

Non-homogeneous semilinear elliptic equations involving critical Sobolev exponent

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Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$ with $N \geq 3$. We consider the existence of multiple positive solutions of the following semilinear elliptic equations

$$(1.1) \quad \begin{cases} -\Delta u + \kappa u = u^p + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\kappa \in \mathbf{R}$, $\lambda > 0$ are parameters, p is the critical Sobolev exponent $p = (N+2)/(N-2)$, and $f(x)$ is a non-homogeneous perturbation satisfying

$$(1.2) \quad f \in H^{-1}(\Omega), \quad f \geq 0, \quad f \not\equiv 0 \quad \text{a.e. in } \Omega.$$

Since p is a critical Sobolev exponent for which the embedding $W^{1,2}(\Omega) \subset L^{2N/(N-2)}(\Omega)$ is not compact, we encounter serious difficulties in applying variational methods to the problem (1.1).

Let us recall the results for the case $f \equiv 0$;

$$(1.3) \quad \begin{cases} -\Delta u + \kappa u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this case, by using the Pohozaev identity, it can be shown that (1.3) admits no nontrivial solutions for each $\kappa \geq 0$, provided that Ω is star-shaped. On the other hand, Brezis and Nirenberg [1] obtained the following results when $\kappa < 0$: let κ_1 be the first eigenvalue of $-\Delta$ with zero Dirichlet condition on Ω ; then

- (i) if $N \geq 4$, then for every $\kappa \in (-\kappa_1, 0)$, there exists a positive solution;
- (ii) if $N = 3$ and Ω is a ball, then there exists a positive solution if and only if $\kappa \in (-\kappa_1, -\kappa_1/4)$.

Let us consider the case where f satisfies (1.2). Tarantello [6] considered the problem with $\kappa = 0$;

$$(1.4) \quad \begin{cases} -\Delta u = u^p + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and showed that (1.4) has at least two positive solution if λ is small enough. The main idea is to divide the Nehari manifold $\Lambda = \{u \in H_0^1(\Omega) : \langle I'(u), u \rangle = 0\}$ into three parts Λ^+ , Λ^- and Λ_0 , and to use the Ekeland principle to get one solution for Λ^+ and another solution for Λ^- . We note here that no positive solution exists if λ is sufficiently large.

The existence of two nontrivial solutions for more general problem

$$\begin{cases} -\Delta u = u^p + g(x, u) + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g(x, u)$ is a suitable lower-order perturbation of u^p , was proved by Cao and Zhou [2]. These achievements have been extended to the p -Laplace equation by Chabrouski [3] and Zhou [7], and to more general problems by Squassina [5].

In this paper we will consider the problem (1.1) with $\kappa \in \mathbf{R}$ in the case where f satisfies (1.2), and show that, when $\kappa > 0$, the situation is drastically different between the cases $N = 3, 4, 5$ and $N \geq 6$.

We call a positive minimal solution $\underline{u}_\lambda$ of $(1.1)_\lambda$, if $\underline{u}_\lambda$ satisfies $\underline{u}_\lambda \leq u$ in Ω for any positive solution u of $(1.1)_\lambda$. Our main results are stated as following theorems.

Theorem 1. *Assume that $\kappa > -\kappa_1$. Then there exists $\bar{\lambda} \in (0, \infty)$ such that*

- (i) *if $0 < \lambda < \bar{\lambda}$ then the problem $(1.1)_\lambda$ has a positive minimal solution $\underline{u}_\lambda \in H_0^1(\Omega)$. Furthermore, if $0 < \lambda < \bar{\lambda} < \bar{\lambda}$ then $\underline{u}_\lambda < \underline{u}_{\bar{\lambda}}$ a.e. in Ω ;*
- (ii) *if $\lambda > \bar{\lambda}$ then the problem (1.1) has no positive solution $u \in H_0^1(\Omega)$.*

Remark. There is no positive solution of (1.1) with $\kappa \leq -\kappa_1$. Assume to the contrary that there exists a positive solution u of (1.1) with $\kappa \leq -\kappa_1$. Let ϕ_1 be the eigenfunction of $-\Delta$ corresponding to κ_1 with $\phi_1 > 0$ on Ω . Then we have

$$0 = \int_{\Omega} \nabla u \cdot \nabla \phi_1 - \kappa_1 u \phi_1 dx \geq \int_{\Omega} \nabla u \cdot \nabla \phi_1 + \kappa u \phi_1 dx = \int_{\Omega} u^p \phi_1 + \lambda f \phi_1 dx > 0.$$

This is a contradiction.

