have an isomorphism of  $(Q_g)_N \rightarrow Q_g$  in which  $X_i$  maps to  $X_{N+i}$  and  $Y_i \rightarrow X_i$  for  $i = 1, \dots, N$ . It follows that  $Q_g/\mathcal{O}$  is isomorphic to

$(Q_g)_N/\alpha^*$ , where  $\alpha^*$  is an ideal of  $(Q_g)_N$  meeting  $Q_g$  in  $(Q \cap \alpha)Q_g$ . The case that will concern us is when  $\alpha$  is generated by general elements  $x_1, \dots, x_{d-1}$  of  $\mathfrak{m}$ -primary ideals  $\alpha_1, \dots, \alpha_{d-1}$  of  $Q$ . For simplicity of exposition, we will restrict ourselves to the case when  $Q$  is a domain. Then  $Q_g/(x_1, \dots, x_{d-1})$  is a local ring of dimension 1. Now suppose  $y_i, z_i$  ( $i = 1, \dots, d-1$ ) is a set of independent general elements of the ideals  $\alpha_1, \alpha_1, \dots, \alpha_{d-1}, \alpha_{d-1}$ . Now choose  $N$  so that the elements  $y_i, z_i$  ( $i = 1, \dots, d-1$ ) are all contained in the sub-ring  $Q_N$  of  $Q_g$ . Then it is not difficult to prove that the elements  $w_i = y_i - x_{N+i}z_i$  ( $i = 1, \dots, d-1$ ) form a set of independent general elements of  $\alpha_1, \dots, \alpha_{d-1}$ . We further note that for each  $i$ , the elements  $y_i, z_1, \dots, z_{d-1}$  generate an  $\mathfrak{m}Q_g$ -primary ideal of  $Q_g$ . We now quote a general result which will be proved in an appendix:

Let  $Q$  be a local domain of dimension  $d$ , and let  $y_i, z_i$  ( $i = 1, \dots, d-1$ ) be elements of  $Q$  such that  $y_i, z_1, \dots, z_{d-1}$  generate an  $\mathfrak{m}$ -primary ideal for each  $i$ . Then, if  $B$  is the ring

$$Q[y_1/z_1, \dots, y_{d-1}/z_{d-1}],$$

i)  $B/\mathfrak{m}B$  is isomorphic to  $k[X_1, \dots, X_{d-1}]$ , where  $k = Q/\mathfrak{m}$ , and  $X_1, \dots, X_{d-1}$  are indeterminates over  $k$ ;

ii) if  $L$  denotes  $B$  localised at the prime ideal  $\mathfrak{m}[y_1/z_1; \dots, y_{d-1}/z_{d-1}]$ , and  $Q(X)$  denotes the ring  $Q[X_1, \dots, X_{d-1}]$  localised at  $\mathfrak{m}[X_1, \dots, X_{d-1}]$ , where  $X_1, \dots, X_{d-1}$  are indeterminates over  $Q$ , then the kernel of the homomorphism of  $Q(X)$  onto  $L$  in which  $X_i \mapsto y_i/z_i$  ( $i = 1, \dots, d-1$ ) is a prime ideal  $\mathfrak{P}$  containing the ideal  $\mathfrak{X} = (y_1 - z_1 X_1, \dots, y_{d-1} - z_{d-1} X_{d-1})$  and  $\mathfrak{P}/\mathfrak{X}$  is annihilated by a power of  $\mathfrak{m}$ .

Applying this result, we see that, replacing  $Q$  by  $Q_g$  and giving  $y_i, z_i$  their original meaning, the ring  $L$  obtained in this situation is isomorphic to  $Q_g/(x_1, \dots, x_{d-1}) : \mathfrak{m}^n$  if  $n$  is large enough.