We consider the existence of the solutions of (1.1) at the extremal value $\lambda = \bar{\lambda}$, so called extremal solutions.

Theorem 2. *Let $\kappa > -\kappa_1$. If $\lambda = \bar{\lambda}$ then the problem (1.1) has a unique positive solution in $H_0^1(\Omega)$.*

Next, let us consider the existence and nonexistence of second positive solutions to (1.1) for $0 < \lambda < \bar{\lambda}$.

Theorem 3. *Assume that either (i) or (ii) holds.*

- (i) $\kappa \in (-\kappa_1, 0]$ and $N \geq 3$; (ii) $\kappa > 0$ and $N = 3, 4, 5$.

If $0 < \lambda < \bar{\lambda}$ then (1.1) has a positive solution $\bar{u}_\lambda \in H_0^1(\Omega)$ satisfying $\bar{u}_\lambda > \underline{u}_\lambda$.

Theorem 4. *Assume that $\kappa > 0$ and $N \geq 6$.*

- (i) *There exists $\lambda^* = \lambda^*(\kappa) \in (0, \bar{\lambda})$ such that if $\lambda^* < \lambda < \bar{\lambda}$ then the problem (1.1) has a positive solution $\bar{u}_\lambda \in H_0^1(\Omega)$ satisfying $\bar{u}_\lambda > \underline{u}_\lambda$.*
- (i) *Let $\Omega = \{x \in \mathbf{R}^N : |x| < R\}$ with some $R > 0$, and let $f = f(|x|)$ be radially symmetric about the origin. Assume that $f \in C^\alpha([0, R])$ with some $0 < \alpha < 1$, and $f(r)$ is nonincreasing in $r \in (0, R)$. Then there exists $\lambda_* \in (0, \lambda^*)$ such that (1.1) $_\lambda$ has a unique positive solution $\underline{u}_\lambda$ for $\lambda \in (0, \lambda_*]$.*

In the proof of Theorem 1, we will employ the bifurcation results and the comparison argument for solutions of (1.1) to obtain the minimal solutions. We will prove Theorem 2 by establishing a priori bound for the solutions of (1.1) at $\lambda = \bar{\lambda}$.

In order to find a second positive solution of (1.1), we introduce the problem

$$(1.5) \quad -\Delta v + \kappa v = (v + \underline{u}_\lambda)^p - \underline{u}_\lambda^p \quad \text{in } \Omega, \quad v \in H_0^1(\Omega),$$

where $\underline{u}_\lambda$ is the minimal positive solution of (1.1) for $\lambda \in (0, \bar{\lambda})$ obtained in Theorem 1. In fact, assume that (1.5) has a positive solution v , and put $\bar{u}_\lambda = v + \underline{u}_\lambda$. Then $\bar{u}_\lambda \in H_0^1(\Omega)$ and solves (1.1) and satisfies $\bar{u}_\lambda > \underline{u}_\lambda$ in Ω . In the proof of Theorem 3, we will show the existence of solutions of (1.5) by using a variational method. To this end we define the corresponding variational functional of (1.5) by

$$I_\kappa(v) = \frac{1}{2} \int_\Omega (|\nabla v|^2 + \kappa v^2) dx - \int_{\mathbf{R}^N} G(v, \underline{u}_\lambda) dx$$

for $v \in H_0^1(\Omega)$, where

$$G(t, s) = \frac{1}{p+1} (t_+ + s)^{p+1} - \frac{1}{p+1} s^{p+1} - s^p t_+.$$

It is easy to see that $I_\kappa : H_0^1(\Omega) \rightarrow \mathbf{R}$ is C^1 and the critical point $v_0 \in H_0^1(\Omega)$ satisfies

$$\int_\Omega (\nabla v_0 \cdot \nabla \psi + \kappa v_0 \psi + g(v_0, \underline{u}_\lambda) \psi) dx = 0$$

for any $\psi \in H_0^1(\Omega)$, where

$$g(t, s) = (t_+ + s)^p - s^p.$$

Denote by S the best Sobolev constant of the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$, which is given by

$$S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\left(\int_{\Omega} |u|^{p+1} dx \right)^{2/(p+1)}}.$$

We will obtain Theorem 3 as a consequence of the following two propositions.

Proposition 5. *Let $\lambda \in (0, \lambda^*)$. Assume that there exists $v_0 \in H_0^1(\Omega)$ with $v_0 \geq 0$, $v_0 \not\equiv 0$ such that*

$$(1.6) \quad \sup_{t>0} I_{\kappa}(tv_0) < \frac{1}{N} S^{N/2}.$$

Then there exists a positive solution $v \in H_0^1(\Omega)$ of (1.5).

Proposition 6. *Assume that either (i) or (ii) holds.*

$$(i) \ \kappa \in (-\kappa_1, 0] \text{ and } N \geq 3; \quad (ii) \ \kappa > 0 \text{ and } N = 3, 4, 5.$$

Then there exists a positive function $v_0 \in H_0^1(\Omega)$ such that (1.6) holds.

In the proof of Proposition 5, we will derive some estimates to establish inequalities relating certain minimizing sequences. In order to prove Proposition 6, for $\varepsilon > 0$, we will set

$$u_{\varepsilon}(x) = \frac{\phi(x)}{(\varepsilon + |x|^2)^{(N-2)/2}},$$

where $\phi \in C_0^{\infty}(\mathbf{R}^N)$, $0 \leq \phi \leq 1$, is a cut off function, and will show that (1.6) holds with $v_0 = u_{\varepsilon}$ for sufficiently small $\varepsilon > 0$.

In the proof of Theorem 4 (ii), we will verify the nonexistence of positive solutions of (1.5) in the radial case by the Pohozaev type argument for the associated ODE. In fact, by [4], the solution v of (1.5) must be radially symmetric, and $v = v(r)$, $r = |x|$, satisfies the problem of the following ordinary differential equation

$$(1.7) \quad \begin{cases} (r^{N-1}v_r)_r - \kappa r^{N-1}v + r^{N-1}g(v, \underline{u}_{\lambda}) = 0, & 0 < r < R, \\ v_r(0) = v(R) = 0. \end{cases}$$

For the solution v to (1.7), we will obtain the following Pohozaev type identity:

$$\begin{aligned} \int_0^R r^{N-1} \left[\frac{2N}{N-2} G(u, \underline{u}_{\lambda}) - g(u, \underline{u}_{\lambda})u \right] dr + \frac{2}{N-2} \int_0^R r^N G_s(u, \underline{u}_{\lambda}) \underline{u}'_{\lambda} dr \\ + \frac{2\kappa}{N-2} \int_0^{\infty} r^{N-1} u^2 dr = \frac{1}{N-2} R^N v_r(R)^2. \end{aligned}$$

In the proofs of Theorems 2, 3 and 4, the results concerning the eigenvalue problems to the linearized equations around the minimal solutions

$$-\Delta\phi + \phi = \mu p(\underline{u}_\lambda)^{p-1}\phi \quad \text{in } \Omega, \quad \phi \in H_0^1(\Omega).$$

play a crucial role.

References

- [1] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent, *Comm. Pure Appl. Math.* 36 (1983) 437–477.
- [2] D.-M. Cao and H.-S. Zhou, On the existence of multiple solutions of nonhomogeneous elliptic equations involving critical Sobolev exponents, *Z. Angew. Math. Phys.* 47 (1996) 89–96.
- [3] J. Chabrowski, On multiple solutions for the nonhomogeneous p-Laplacian with a critical Sobolev exponent, *Differential Integral Equations* 8 (1995) 705–716.
- [4] B. Gidas, W.-M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* 68 (1979) 209–243.
- [5] M. Squassina, Two solutions for inhomogeneous nonlinear elliptic equations at critical growth, *NoDEA Nonlinear Differential Equations Appl.* 11 (2004) 53–71.
- [6] G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 9 (1992) 281–304.
- [7] H. S. Zhou, Solutions for a quasilinear elliptic equation with critical sobolev exponent and perturbations on R^n , *Differential Integral Equations* 13 (2000) 595–612.