It follows that we can consider  $L$  in two ways, first as a homomorphic image of  $Q_g$ , and second as a local ring containing  $Q_g$  and contained in its field of fractions  $F_g$ . Further the maximal ideal of  $L$  is  $\mathfrak{m}L$  and  $\mathfrak{m}L \cap Q_g = \mathfrak{m}Q_g$ . Now  $L$  is 1-dimensional. Hence, by the Krull-Akizuki theorem, the integral closure  $L^*$  of  $L$  in  $F_g$  is the intersection of a finite set of discrete valuation rings. Let the associated valuations be  $V_1, \dots, V_q$  and let their restriction to the field of fractions  $F$  of  $Q$  be  $v_1, \dots, v_q$ . Then  $v_1, \dots, v_q$  are independent of the choice of the elements  $y_i, z_i$ .

Now we must digress to consider valuations on  $Q_g$ . Suppose that  $V$  is a valuation  $\geq 0$  on  $Q_g$ , and  $> 0$  on  $\mathfrak{m}Q_g$ , and taking integer values. If  $K_V$  is the residue field of  $V$ , then  $K_V$  is an extension of  $k_g$ , and an old result of Zariski states that  $\text{tr.deg}_{k_g} K_V \leq d-1$ . Now let  $v$  be the restriction of  $V$  to  $F$ . Then it is quite easy to prove that

$$\text{tr.deg}_{k_g} K_V \geq \text{tr.deg}_k K_v.$$

Now I recall another old result, due in this case to Northcott. Let  $K$  denote the residue field of  $L$  (which is a pure transcendental extension of  $k_g$  of transcendence degree  $d-1$ ). Now the valuations  $V_i$  already referred to have an extension to

the completion  $\bar{L}$  of  $L$  which we denote by  $\bar{V}_i$ , and each such extension  $\bar{V}_i$  takes the value  $\infty$  on a minimal prime ideal  $\mathfrak{P}_i$  of  $\bar{L}$ . Let  $\delta_i$  denote the length of the primary component of  $(0)$  in  $\bar{L}$  with associated prime  $\mathfrak{P}_i$ . Then if  $x \in L$ ,

$$e(xL) = \ell(L/xL) = \sum_{i=1}^q \delta_i [K_{V_i} : K] v_i(x)$$

where  $e(\cdot)$  is the multiplicity.

Now we turn to multiplicities and degree functions. Following Teissier, we will use mixed multiplicities. Let  $\alpha_1, \dots, \alpha_d$  be  $d$   $\mathcal{M}$ -primary ideals of  $Q$ , and let  $M$  be a finitely generated  $Q$ -module. Then we define  $e(\alpha_1, \dots, \alpha_d; M)$  as  $e(x_1, \dots, x_d; M)$  where  $x_1, \dots, x_d$  are independent general elements of  $\alpha_1, \dots, \alpha_d$ . Then we have the result that if  $L$  is as described earlier,

$$e(\alpha_1, \dots, \alpha_d) = e(x_d L) = e(\alpha_d L),$$

the latter following since  $x_d L$  is a reduction of  $\alpha_d L$ . Further this latter remark also implies that, if  $V_i, v_i$  have the meanings given earlier, then  $V_i(x_d) = v_i(\alpha_d)$  where the latter denotes the minimum value of  $v_i(x)$  on  $\alpha_d$ . We further note that

$e(\alpha_1, \dots, \alpha_d; M)$  is a symmetric function of  $\alpha_1, \dots, \alpha_d$  and, if  $\alpha'_d$  is another  $\mathcal{M}$ -primary ideal of  $Q$ , then

$$e(\alpha_1, \dots, \alpha_d, \alpha'_d; M) = e(\alpha_1, \dots, \alpha_d; M) + e(\alpha_1, \dots, \alpha'_d; M)$$

we can now write down a formula for the multiplicity symbol

$$e(\alpha_1, \dots, \alpha_d; Q) = \sum_{i=1}^q \delta_i [K_{V_i} : K] v_i(\alpha_d)$$

and similar formulae arising from the symmetry of the symbol.

However this formula attains its full force if we introduce

degree functions. We define the degree function  $d(\alpha_1, \dots, \alpha_{d-1}; x)$  where  $x$  is an element of  $Q$  to be  $e(\alpha'_1, \dots, \alpha'_{d-1}; Q')$  where  $Q' = Q/x$  and  $\alpha'_i = (\alpha_i + xQ)/xQ$ . If  $Q$  is a domain, this can also be written as  $e(x_1, \dots, x_{d-1}, x; Q)$  and we obtain the expression

$$d(\alpha_1, \dots, \alpha_{d-1}; x) = \sum_{i=1}^q \delta_i [K_{V_i} : K] v_i(x).$$

### APPENDIX

First we prove a lemma which is well known.

LEMMA. Let  $B$  be a noether domain,  $y, z$  elements of  $B$  such that  $(y, z)$  has height 2. Let  $B'$  be the ring  $B[y/z]$  and let  $\mathfrak{P}$  be the kernel of the map  $B[Y] \rightarrow B'$  in which  $Y \mapsto y/z$ . Then  $\mathfrak{P}$  contains  $w = zY - y$ , and

$$wB[Y] : (z^m, y^m) = \mathfrak{P}$$

if  $m$  is sufficiently large. Further, if  $\mathfrak{m}$  is any prime ideal of  $B$  containing  $(y, z)$ , then  $B'/\mathfrak{m}B' \cong (B/\mathfrak{m})[X]$ , where  $X$  is an indeterminate over  $B'/\mathfrak{m}B'$ .

Proof. Let  $f(Y)$  be a polynomial of degree  $r$  over  $B$  such that  $f(y/z) = 0$ . Then we can write  $f(Y) = F(Y, 1)$  where  $F(Y, Z)$  is a homogeneous polynomial over  $B$  of degree  $r$  such that  $F(y, z) = 0$ . Then

$$\begin{aligned} z^r F(Y, Z) &= F(zY, zZ) = F(yZ + (zY - yZ), zZ) \\ &= F(yZ, zZ) + (zY - yZ)G(Y, Z) \quad \text{by Taylor's Theorem} \\ &= z^r F(y, z) + (zY - yZ)G(Y, Z) \end{aligned}$$

whence, by putting  $Z = 1$ , we see that  $z^r f(Y) \in wB[Y]$ . Also,

$$y^r f(Y) = (y^r - z^r Y^r) f(Y) + Y^r z^r f(Y) \in wB[Y].$$

But as the ascending sequence of ideals  $wB[Y] : (y^r, z^r)$  becomes stationary for large  $r$ , it follows that

$$\mathfrak{P} = wB[Y] : (y^m, z^m) \quad m \text{ large.}$$

Hence  $\mathfrak{P}$  is the radical of  $wB[Y]$  and since  $y, z \in \mathfrak{M}$ ,  $w \in \mathfrak{M}B[Y]$ , i.e.  $\mathfrak{P} \subset \mathfrak{M}B[Y]$ , which proves the result.

We now come to the main result of this appendix.

**THEOREM.** Let  $(Q, \mathfrak{M}, k)$  be a local domain of dimension  $d \geq 2$ , and let  $y_i, z_i$  ( $i = 1, \dots, d-1$ ) be elements of  $\mathfrak{M}$  such that  $(y_i, z_1, \dots, z_{d-1})$  is  $\mathfrak{M}$ -primary for  $i = 1, \dots, d-1$ . Let  $u_i = y_i/z_i$  and  $B = Q[u_1, \dots, u_{d-1}]$ . Then

$$B/\mathfrak{M}B \cong k[X_1, \dots, X_{d-1}]$$

where  $X_1, \dots, X_{d-1}$  are indeterminates over  $k$ , implying that  $\mathfrak{M}B$  is prime.

Further let  $L = B_{\mathfrak{M}B}$  and let  $Q_{d-1}$  denote  $Q[X_1, \dots, X_{d-1}]$  localised at  $\mathfrak{M}[X_1, \dots, X_{d-1}]$ . Let  $\mathfrak{P}$  denote the kernel of the homomorphism  $Q_{d-1} \rightarrow L$  in which  $X_i \rightarrow u_i$ . Let  $w_i = z_i X_i - y_i$  and let  $\mathfrak{X}$  be the ideal  $(w_1, \dots, w_{d-1})$ . Then for  $r$  large,

$$\mathfrak{M}^r \mathfrak{P} \subset \mathfrak{X}.$$

Proof. The proof will be by induction on  $d$ , the case  $d=2$  following from the lemma. Now suppose that  $d > 2$ . Write  $Q'$  for  $Q[u_{d-1}]$  localised at  $\mathfrak{M}[u_{d-1}]$ , which is prime by the lemma. We first prove that  $(y_i, z_1, \dots, z_{d-2})Q'$  is  $\mathfrak{M}Q'$ -primary for  $i = 1, \dots, d-2$ . Now, by the lemma,  $Q' \cong Q(X_{d-1})/\mathfrak{P}'$ , where  $Q(X_{d-1})$  denotes  $Q[X_{d-1}]$  localised at  $\mathfrak{M}[X_{d-1}]$ , and  $\mathfrak{P}'$  is the radical of  $w_{d-1}Q(X_{d-1})$ . Hence it will be sufficient to show that  $(w_{d-1}, y_i, z_1, \dots, z_{d-2})$  is  $\mathfrak{M}Q(X_{d-1})$ -primary. Write

$$C_i = y_i Q(X_{d-1}) + z_1 Q(X_{d-1}) + \dots + z_{d-2} Q(X_{d-1}).$$

Then the minimal prime ideals of  $C_i$  are generated by elements of  $Q$  and so can only contain  $w_{d-1}$  if it contains  $y_{d-1}, z_{d-1}$ . Since  $C_i + z_{d-1} Q(X_{d-1})$  is  $\mathfrak{m}$ -primary,  $\dim C_i = 1$ , and since  $w_{d-1}$  belongs to no minimal prime of  $C_i$ , the result now follows.

Now we consider the first statement of the theorem. It is clearly equivalent to the statement that if  $f(X_1, \dots, X_{d-1})$  is a polynomial over  $Q$  such that  $f(u_1, \dots, u_{d-1}) = 0$ , then all the coefficients of  $f$  belong to  $\mathfrak{m}$ . Suppose there is a coefficient of  $f$  not in  $\mathfrak{m}$ . Then if we consider the polynomial  $f(X_1, \dots, X_{d-2}, u_{d-1})$  as a polynomial with coefficients in  $Q'$ , then the lemma implies that this has a coefficient not in  $\mathfrak{m}Q'$ . But  $Q'$  has dimension  $d-1$  and the conditions of the theorem apply. Hence by our inductive hypothesis  $f(u_1, \dots, u_{d-1}) \neq 0$ .

We are now in a position to construct  $L$ . Consider the homomorphism  $Q_{d-1} \rightarrow L$ . This can be factored as the product of the homomorphism  $Q_{d-1} \rightarrow Q'_{d-2}$  in which  $X_{d-1} \rightarrow u_{d-1}$  and the homomorphism  $Q'_{d-2} \rightarrow L$ . Denote by  $\mathcal{O}$  the kernel of the homomorphism  $Q_{d-1} \rightarrow Q'_{d-2}$ . Applying the inductive hypothesis to the second factor, we see that, for  $r$  large,

$$\mathfrak{m}^r \not\subset \mathcal{O} + (w_1, \dots, w_{d-2})$$

while, by the lemma,

$$(y_{d-1}^m, z_{d-1}^m) \mathcal{O} \subset w_{d-1} Q_{d-1}.$$

Hence

$$(y_{d-1}^m, z_{d-1}^m) \mathfrak{m}^r \not\subset (w_1, \dots, w_{d-1}) = \mathcal{X}.$$

But by reordering the suffixes  $1, \dots, d-1$ , we can replace  $d-1$  on the left hand side by  $i$  ( $i = 1, \dots, d-2$ ). Hence if  $m, r$  are

large enough,

$$(y_1^m, \dots, y_{d-1}^m, z_1^m, \dots, z_{d-1}^m)^{m^r} \mathfrak{p} \subset \mathfrak{I}$$

and the result follows since the first factor is  $m$ -primary.